Waving Marta goodbye

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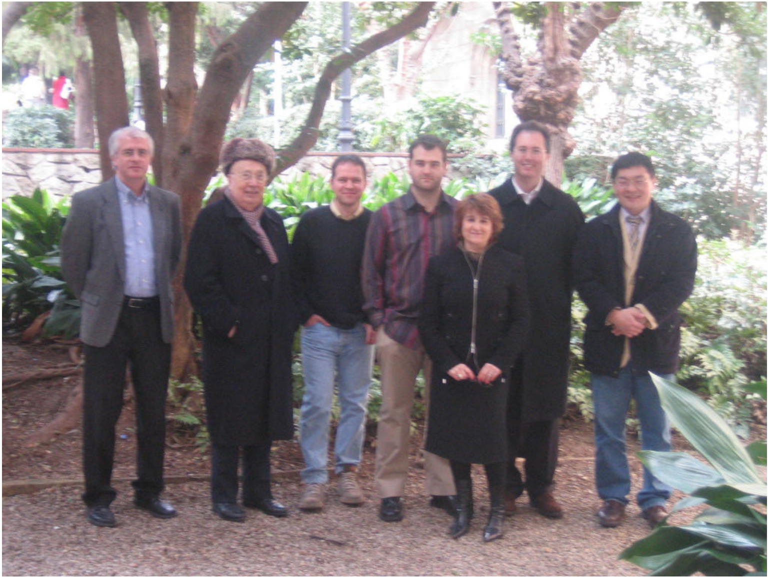
Stochastic Analysis and Stochastic PDEs

Outline

Some memories



Outline

Equation under consideration

Equation:

Stochastic heat equation on $\mathbb{R}^d$:

$$\partial_t u_t(x) = \frac{1}{2}\Delta u_t(x) + u_t(x)\,\dot{W}_t(x), \tag{1}$$

with

- $t \geq 0$, $x \in \mathbb{R}^d$.
- $\dot{W}$ Gaussian noise such that
  - $\dot{W}$ white noise in time
  - $\dot{W}$ has a certain space-time covariance structure.
- $u_t(x)\,\dot{W}_t(x)$ differential: Stratonovich or Itô sense.

Noisy part

General structure of the noise: $\dot{W}_t(x)$ is

- Gaussian process
- Distribution valued
- More about covariance structure later

Main challenge:

- Understand term $u_t(x)\,\dot{W}_t(x)$ in (1)

Motivation 1: Homogenization

Problem: Asymptotic regime for

$$\partial_t u_t(x)^\varepsilon = \frac{1}{2}\Delta u_t(x)^\varepsilon + \varepsilon^{-\beta} u_t(x)\, V_{\frac{t}{\varepsilon^\alpha}}(x/\varepsilon)\,,$$

where V stationary random field.

Link with SPDEs:

- Under certain regimes for α, β, V we have $u^\varepsilon \to u$.
- Analysis through Feynman-Kac formula.
- See Iftimie-Pardoux-Piatnitski, Bal-Gu.

Phenomenon:

- Highly oscillating PDE converging to stochastic PDE.

Motivation 2: Intermittency

Equation: $\partial_t u_t(x) = \frac{1}{2}\Delta u_t(x) + \lambda\, u_t(x)\, \dot{W}_t(x)$

Phenomenon: The solution u concentrates its energy in high peaks.

Characterization: through moments
$\hookrightarrow$ Easy possible definition of intermittency: for all $k_1 > k_2 \geq 1$

