Introduction to Singular Foliations

(And mainly to its Geometry)

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UNIVERSITÉ DE LORRAINE

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Schedule :

1. Tuesday : What are singular foliations ?
2. Wednesday : What structures do they hide ?
3. Thursday : exercises, symmetries of a subset.
4. Friday : More (higher) structures they hide + open questions.

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There is a (not totally finished) handout on-line.

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Too many definitions ?

What is a singular foliation ? A first attempt

A first attempt to define singular foliations on M :

Definition

A *partitionifold* of M is a partition of M into connected immersed submanifolds [a], called leaves.

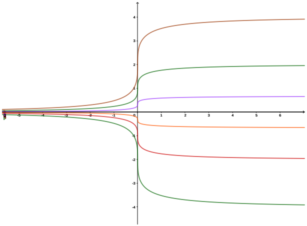
a. From now on, "submanifold" means by default "immersed submanifolds".

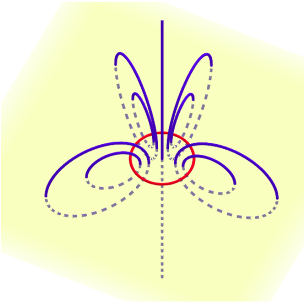
Notation $L_\bullet : m \mapsto L_m$.

Question

Should we take it as a definition of singular foliation ?

Annoying examples

Pinch 

Magnetic 

One has to make a choice

A choice has to be made, What do we wish to study ?	
Isolated lasagna in a spaghetti dish ?	Isolated spaghetti in a lasagna dish ?
No	Yes
Defined with forms	Defined with tangent vector

Other problems : magnetic or pinch partitionifolds have little interesting geometry : we need one more assumption !

A second attempt : smooth partitionifolds

Definition

A partitionifold $L_\bullet$ is said to be smooth if for every $\ell \in M$ and every tangent vector $u \in T_\ell L_\ell$, there exists a vector field X through u which is tangent to all leaves.

This forbids isolated lasagnas, magnetic or pinch-partitioniolds. It is better.

Question

Should we take it as a definition of singular foliation ?

Smooth partitionifolds are fine (I)

The flow of a vector field tangent to all leaves preserves $L_\bullet$.

Proposition

Let $L_\bullet$ be a smooth partitionifold.

1. *Travelling along a leaf is boring*
2. *Every leaf has a transverse structure*
3. *Which is unique*
4. *And there is a Weinstein-splitting-like theorem.*

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1. *Two points on the same leaves have open neighborhoods on which $L_\bullet$ are isomorphic.*
2. *For Σ transverse to L, $m \mapsto (\Sigma \cap L_m)_0$ is a smooth partitionifold on a neighborhood of $L \cap \Sigma$.*
3. *Which is unique*
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3. *And any two such transverse smooth partitionifolds have isomorphic germs.*
4. *And near any point m, $L_\bullet$ is a isomorphic to the direct product of the leaf by any representative of the transverse structure*

Smooth partitionifolds are fine (II)

For any smooth partitionifold $L_\bullet$:

1. The singular distribution :

$$m \mapsto T_m L_m$$

   is involutive, integrable, any of its section has a flow that preserves it.
2. has a upper-semi-continuous dimension,
3. and on the open dense subset where this rank is locally maximum, we obtain a "good old" regular foliation.

(So there is a dense open subset where it is a regular foliation + some singularities where leaves are strictly smaller in dimension.)

Question

So, is it a good definition of a singular foliation ?

Yes, but it has lost

Here is the consensus definition of what a singular foliation is.

Definition

A singular foliation on a smooth manifold M is a subspace $\mathcal{F} \subset \mathfrak{X}_c(M)$ which

(α) is involutive,

(β) is a $\mathcal{C}^\infty(M)$-module

(γ) is locally finitely generated.

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(β) For all $F \in \mathcal{C}^\infty(M)$, $X\mathcal{F} \implies FX \in \mathcal{F}$.

(γ) For every point $m \in M$ there exists $X_1, \ldots, X_r \in \mathcal{F}$ and an open neighborhood $\mathcal{U}$ of m such that every for every $X \in \mathcal{F}$ there exists $f_1, \ldots, f_r \in C^\infty(M)$ such that $X - \sum_{i=1}^r f_i X_i$ is zero on $\mathcal{U}_m$.

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Complain

Come on ! How do you dare to call foliation something which has no leaves !

The holomophic setting, and a bit of algebraic geometry

If you hate compactly supported, and like sheaves, here is an equivalent definition on a smooth manifold M :

Definition

A singular foliation on a smooth manifold M is a subsheaf

$$\mathcal{F}_\bullet : \mathcal{U} \mapsto \mathcal{F}_\mathcal{U}$$

of the sheaf $\mathfrak{X}_\bullet$ of vector fields on M such that

(α) $\mathcal{F}_\bullet$ is involutive,

(β) is a sub-sheaf of $\mathcal{C}^\infty_\bullet$-modules ,

(γ) is locally finitely generated.

