Poisson 2022, Poisson geometry minicourse: problem set

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Problem 1: Let π be a bivector field on M, consider the corresponding bracket on smooth functions given by $\{f, g\} = \pi(df, dg)$ and the trilinear operation $Jac(f, g, h) = \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\}$. Verify that

$$Jac(f, g, h) = \mathcal{L}_{[X_f, X_g]} h - \mathcal{L}_{X_{\{f,g\}}} h = (\mathcal{L}_{X_f} \pi)(dg, dh).$$

For a regular Poisson manifold (M, π), use the first equality to show that the characteristic distribution $D = \pi^\sharp(T^*M) \subseteq TM$ is involutive, and hence integrable.

Problem 2: Let (M_1, π_1) and (M_2, π_2) be Poisson manifolds. Show that $\varphi : M_1 \to M_2$ is a Poisson map iff $X_{\varphi^* f}$ is φ-related to X_f for all $f \in C^\infty(M)$ iff $\pi_2^\sharp|_{\varphi(x)} = T_x\varphi \circ \pi_1^\sharp|_x \circ T_x^*\varphi$ for all $x \in M_1$.

Problem 3: Consider symplectic manifolds (M_i, ω_i), with corresponding Poisson brackets $\{\cdot, \cdot\}_i$, $i = 1, 2$, and let $\phi : M_1 \to M_2$ be a smooth map.

(a) Show that if ϕ preserves symplectic forms (i.e., $\phi^*\omega_2 = \omega_1$), then it must be an immersion, and that if ϕ is a Poisson map, then it must be a submersion.

(b) Prove that, if ϕ is a (local) diffeomorphism, then it is a Poisson map if and only if it preserves symplectic forms.

(c) Find examples of M_1, M_2 and $\phi : M_1 \to M_2$ such that (1) ϕ is a Poisson map but does not preserve symplectic forms; (2) ϕ preserves symplectic forms but is not a Poisson map.

Problem 4: Let (M, π) be a Poisson manifold. A submanifold $N \hookrightarrow M$ is called *coisotropic* if $\pi^\sharp(\mathrm{Ann}(TN)) \subseteq TN$. Let $I_N \subseteq C^\infty(M)$ be the vanishing ideal of N. (a) Show that if N is coisotropic then $\{I_N, I_N\} \subseteq I_N$, and that the converse holds as long as N is embedded. (b) Show that a map $\varphi : M_1 \to M_2$ between Poisson manifolds is a Poisson map iff its graph is a coisotropic submanifold of $M_1 \times \overline{M_2}$, where $\overline{M_2}$ has minus the Poisson structure of M_2.

Problem 5: We say that a submanifold $N \hookrightarrow M$ is a *Poisson submanifold* if the inclusion is a Poisson map. (a) Show that this is the case iff $\pi^\sharp(\mathrm{Ann}(TN)) = 0$ iff every hamiltonian vector field on M is tangent to N. (b) When N is embedded, verify that this is equivalent to the vanishing ideal I_N being a Lie-ideal, i.e., $\{C^\infty(M), I_N\} \subseteq I_N$. (c) Show that a complete Poisson submanifold (i.e., a Poisson submanifold for which the inclusion map is complete) is a union of symplectic leaves.

Problem 6: Let $\mathfrak{g}$ be a Lie algebra and consider its dual $\mathfrak{g}^*$ with the corresponding linear Poisson structure. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a subspace. Then $\mathfrak{h}$ is a Lie subalgebra (resp. ideal) iff $\mathrm{Ann}(\mathfrak{h}) \subseteq \mathfrak{g}^*$ is a coisotropic (resp. Poisson) submanifold.

Problem 7: Consider Poisson manifolds $(M_1, \{\cdot,\cdot\}_1)$ and $(M_2, \{\cdot,\cdot\}_2)$, let $M = M_1 \times M_2$. Show that the formula[1]

$$\{f,g\}(x_1,x_2) = \{f_{x_2}, g_{x_2}\}_1(x_1) + \{f_{x_1}, g_{x_1}\}_2(x_2)$$

defines a Poisson structure on M, for which the projections $p_i : M \to M_i$ are Poisson maps and $\{p_1^*C^\infty(M_1), p_2^*C^\infty(M_2)\} = 0$. (Considering the category of Poisson manifolds and Poisson morphisms, is this product "categorical´´?)

