Renormalisation: from Quantum Field Theory to SPDEs

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2nd of June 2022
Conference in honour of Marta Sanz-Solé

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Some books

- G. Folland, *Quantum Field Theory, A Tourist guide for mathematicians*
- M. Talagrand, *What is a Quantum Field Theory, A first introduction*
- A. Connes and M. Marcolli, *Noncommutative geometry, Quantum Fields and Motives*

The aim of QFT is to combine Quantum Mechanics and Relativity

From the introduction of Folland's book

"In spite of its mathematical incompleteness, quantum field theory has been an enormous success for physics. It has yielded profound advances in our understanding of how the universe works at the submicroscopic level, and Quantum Electrodynamics (QED) in particular has stood up to extremely stringent experimental tests of its validity."

"For example, the theoretical and experimental values of the magnetic moment of the electron agree to within one part in 10^{10}, which is like determining the distance from the Empire State Building to the Eiffel Tower to within a millimeter."

Renormalisation in Physics

One of the main puzzles in QFT is that it requires the mysterious procedure of renormalisation:

Computations involve infinite quantities which have to be subtracted off in order to obtain meaningful (and finite) results.

Amazingly, these are the results which then are successfully compared with experiments.

Between the 40s and the 80s physicists have developed a number of methods, some more rigorous, some less so, to understand all this. Their approach is mainly based on perturbation theory.

Renormalisation in Mathematics

In the 70s and in the 80s, a group of mathematical physicists proposed a rigorous method called constructive QFT and which is non-perturbative.

This is based on techniques like the cluster expansion, which however impose restrictions on the parameters of the theory.

Another non-perturbative approach based on stochastic dynamics was proposed in the 80s. This approach has inspired the recent wave of interest.

Renormalisation of the dynamics (SPDE) still plays a crucial but different role.

What puzzles me is that there is still little understanding of how the physical and the more mathematical approaches are related to each other.

The physicists' setting

One of the main ingredients is the following Feynman path integral

$$\frac{1}{Z} \exp\left(i \, \frac{S(\phi)}{\hbar} \right) [\mathrm{d}\phi]$$

where

- $\mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}$ is space-time (so the physical case is $d = 4$)
- $\phi : \mathbb{R}^d \to \mathbb{R}^k$ is a function, possibly a distribution
- $S(\phi)$ is an action (from classical Lagrangian mechanics); in particular, a real-valued non-linear functional
- $i^2 = -1$
- $\hbar$ is Planck's constant; one assumes $\hbar = 1 = c$
- $[\mathrm{d}\phi]$ is a formal infinite-dimensional Lebesgue measure on the space of ϕ

The Feynman path integral is to the Wiener measure what the Schrödinger equation is to the heat equation.

Correlations functions

Unfortunately the Feynman path integral is still mathematically ill-defined.

Among the quantities of plysical interest are the correlation functions

$$(\mathbb{R}^d)^n \ni (x_1, \ldots, x_n) \mapsto \int \frac{1}{Z} \phi(x_1) \cdots \phi(x_n) \exp\left(i\, S(\phi)\right) \mathrm{d}\phi.$$

This allows to predict values for quantities which can be measured in experiments.

QFT gives formulae to "approximate" such functions.

Perturbation theory

We assume the following structure for the action S:

$$S = S_{\text{free}} + \lambda\, S_{\text{interaction}}$$

where $\lambda \in \mathbb{R}$. In the free case $\lambda = 0$ we can compute everything.

Then one writes the correlation function

$$\begin{aligned} f(x_1, \dots, x_n) &= \int \frac{1}{Z} \phi(x_1) \cdots \phi(x_n) \exp\left(i\, S(\phi)\right) \mathrm{d}\phi \\ &= \int \frac{1}{Z} \phi(x_1) \cdots \phi(x_n) \exp\left(i\left(S_{\text{free}} + \lambda\, S_{\text{interaction}}\right)(\phi)\right) \mathrm{d}\phi \\ &= \sum_{k \geq 0} \frac{(i\lambda)^k}{k!} \int \frac{1}{Z} \phi(x_1) \cdots \phi(x_n) \left(S_{\text{interaction}}(\phi)\right)^k \exp\left(i\, S_{\text{free}}(\phi)\right) \mathrm{d}\phi \end{aligned}$$

Perturbation theory

We obtain

$$f(x_1,\ldots,x_n) = \sum_{k\geq 0} \frac{(i\lambda)^k}{k!}\,[\text{A big finite dimensional integral}(k)]$$

where the precise form of the $\text{Integrals}(k)$ depend on the precise form of S_{free} and $S_{\text{interaction}}$.