The holomophic setting, and a bit of algebraic geometry

If you hate compactly supported, and like sheaves, here is the definition for a complex manifold M with holomorphic functions $\mathcal{O}_\bullet$.

Definition

A singular foliation on a ~~smooth~~ complex manifold M is a subsheaf

$$\mathcal{F}_\bullet : \mathcal{U} \mapsto \mathcal{F}_\mathcal{U}$$

of the sheaf $\mathfrak{X}_\bullet$ of vector fields on M such that

- (α) $\mathcal{F}_\bullet$ is involutive,
- (β) is a sub-sheaf of ~~$\mathcal{C}^\infty_\bullet$-modules~~ $\mathcal{O}_\bullet$-modules,
- (γ) ~~is locally finitely generated~~ - forget (γ), germs of holomorphic functions are Noetherian anyway

"Consensus" Definition

Definition

A singular foliation on a smooth manifold M is a subsheaf of the sheaf $\mathfrak{X}$ of vector fields on M subspace of the space $\mathfrak{X}$ of compactly supported vector fields on M such that

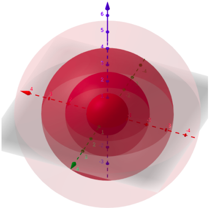
(α) $\mathcal{F}$ is involutive,

(β) $\mathcal{F}$ is a modules over smooth functions,

(γ) is locally finitely generated.

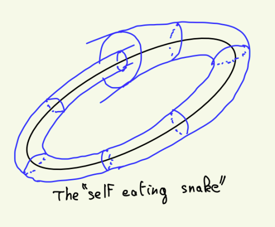
Examples

1. Image through anchor map of a Lie algebroids :
   1. The module generated by Hamiltonian vector fields of a Poisson structure,
   2. Infinitesimal actions of Lie group actions.
2. Vector fields tangent to a (reasonable) subset, or that "kill" prescribed functions.
3. Vector fields vanishing at prescribed order at prescribed points.
4. Representations.



Some natural operations

1. Direct product,
2. Pull-back through a transverse map. Includes :
   1. Pull-back through submersions.
   2. Restriction to a transverse submanifold Σ (i.e. $T\Sigma + T\mathcal{F} = TM$).
3. Push-forward (sometimes).
4. Suspension through a symmetry.
5. Blow-up along a leaf.



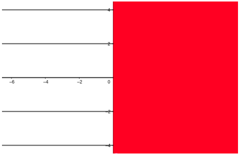
What are leaves ? And why finitely generated

Definition

Let $\mathcal{F}$ be a singular foliation on M. Choose $m \in M$

1. the <u>R-leaf</u> through m is the set of points reachable from m by following finitely many flows of vector fields in $\mathcal{F}$.
2. We call <u>T-leaf</u> a submanifold L :
   1. containing m
   2. such that $T_x L = T_x \mathcal{F}$ for all $x \in M$
   3. and maximal among those.

A problem, the infinite comb.



Structure of the proof.

Proposition

The flow of a vector field in $\mathcal{F}$ is a symmetry of $\mathcal{F}$.

Démonstration.

☠ This is not easy. Better proof tomorrow. □

Theorem

Near a point m, a singular foliation is the direct product of :

1. *the singular foliation of all vector fields on $\mathbb{R}^a$, with $a = \dim(T_m\mathcal{F})$.*
2. *some singular foliation on $\mathbb{R}^b$ made of vector fields that vanish at 0.*

Corollary

T-leaves = R-leaves form a smooth partitionifold.

Consequences for leaves.

Proposition

Let $\mathcal{F}$ be a singular foliation.

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2. *Every leaf has a transverse structure*
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2. *For Σ transverse to L, $\mathfrak{F}_\Sigma$ is a singular foliation on a neighborhood of $L \cap \Sigma$.*
3. *And any two such transverse singular foliations have isomorphic germs.*
4. *And near any point m, $\mathcal{F}$ is a isomorphic to the direct product of the leaf by any representative of the transverse structure*

   (Hermann, Nagoya, Cerveau, Dazord, Androulidakis-Skandalis, H Bursztyn, H Lima, E Meinrenken, Garmendia-Villatoro).

Lecture number 2

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What kind of structures is hidden ?

What are the hidden structures ?

What Poisson geometers would like.	What we get.
a Lie algebroid	an almost Lie algebroid
its isotropy Lie algebras	✓
transitive Lie algebroid on each leaf	✓
a Lie groupoid.	AS holonomy groupoid.

AS := Androulidakis-Skandalis.

