GEOMETRIC QUANTIZATION, OLD AND NEW. LIST OF PROBLEMS

JONATHAN WEITSMAN

1. List of problems

Exercise 1.1. Consider $M = T^*\mathbb{R}^n = \mathbb{C}^n$ with $\omega = \frac{i}{2}\sum_i d\bar{z}_i \wedge dz_i$. $\left(S^1\right)^n$ acts on M by

$$(e^{i\theta_1}, \cdots, e^{i\theta_n}) \cdot (z_1, \ldots, z_n) = (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n).$$

Compute the moment map of this action and find the reduced space (the symplectic quotient) by (a subgroup of) $(S^1)^n$.

Exercise 1.2. The coupling of two harmonic oscillators gives can be modelled in $T^*(\mathbb{R}^2)$. Check that, in this system, the energy function

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(x_1^2 + x_2^2)$$

and the angular momentum function

$$L = x_1 y_2 - x_2 y_1$$

define a polarization. I.e. check that $\{H, L\} = 0$.

Exercise 1.3. Check that the complex projective line with line bundle $L^k = \{\mathbb{C}^2 \setminus \{0\} \times \mathbb{C}_{(k)}\}/\mathbb{C}^* \longrightarrow \{\mathbb{C}^2 \setminus \{0\}\}/\mathbb{C}^*$ satisfies that

$$\dim H^0(P^1, L^k) = \dim \mathcal{H}_{BS}.$$

Exercise 1.4. Consider the torus T^2 with symplectic form $\omega = dp \wedge dq$. It is a Kähler manifold with Line bundle $L =$ "θ-line bundle". Prove that the dimension of $H^0(S^1 \times S^1, L^k)$ is equal to $\dim \mathcal{H}_{BS}$.

Exercise 1.5. Compute the Bohr-Sommerfeld leaves of the cylinder $\mathbb{R} \times S^1$ with respect to the real polarization given by vectors tangent to the S^1 directions.

Exercise 1.6. Generalizing the previous exercise, compute the Bohr-Sommerfeld leaves of a symplectic manifold polarized by an integrable system with global action-angle coordinates.

Exercise 1.7. Compute the moment map of the toric action $\mathbb{T}^2$ on $\mathbb{C}P^2$ given by $((\theta_1, \theta_2), [z_0 : z_1 : z_2]) \mapsto ([z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2])$. Then, construct a b-toric manifold applying symplectic blow-up and the Gompf sum on $\mathbb{C}P^2$ such that:

- it has 6 fixed points, or
- it has 12 fixed points.

What you will obtain is a Hirzebruch surface.

Exercise 1.8. Take S^2 with the b-Poisson structure $\Pi_1 = h\frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta}$ and the symplectic torus $\mathbb{T}^2$ with dual Poisson structure $\Pi_2 = \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial \theta_2}$. Prove that

$$\hat{\Pi} = h\frac{\partial}{\partial h} \wedge (\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \theta_1}) + \Pi_2$$

is a b-Poisson structure on $S^2 \times \mathbb{T}^2$.

Exercise 1.9. Let (N^{2n+1}, π) be a regular corank-1 Poisson manifold, X be a Poisson vector field and $f : S^1 \to \mathbb{R}$ a smooth function. Prove that the bivector field

$$\Pi = f(\theta)\frac{\partial}{\partial \theta} \wedge X + \pi$$

is a b-Poisson structure on $S^1 \times N$ if the function f vanishes linearly and the vector field X is transverse to the symplectic leaves of N.

Exercise 1.10. Consider the b-symplectic manifold $(S^2, Z = \{h = 0\}, \omega = \frac{dh}{h} \wedge d\theta)$, where the coordinates on the sphere are $h \in [-1, 1]$ and $\theta \in [0, 2\pi]$. Compute a moment map of the S^1-action given by the flow of $-\frac{\partial}{\partial \theta}$ and draw its image.

Exercise 1.11. Consider the b-symplectic manifold

$$(\mathbb{T}^2, Z = \{\theta_1 \in \{0, \pi\}\}, \omega = \frac{d\theta_1}{\sin \theta_1} \wedge d\theta_2),$$

where the coordinates on the torus are $\theta_1, \theta_2 \in [0, 2\pi]$. Find the ${}^bC^\infty$ Hamiltonian function of the circle action of rotation on the θ_2 coordinate and draw it.

Exercise 1.12. The moment image of a $2n$-dimensional b-symplectic toric manifold is represented by an n-dimensional polytope P, and the corresponding extremal polytope Δ_P is an $(n-1)$-dimensional Delzant polytope. Describe the extremal polytope for $n = 1$ and $n = 2$.

Exercise 1.13. Compute the Bohr-Sommerfeld quantization with sign of a b-sphere, a b-torus and higher dimensional b-symplectic toric manifold.

Jonathan Weitsman, Department of Mathematics, Northeastern University, Boston, MA 02115

Email address: `j.weitsman@neu.edu`