Problem 8: Let V be a (real) vector space and $\pi \in \wedge^2 V$ (a constant Poisson structure). Consider $\pi^\sharp : V^* \to V$ defined by $\beta(\pi^\sharp(\alpha)) = \pi(\alpha,\beta)$, and let $R = \pi^\sharp(V^*) \subseteq V$.

(a) Show that there is a unique nondegenerate skew-symmetric bilinear form Ω on R given by $\Omega(u,v) = -\pi(\alpha,\beta)$, for $u = \pi^\sharp(\alpha)$ and $v = \pi^\sharp(\beta)$. Conversely, show that given a pair (R,Ω), where $R \subseteq V$ is a subspace and $\Omega \in \wedge^2 R^*$ is nondegenerate, there is a unique Poisson structure π on V such that $R = \pi^\sharp(V^*)$ and Ω is defined as before.

(b) For a subspace $W \subseteq V$, let $W^\pi = \pi^\sharp(\mathrm{Ann}(W))$. Show that $W^\pi = (W \cap R)^\Omega$ (symplectic orthogonal in (R,Ω)), and hence $(W^\pi)^\pi = W \cap R$.

Problem 9: Let π be a bivector field on a manifold M, consider the distribution $R = \pi^\sharp(T^*M) \subseteq TM$ and suppose that R is integrable. Show that any leaf $S \hookrightarrow M$ carries a smooth, nondegenerate 2-form $\omega_S \in \Omega^2(S)$ given, at each point, as in the previous problem. Prove that π is Poisson iff each ω_S is closed. Moreover, the inclusion of each leaf is a Poisson map. (In particular, if R is integrable and π has rank at most 2, it must be Poisson.)

Problem 10: Consider a regular m-dimensional Poisson manifold (M,π), with $\mathrm{rank}(\pi) = k$. Suppose that $f_1, \ldots, f_{m-k}$ are Casimir functions (i.e., $X_{f_i} = 0$, $i = 1, \ldots, m-k$) such that $df_1, \ldots, df_{m-k}$ are linearly independent at all points. Show that the symplectic leaves are given by connected components of the level sets of $(f_1, \ldots, f_{m-k}) : M \to \mathbb{R}^{m-k}$.

Problem 11: Let (M,π) be a Poisson manifold and $B \in \Omega^2(M)$. Let $B^\flat : TM \to T^*M$, $B^\flat(X) = i_X B$, and consider the map $Id + B^\flat \circ \pi^\sharp : T^*M \to T^*M$.

(a) Check that $Id + B^\flat \circ \pi^\sharp$ is an isomorphism iff, on each leaf S of (M,π), the 2-form $\omega_S + B_S$ is nondegenerate (here ω_S is the symplectic structure on S and B_S is the pullback of B to S).

(b) If $Id + B^\flat \circ \pi^\sharp$ is an isomorphism, show that there is a bivector field π_B on M such that $\pi_B^\sharp = \pi^\sharp \circ (Id + B^\flat \circ \pi^\sharp)^{-1}$. Note that the characteristic distribution of π_B coincides with that of π, so they have the same leaves. Verify that, for each leaf S, the 2-form on S induced by π_B (cf. Problem 10) is given by $\omega_S + B_S$. Conclude (from Problem 10) that π_B is Poisson iff the pullback of dB to each leaf vanishes.

For a closed 2-form B, we call the Poisson structure π_B the *gauge transformation* of π.

Problem 12: let (M,π) be a Poisson manifold. A submanifold $N \hookrightarrow M$ is called a *Poisson transversal* (a.k.a. *cosymplectic submanifold*) if $T_yM = T_yN \oplus (T_yN)^\pi$ for all $y \in N$ [2]. Suppose that N is a Poisson transversal, let $p_N : TM|_N \to TN$ be the projection along $(TN)^\pi$.