This is an example of perturbative series.

There are two major problems:

- Some (or many) of the $\text{Integrals}(k)$ can diverge.
- Even in very simple situations, where the integrals are convergent, or after renormalisation, the series is known to be non convergent.

The problem of non-convergence of the series is solved by physicists with a truncation, which seems impossible to justify mathematically. (Note that λ is not assumed to be small).

Feynman integrals

The $\text{Integrals}(k)$ in the perturbative series are known as Feynman integrals.

They diverge typically because they can be reduced to expressions like

$$C \int_0^\infty \frac{r^{d+1}}{(r^2+1)^3} \mathrm{d}r = C \frac{\Gamma(1+\frac{d}{2})\Gamma(2-\frac{d}{2})}{2}$$

which is indeed divergent in the physical case $d = 4$.

Renormalisation here is a combinatorial procedure to modify these integrals in order to make them convergent.

Of course this can not be done arbitrarily and there are very precise rules for this procedure.

Probability measures

Formally, if one performs an analytic continuation in time $t \mapsto it$ (or $x_0 \mapsto ix_0$) then the Feynman path integral becomes a probability measure

$$\frac{1}{Z} \exp\left(-\frac{1}{2}\int_{\mathbb{R}^d} |\nabla\phi|^2 + R\phi^2 + V(\phi)\right) \mathrm{d}\phi$$

where V is the potential and R is a parameter. If $R = m^2$ then m plays the role of a mass. (From Schrödinger to heat equation)

This probabilistic approach (the so-called Euclidean QFT) is so far necessary for a rigorous treatment (see e.g. the book by Glimm and Jaffe).

This is less ill-defined than the Feynman path integral, but still we expect ϕ to be a distribution, so that $V(\phi)$ remains problematic.

Note that this method is non-perturbative, but in order to go back to predictions for real measurements one need to reverse the analytic continuation.

Free model

If the potential $V \equiv 0$ is zero, then the above probability measure is Gaussian

$$\frac{1}{Z} \exp\left(-\frac{1}{2}\int_{\mathbb{R}^d} |\nabla\phi|^2 + m^2\phi^2\right) \mathrm{d}\phi = \mathcal{N}\left(0, (m^2 - \Delta)^{-1}\right).$$

It is well known that indeed this measure is supported by distributions as soon as $d > 1$.

Therefore

$$\frac{1}{Z} \exp\left(-\frac{1}{2}\int_{\mathbb{R}^d} |\nabla\phi|^2 + m^2\phi^2 + V(\phi)\right) \mathrm{d}\phi = \frac{1}{Z} \exp\left(-\frac{1}{2}\int_{\mathbb{R}^d} V(\phi)\right) \mathcal{N}\left(0, (m^2 - \Delta)^{-1}\right)(\mathrm{d}\phi)$$

is still ill-defined. This is called a ultra-violet divergence.

Regularisation

Let us call $\mu_R := \mathcal{N}\left(0, (R-\Delta)^{-1}\right)$ and $\mu_{R,\varepsilon} := \mathcal{N}\left(0, (R-\Delta_\varepsilon)^{-1}\right)$ a Gaussian measure such that

- $\mu_{R,\varepsilon}$ is supported by functions
- $\mu_{R,\varepsilon} \to \mu_R$ as $\varepsilon \downarrow 0$.

Now

$$\frac{1}{Z_\varepsilon} \exp\left(-\frac{1}{2}\int_{\mathbb{R}^d} V(\phi)\right) \mathcal{N}\left(0, (R-\Delta_\varepsilon)^{-1}\right)(\mathrm{d}\phi)$$

is well-defined.

What happens as $\varepsilon \to 0$?

Example: ϕ^4

A toy model in this framework is $V(\phi) = \phi^4$.

Now a Wiener chaos expansion shows that under $\mu_{R,\varepsilon}$

$$\phi^4 = H_4(\phi) + C_\varepsilon H_2(\phi) + K_\varepsilon$$

where H_n is a Hermite polynomial of degree n and $C_\varepsilon > 0, K_\varepsilon$ are constants.

Then our measure becomes

$$\frac{1}{Z'_\varepsilon} \exp\left(-\frac{1}{2}\int_{\mathbb{R}^d} H_4(\phi)\right) \mathcal{N}\left(0, (R + C_\varepsilon - \Delta_\varepsilon)^{-1}\right)(\mathrm{d}\phi).$$

In fact, $C_\varepsilon \to +\infty$ as $\varepsilon \to 0$. (The asymptotics depends on the dimension).