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Anchored bundles

Definition

An anchored vector bundle over $\mathcal{F}$ is a pair (A, ρ) made of a vector bundle $A \to M$, and a vector bundle morphism called its anchor map.

$$\begin{array}{ccc} A & \xrightarrow{\rho} & TM \\ \downarrow & & \downarrow \\ M & = & M \end{array}$$

such that $\rho(\Gamma(A)) = \mathcal{F}$.

Proposition

A smooth singular foliation admits an anchored bundle over it if and only if it is globally finitely generated.

Equivalence of anchored bundles

Definition

Let (A_1, ρ_1) and (A_2, ρ_2) be two anchored bundles over $\mathcal{F}$.

1. We call morphism of anchored bundles any vector bundle morphism $\Phi \colon A_1 \longrightarrow A_2$ over id_M s.t. :

$$\begin{array}{ccc} A_{\mathbf{1}} & \xrightarrow{\Phi} & A_{\mathbf{2}} \\ {\scriptstyle \rho_{\mathbf{2}}}\downarrow & & \downarrow{\scriptstyle \rho_{\mathbf{1}}} \\ TM & \longrightarrow & TM \end{array} \tag{1}$$

2. Two morphisms of anchored bundles Φ, Φ' as in item 1 are said to be equivalent if $\rho \circ (\Phi - \Phi') = 0$.
3. An equivalence of anchored bundles is a pair of anchored bundle morphisms $A_1 \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} A_2$ such that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are equivalent to the identities of A_1 and A_2.

Any anchored bundles are equivalent

Proposition

Any two anchored bundles over the same singular foliation are equivalent.

Did you know that given a vector bundle morphism over id_M

$$\Phi \colon A \to B$$

there are two notions of kernel at a point :

1. weak kernel, just the ordinary one
2. strong kernel ?

And the quotient

$$\frac{\ker(\Phi_m)}{\mathrm{Sker}(\Phi, m)}$$

is zero on a dense open subset ?

Isotropy vector space

Call isotropy space at m the quotient :

$$\mathfrak{g}_m(\mathcal{F}) = \frac{\ker(\rho_{\mathrm{m}})}{\mathrm{Sker}(\rho, \mathrm{m})}.$$

Proposition

The isotropy vector space at m does not depend on any choice made in the construction, and

$$\mathrm{rk}_m(\mathcal{F}) = \dim\left(\mathfrak{g}_m(\mathcal{F})\right) + \dim(L_m).$$

Almost

Question

What kind a structure may we have on anchored bundle over $\mathcal{F}$?

1. **Dream :** a Lie algebroid,

Definition

Let (A, ρ) be an anchored vector bundle. We call Lie algebroid bracket a skew-symmetric bilinear (over $\mathbb{K}$) map

$$[\cdot\,,\cdot]_A : \Gamma(A) \wedge \Gamma(A) \longrightarrow \Gamma(A)$$

that satisfies the Leibniz identity and the Jacobi condition :,

$$[x, fy]_A = \rho(x)[f]y + f[x, y]_A, \quad \text{for all } x, y \in \Gamma(A), f \in \mathcal{C}^\infty(M)$$

$$[[x, y]_A, z]_A + [[y, z]_A, x]_A + [[z, x]_A, y]_A = 0, \quad \text{for all } x, y, z \in \Gamma(A).$$

Almost

Question

What kind a structure may we have on anchored bundle over $\mathcal{F}$?

1. **Truth :** an almost-Lie algebroid

Definition

Let (A, ρ) be an anchored vector bundle. We call almost-Lie algebroid bracket a skew-symmetric bilinear (over $\mathbb{K}$) map

$$[\cdot\,,\cdot]_A : \Gamma(A) \wedge \Gamma(A) \longrightarrow \Gamma(A)$$

that satisfies the Leibniz identity and the anchor condition :,

$$[x, fy]_A = \rho(x)[f]y + f[x, y]_A, \quad \text{for all } x, y \in \Gamma(A), f \in \mathcal{C}^\infty(M)$$

$$\rho([x, y]_A) = [\rho(x), \rho(y)], \quad \text{for all } x, y \in \Gamma(A). \tag{2}$$

Consequences

Consequences. This almost Lie algebroid is not much, but is still useful.

1. Vector fields in $\mathcal{F}$ have flows that are symmetries of $\mathcal{F}$.
2. The isotropy vector space comes equipped with a Lie bracket.
3. Which coincides with **AS** isotropy Lie algebra :

$$\mathfrak{g}_m(\mathcal{F}) \simeq \frac{\mathcal{F}(m)}{\mathcal{I}_m \mathcal{F}}$$

4. For every leaf, the quotient $A_L := A/SKer(\rho)$ comes equipped with a transitive Lie algebroid.
5. Which coincides with the one in **AS** :

$$\Gamma(A_L) \simeq \frac{\mathcal{F}}{\mathcal{I}_L \mathcal{F}}$$