(a) Observe that N inherits a bivector field π_N satisfying $\pi_N^\sharp = p_N \circ \pi^\sharp \circ p_N^*$. Noticing that $p_N^*(T^*N) = \mathrm{Ann}((TN)^\pi)$ and using Problem 9(b), check that $\pi_N^\sharp(T^*N) = TN \cap R$, where $R = \mathrm{Im}(\pi^\sharp)$.

(b) Show that N intersects each leaf S of (M,π) transversally (i.e., for $y \in N$ and S leaf through y, $T_yM = T_yS + T_yN$), so $N \cap S$ is a submanifold with $T(N \cap S) = TN \cap TS$. By (a), the characteristic distribution of π_N is integrable.

(c) Let (S,ω_S) be the symplectic leaf of (M,π) through $y \in N$. Show that the (nondegenerate) 2-form on $N \cap S$ induced by π_N (as in Problem 10) coincides with the restriction of ω_S. It follows

[1] For $f \in C^\infty(M_1 \times M_2)$ and $(x_1,x_2) \in M_1 \times M_2$, denote by $f_{x_1} \in C^\infty(M_2)$ the function $y \mapsto f(x_1,y)$, analogously for f_{x_2}.

[2] Note that $\dim((T_yN)^\pi) \leq \dim(M) - \dim(N)$, so it is enough to require that $T_yM = T_yN + (T_yN)^\pi$

from Problem 10 that π_N is a Poisson structure (its symplectic leaves are symplectic submanifolds of the symplectic leaves of π given by the connected components of their intersections with N).

(d) Take $x \in M$, let S be the symplectic leaf cointaining x and $N \subseteq M$ be a submanifold such that $T_xM = T_xN \oplus T_xS$. Show that there is a neighborhood of x in N that is a Poisson transversal.

(e) Let $M = S \times N$ be the product of two Poisson manifolds S and N, with S symplectic, let $x \in S$. Show that $\{x\} \times N$ is a Poisson transversal in M and its Poisson structure coincides with the one inherited as such.

Problem 13: Consider a Weinstein splitting chart $S \times N$, where S is a neighborhood of 0 in $(\mathbb{R}^{2k}, \omega_{can})$ and N a neighborhood of 0 in $(\mathbb{R}^s, \pi_N)$, with $\pi_N|_0 = 0$. Consider a smooth map $\phi : N \to S$, $\phi(0) = 0$, and the submanifold $N_1 = \{(\phi(x), x),\ x \in N\}$ in M. Show that if N is small enough, N_1 is a Poisson transversal (use Problem 12(d)), hence inherits a Poisson structure π_{N_1}. Let now π' be the Poisson structure on N corresponding to π_{N_1} under the diffeomorphism $N_1 \cong N$ given by the second projection. Prove that π' is a gauge transformation of π_N by $B = \phi^*\omega_{can}$. [Since in this case B is exact, one can use Moser's method to conclude that N and N_1 are Poisson diffeomorphic near the origin.]

Problem 14:[Dirac bracket] Let (M, π). Consider a submersion $\Psi = (\psi^1, \ldots, \psi^k) : M \to \mathbb{R}^k$ and the submanifold $N = \Psi^{-1}(0)$. Let (c^{ij}) be the matrix with entries $c^{ij} = \{\psi^i, \psi^j\}$. Show that N is a Poisson transversal iff the matrix (c^{ij}) is invertible. Let (c_{ij}) be the inverse matrix. Prove that the Poisson bracket on N is given by

$$\{f, g\}_N = (\{F, G\} - \sum_{i,j} \{F, \psi^i\} c_{ij} \{\psi^j, G\})|_N,$$

where F and G are extensions of f and g, respectively, to M.

Problem 15: A vector field X on a Poisson manifold (M, π) is a *Poisson vector field* if $\mathcal{L}_X \pi = 0$, or, equivalently,

$$\mathcal{L}_X\{f, g\} = \{\mathcal{L}_X f, g\} + \{f, \mathcal{L}_X g\}$$

for all $f, g \in C^\infty(M)$. Any hamiltonian vector field is Poisson (cf. Problem 1). Give an example of a Poisson manifold (it is enough to consider $\mathbb{R}^2$...) with a Poisson vector field that is not tangent to the symplectic leaves (in particular, not hamiltonian).