Renormalisation

Here we see one of the main mysteries of this theory: in order to have any hope of a convergent limit, we must assume that $R = R_\varepsilon$

$$\frac{1}{Z'_\varepsilon} \exp\left(-\frac{1}{2}\int_{\mathbb{R}^d} H_4(\phi)\right) \mathcal{N}\left(0, (R_\varepsilon + C_\varepsilon - \Delta_\varepsilon)^{-1}\right)(\mathrm{d}\phi).$$

At the start, I said that if $R = m^2$ then m was a mass, therefore a parameter with a physical sense.

Now it appears that $R = R_\varepsilon$ depends on a regularisation parameter.

Moreover, since $C_\varepsilon \to +\infty$, we must assume that $R_\varepsilon \to -\infty$ in such a way that $R_\varepsilon + C_\varepsilon$ converges.

Renormalisation

One says that

- R is the bare parameter
- R_ε is the renormalized parameter
- the physical parameter $\hat{m}$ must be extracted from some observable which can be measured.

Our renormalised measure is

$$\frac{1}{Z'_\varepsilon} \exp\left(-\frac{1}{2}\int_{\mathbb{R}^d} H_4(\phi)\right) \mathcal{N}\left(0, (R_\varepsilon + C_\varepsilon - \Delta_\varepsilon)^{-1}\right)(\mathrm{d}\phi),$$

and this does converge to a well-defined probability measure if R_ε is correctly chosen.

Back to Feynman diagrams

The above discussion on renormalisation of Feynman diagrams can be now interpreted as follows:

- First one "regularises" the integrals (one replaces μ_R with $\mu_{R,\varepsilon}$).
- Then the bare parameters are replaced with the renormalised parameters (one replaces $\mu_{R,\varepsilon}$ with $\mu_{R_\varepsilon,\varepsilon}$).
- Finally the regularisation is removed ($\varepsilon \to 0$).

After these operations, the coefficients in the perturbative series should be well-defined (though the series is in general still non-convergent).

SPDEs

In the 80s some theoretical physicists (Giorgio Parisi, Gianni Jona-Lasinio), introduced SPDEs in their models, e.g.

$$(\mathrm{KPZ}) \qquad \partial_t u = \Delta u + (\partial_x u)^2 + \xi, \quad x \in \mathbb{R},$$

$$(\Phi^4_3) \qquad \partial_t u = \Delta u - u^3 + \xi, \quad x \in \mathbb{R}^3,$$

where ξ is a space-time white noise, namely the random distribution on $\mathbb{R}_+ \times \mathbb{R}^d$

$$\xi(t,x) = \sum_k \dot{B}_k(t)\, e_k(x), \qquad x \in \mathbb{R}^d,$$

with $(e_k)_k$ a complete orthonormal system in $L^2(\mathbb{R}^d)$ and $(B_k)_k$ independent standard Brownian motions.

The physical models: Φ^4_3 from Quantum Field Theory

The Parisi-Wu stochastic quantisation in space-dimension 3:

$$(\Phi^4_3) \qquad \partial_t u = \Delta u - u^3 + \xi, \quad x \in \mathbb{R}^3/\mathbb{Z}^3 = \mathbb{T}^3.$$

Here u is a distribution.

Therefore u^3 is ill-defined.

The measure we discussed in the first part of this talk is supposed to be the invariant measure of this dynamics, or the limit in law of $u(t,\cdot)$ as $t \to +\infty$.

More general equations

More generally

$$\partial_t u = \Delta u + F(u, \nabla u, \xi).$$

Even for polynomial non-linearities, we do not know how to properly define products of (random) distributions.

Regularity structures give a theory of well-posedness for a class of these equations.

This theory develops in a far-reaching way ideas of, among others, Terry Lyons and Massimiliano Gubinelli on Rough Paths.

Regularity structures (due to M. Hairer) use local generalised Taylor expansions. Another approach, due to M. Gubinelli and coauthors, uses a stochastic version of para-differential calculus.