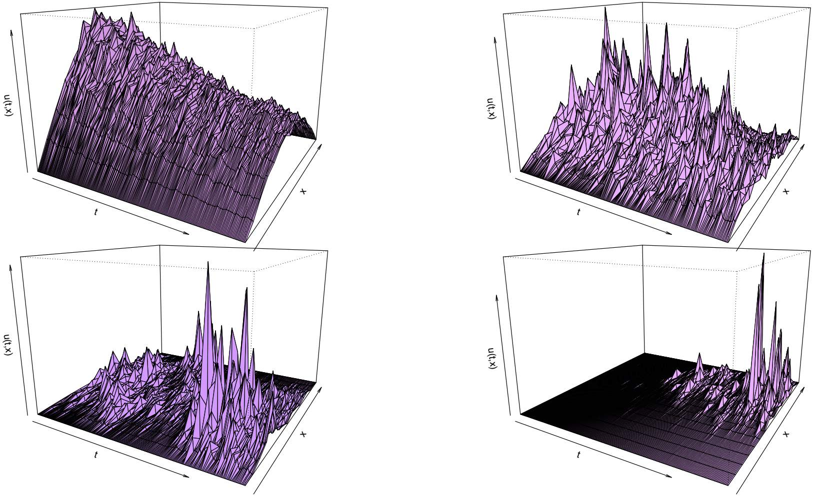
$$\lim_{t\to\infty} \frac{\mathbf{E}^{1/k_1}\left[|u_t(x)|^{k_1}\right]}{\mathbf{E}^{1/k_2}\left[|u_t(x)|^{k_2}\right]} = \infty\,.$$

Results:

- White noise in time: Khoshnevisan, Foondun, Conus, Joseph
- Fractional noise in time: Balan-Conus, Hu-Huang-Nualart-T

Intemittency: illustration (by Daniel Conus)

Simulations: for $\lambda = 0.1$, 0.5, 1 and 2.



Motivation 3: KPZ equation

KPZ equation:

$$\partial_t h_t(x) = \Delta h_t(x) + (\partial_x h)^2 + \dot{W}_t(x) - \infty,$$

Motivation:

- Standard growth model

Resolution:

- Very challenging problem, solved thanks to 2 different methods
- Regularity structures (Hairer, Zambotti)
- Paracontrolled calculus (Gubinelli-Imkeller-Perkowski)

Link with stochastic heat equation:

- Through Hopf-Cole transform
- Morally, $u = \exp(h)$

Outline

Equation under consideration

Equation:

Stochastic wave equation on $\mathbb{R}^d$:

$$\partial^2_{tt} u_t(x) = \frac{1}{2}\Delta u_t(x) + \sigma\left(u_t(x)\right)\, \dot{W}_t(x), \tag{2}$$

with

- $t \geq 0$, $x \in \mathbb{R}^d$.
- $\dot{W}$ Gaussian noise such that
  - $\dot{W}$ white noise in time
  - $\dot{W}$ has a certain space-time covariance structure.
- $u_t(x)\, \dot{W}_t(x)$ differential: Stratonovich or Itô sense.

Description of the noise (1)

Encoding the noise as a random distribution:
Since W is distribution-valued, we set

$$W = \{W(\varphi);\, \varphi \in \mathcal{H}\} \quad \text{centered Gaussian family}$$

Covariance function: We have

$$\mathbf{E}\left[W(\varphi)\, W(\psi)\right] = \langle \varphi, \psi \rangle_{\mathcal{H}}$$

with

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}_+ \times \mathbb{R}^{2d}} \varphi(t, x)\, \psi(t, y)\, \Lambda(x - y)\, dx\, dy\, dt$$

Description of the noise (2)

Observation ('98 approx, due to Peszat-Zabczyk and Dalang):

- Λ covariance function
  $\hookrightarrow \mathcal{F}\Lambda = \mu$, where μ is a measure
- $\langle \varphi, \psi \rangle_{\mathcal{H}}$ is easier to express in Fourier modes
- White noise corresponds to $\mu =$ Lebesgue measure
- For $d \geq 2$, we need smoother noises than white noise

Covariance in Fourier modes:

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}_+ \times \mathbb{R}^d} \mathcal{F}\varphi(t, \xi) \, \overline{\mathcal{F}\psi}(t, \xi) \, \mu(d\xi) \, dt.$$

Typical examples of noises

Advantage of Fourier:

- One can take φ, ψ distributions
- We need the Fourier transform to be nice enough
  $\hookrightarrow$ Typically true for kernels of PDEs

Most common class of noise:

Covariance	Singularity at 0	FT: sing. at ∞	Roughness
$\gamma(t)$	$\delta_0(t)$	Not used	$\mathcal{B}^{-1/2}$
$\Lambda(x)$	$\lvert x\rvert^{-\eta}$	$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+\lvert\xi\rvert^{\eta}} < \infty$	$\mathcal{B}^{-\eta/2}$

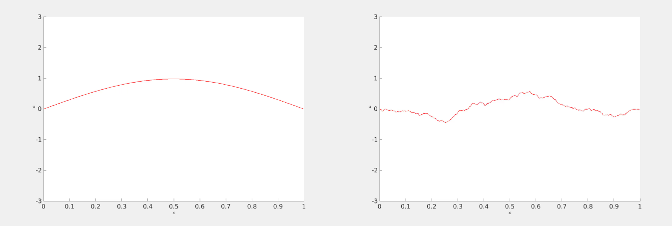
A simulation by David Cohen (Chalmers)

Equation: Stochastic wave equation on $[0,1]$:

$$\partial_{tt}^2 u_t(x) = \frac{1}{2}\Delta u_t(x) + \dot{W}_t(x), \tag{3}$$

with

- $t \geq 0$, $x \in [0,1]$.
- $\dot{W}$ space-time white noise



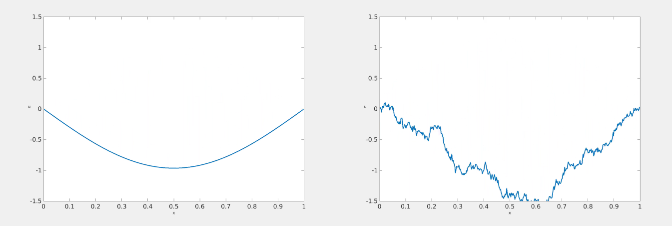
Another simulation by David Cohen

Equation: Stochastic wave equation on $[0,1]$:

$$\partial^2_{tt} u_t(x) = \frac{1}{2}\Delta u_t(x) + u\,\dot{W}_t(x), \tag{4}$$

with

- $t \geq 0$, $x \in [0,1]$.
- $\dot{W}$ space-time white noise



Where is Marta?

Some contributions by Marta:

Millet-Sanz	Wave $d = 2$, smoothness of density AOP 99
Quer-Sanz	Wave $d = 3$, smoothness of density, JFA 04, Bernoulli 05
Sanz	Wave $d = 3$, regularity of $(t, x) \mapsto p_{t,x}(y)$ JFA 08
Sanz-Süß	Wave $d \geq 4$, smoothness of density, EJP 13, ECP 15
Delgado-Sanz	Wave $d = 3$, support theorem Bernoulli 16

Mild formulation

Notation: We set

- $G_t(x) \equiv$ fundamental solution of the wave equation

Duhamel's principle: The solution to

$$\partial^2_{tt} u_t(x) = \frac{1}{2}\Delta u_t(x) + \sigma\left(u_t(x)\right)\, \dot{W}_t(x), \quad u(0,x) = \partial_t u(0,x) = 0$$

can be written as

$$u_t(x) = \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\, \sigma(u_s(y))\, W(\mathrm{d}s, \mathrm{d}y)$$

Fundamental solution

Notation: Set

$$\rho_t = \text{ Uniform measure on sphere with radius } t.$$

Expression for the fundamental solution: We have

$$G_t(x) = \begin{cases} \frac{1}{2}\mathbf{1}_{[|x|<t]} & \text{if } d = 1, \\ \frac{1}{2\pi}\frac{1}{\sqrt{t^2 - |x|^2}}\mathbf{1}_{[|x|<t]} & \text{if } d = 2,, \\ \frac{1}{4\pi t}\,\rho_t(dx) & \text{if } d = 3, \\ \text{Derivatives of } \rho_t & \text{if } d \geq 4. \end{cases}$$

Conclusion: Ugly expressions as d gets large!