Problem 16: (a) Consider $\mathbb{R}^2 = \{(x_1, x_2)\}$ with its canonical Poisson (symplectic) structure π_{can}. Let $f \in C^\infty(\mathbb{R}^2)$ be such that $f(m) = 0$ and $df|_m \neq 0$. Prove that, in a neighborhood of m, there is a smooth function g such that $z_1 = f(x_1, x_2)$, $z_2 = g(x_1, x_2)$ define local coordinates around m satisfying $\{z_1, z_2\}_{can} = 1$. (Hint: straightening-out theorem for X_f).

(b) A Poisson structure on $\mathbb{R}^2$ is always of the form $\pi = f\pi_{can}$, for $f \in C^\infty(\mathbb{R}^2)$. Suppose that it is log-symplectic (note that this means that 0 is a regular value of f). Let $Z = f^{-1}(0)$. Show that any $m \in Z$ admits a neighborhood with coordinates (z_1, z_2) satisfying $\{z_1, z_2\}_\pi = z_1$ (i.e., $\pi = z_1 \partial_{z_1} \wedge \partial_{z_2}$).

(c) Let (M^{2n}, π) be a log-symplectic manifold, with $Z = (\wedge^n \pi)^{-1}(0)$, and $m \in Z$. Check that, in Weinstein splitting coordinates around m, the transverse Poisson factor at m must have dimension 2. Conclude that there are local coordinates $(z_1, z_2, q_1, \ldots, q_{n-1}, p_1, \ldots, p_{n-1})$ around p such that

$$\pi = z_1 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} + \sum_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}$$

Note, in particular, that π restricts to a Poisson structure of rank $2n - 2$ on Z.

Problem 17:[Libermann's theorem] Let (S, ω) be a symplectic manifold and $\mu : S \to M$ be a surjective submersion with connected fibers. Then M admits a (necessarily unique) Poisson structure so that μ is a Poisson map (= symplectic realization) iff the distribution $(\ker(d\mu))^\omega$ is involutive.

Problem 18: Consider a Lie group G acting freely and properly on a Poisson manifold (M, π) by Poisson diffeomorphisms. Show that M/G carries a (unique) Poisson structure for which the quotient map $M \to M/G$ is Poisson. Prove also that this Poisson map is *complete*.

For those familiar with symplectic reduction: When M is symplectic and the action is hamiltonian, with momentum map $\mu : M \to \mathfrak{g}^*$, show that the symplectic leaves of M/G are identified with the (connected components of the) symplectic reduced spaces at different levels.

Problem 19: Let (M, π) be a Poisson manifold, and consider T^*M with the induced Lie algebroid structure. Show thata submanifold $N \hookrightarrow M$ is coisotropic iff $\mathrm{Ann}(TN) \subseteq T^*M$ is a Lie subalgebroid. Use this fact to conclude that there is a natural 1-1 correspondence between coisotropic submanifolds of M and Lie subalgebroids of T^*M that are lagrangian submanifolds (with respect to the canonical symplectic form). [There is also a global version of this correspondence relating lagrangian subgroupoids of a symplectic groupoid and coisotropic submanifolds of the unit manifold.]

Problem 20: Let M be a manifold and $\mathbb{T}M = TM \oplus T^*M$. For a 2-form $B \in \Omega^2(M)$, consider the operation $\tau_B : \mathbb{T}M \to \mathbb{T}M$, $(X, \alpha) \mapsto (X, i_X B + \alpha)$. Note that τ_B preserves the natural symmetric pairing on $\mathbb{T}M$.

(a) Show that τ_B preserves the Courant bracket iff $dB = 0$. In particular, if L is a Dirac structure, so is $\tau_B(L)$.

(b) Suppose that $L_\pi = \mathrm{graph}(\pi)$ is given by a Poisson structure. Show that $\tau_B(L_\pi)$ is again given by a Poisson structure iff $(Id + B^\flat \circ \pi^\sharp) : T^*M \to T^*M$ is an isomorphism, in which case $\tau_B(L_\pi) = L_{\pi_B}$ (see Problem 11).