A general Theorem based on RS

Theorem (M. Hairer)

Let ξ_ε *a regularisation of* ξ *and let* u_ε *solve*

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + F(u_\varepsilon, \nabla u_\varepsilon, \xi_\varepsilon).$$

In general, u_ε *can fail to converge.*

For a class of equations, called subcritical, one can find a renormalised equation

$$\partial_t \hat{u}_\varepsilon = \Delta \hat{u}_\varepsilon + \hat{F}_\varepsilon(\hat{u}_\varepsilon, \nabla \hat{u}_\varepsilon, \xi_\varepsilon)$$

with an explicit $\hat{F}_\varepsilon$*, such that*

$$\hat{u}_\varepsilon \to \hat{u}, \qquad \varepsilon \to 0.$$

Examples

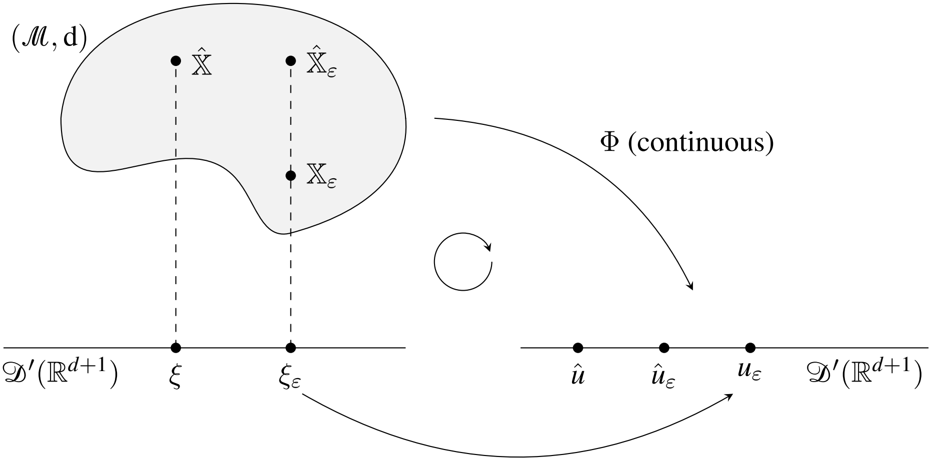
For example, for Φ^4_3:

$$F(u,\xi) = -u^3 + \xi, \qquad \hat{F}_\varepsilon(\hat{u}_\varepsilon, \xi_\varepsilon) = -\hat{u}_\varepsilon^3 + \left(\frac{C_1}{\varepsilon} + C_2 \log \varepsilon\right)\hat{u}_\varepsilon + \xi_\varepsilon.$$

The non-linearity $\hat{F}_\varepsilon$ always has the form F plus explicit terms which can contain diverging constants.

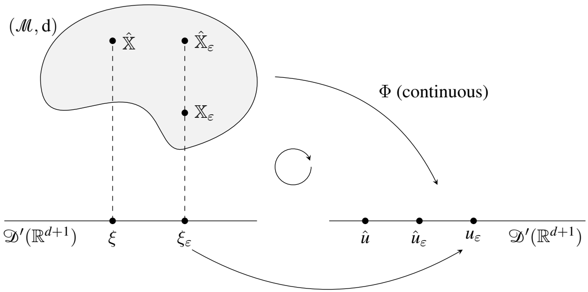
These are the (in)famous counter-terms.

A factorisation



Here $\quad u_\varepsilon = \Phi(\mathbb{X}_\varepsilon), \quad \hat{u}_\varepsilon = \Phi(\hat{\mathbb{X}}_\varepsilon), \quad \hat{u} = \Phi(\hat{\mathbb{X}}).$

A factorisation



The object of (physical) interest is the law of $\hat{u}$.

The random field $\hat{u}$ is realised as a non-linear function of a white noise ξ.

There is an analogy with Itô's introduction of path-space equations (SDEs).

A non-linear function of a noise

Unfortunately the map $\xi \mapsto \hat{u}$ is still very difficult to construct...

In particular it seems quite hard to prove a.s. properties of the sample path.

However there are currently being signicant progresses.

In particular there are recent papers by M. Gubinelli et al. which give direct expressions for the law of $\hat{u}$, and which have allowed to prove genuinely new results on Φ^4_3 (e.g. existence of phase transitions, see H. Weber et al..

Conclusion

We have discussed three methods, all requiring some form of renormalisation:

- the perturbation approach: this is the most useful for prediction of real measurements, but inherently non-rigorous
- the constructive approach: this works on the Euclidean setting (probability measures rather than complex functional integrals) and is non-perturbative, but still it requires some parameter to be small
- the stochastic quantisation: this is non-perturbative and has no smallness assumption, but it is still difficult to work with the constructed objects.

There will be certainly interesting developments in the next years...