Rough idea of the method (1)

Fundamental solution in Fourier modes: We have

$$\mathcal{F}G_t(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}.$$

Basic inequality: We have

$$\mathbf{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\,\sigma(u_s(y))W(\mathrm{d}s,\mathrm{d}y)\right)^2\right]$$
$$\leq \int_0^t \mathbf{E}\left[\sigma(u(s,0))^2\right] \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \left|\mathcal{F}G_{t-s}(\xi+\eta)\right|^2 \mu(\mathrm{d}\xi)\mathrm{d}s$$

Rough idea of the method (2)

Typical assumption: For the spectral measure μ,

$$\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mu(\mathrm{d}\xi)}{1+|\xi+\eta|^2} < \infty. \tag{5}$$

Riesz kernel: If Λ (spatial covariance) is given by

$$\Lambda(x) = |x|^{-\beta}$$

then the assumption (5) satisfied if

$$0 < \beta < \inf\{2, d\}$$

Rough idea of the method (3)

Difficulties:

- Non standard stochastic integration theory
- Intricate estimates for singularities
- For $d = 3$, delicate considerations for $G_t \equiv$ measure
- For $d \geq 4$, lack of L^p-estimates, non positivity of G_t

Marta's solutions:

- Delicate approximation procedures
- Sharp integral estimates
- New advanced Malliavin calculus tools

Outline

An additive case (1)

First equation under consideration:

$$\partial_{tt}^2 u_t(x) = \frac{1}{2}\Delta u_t(x) - u_t^2(x) + \dot{W}_t(x), \tag{6}$$

Approach: Solution as perturbation of the stochastic convolution

$$\Psi_t(x) = \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\, W(\mathrm{d}s, \mathrm{d}y)$$

Equation for $v \equiv u - \Psi$:

$$\partial_{tt}^2 v_t(x) = \frac{1}{2}\Delta v_t(x) - \left(v_t(x) + \Psi_t(x)\right)^2$$

An additive case (2)

Problem: When W rough or dimension high
$\hookrightarrow$ Ψ is a distribution and Ψ^2 ill-defined

Renormalized equation: One considers

- Smooth approximation of the noise W^n
- Family $\{u^n;\, n \geq 1\}$
- $\sigma_n \sim 2^{n\gamma} t$

such that

$$\partial^2_{tt} u^n_t(x) = \frac{1}{2} \Delta u^n_t(x) - \left[(u^n_t(x))^2 - \sigma_n(t) \right] + \dot{W}^n_t(x),$$

Then (Gubinelli-Koch-Oh, Deya) u^n converges
$\hookrightarrow$ to renormalized version of (7)

A multiplicative case (1)

First equation under consideration:

$$\partial_{tt}^2 u_t(x) = \frac{1}{2}\Delta u_t(x) + u_t(x)\dot{W}_t(x), \tag{7}$$

Approach (Chen-Deya-Song-T):

- Mild form of the equation in pathwise sense:

$$u_t(x) = \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\, u_s(y) W(\mathrm{d}s, \mathrm{d}y)$$

- Smoothing effect of wave kernel G
- Young type integration

A multiplicative case (2)

Weighted Besov space: We set

- $\mathcal{B}^\alpha \equiv$ weighted Besov space with exponential weight on $\mathbb{R}^d$
- Parameters μ, p, q in $\mathcal{B}^{\alpha,\mu}_{p,q}$ not specified for simplicity

Strichartz type estimates: For all $t \in [0,1]$, it holds that

$$\left\| \mathcal{G}_t f \right\|_{\mathcal{B}^{\alpha+\rho_d}} \lesssim \| f \|_{\mathcal{B}^\alpha} , \qquad \text{with } \rho_d \equiv \begin{cases} 1 & \text{if } d = 1 \\ \frac{1}{2} & \text{if } d = 2 \end{cases}$$

Remarks:

- Those Strichartz type estimates appear to be new
- They rely on Ryzhkov's version of weighted Besov spaces
- Possibility of regularity structure type expansions

Thanks

Concluding remarks:

- Still plenty to do in 2022 on stochastic wave equations
- Thanks Marta for your contributions!

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