Taming uncertainty and profiting from randomness

Michael Röckner
(Bielefeld University and Academy of Mathematics and Systems Science,
Chinese Academy of Sciences, Beijing)

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Reference: "Generations of stochastic analysts"
Including: Huge contribution by Marta

Contents

1. Introduction

- smashing a vase
- the man/woman and the wall
- the skier and his/her favorite pub

- stochastic resonance

Fact: Roughly every 100,000 years "big" (that is, very cold) ice ages.
One possible explanation: (Klaus Ferdinand Hasselmann, Max-Planck-Institute for Meteorology, Hamburg, 1976)

random influences

(Phenomenon of "**stochastic resonance**").
Using this, in a pioneering paper four physicists Roberti Benzi, Giorgio Parisi, Alfonso Sufera, Angelo Vulpiani, in 1982, succeeded in giving an explanation for the "big" ice ages every 100,000 years.

2. Stochastics dynamical processes and stochastic (partial) differential equations

SDE on $\mathbb{R}^d$

$$\begin{aligned} dX(t) &= b(t, X(t))dt + \sigma(t, X(t))dW(t), \quad t \in [0, T] \\ X(0) &= x \in \mathbb{R}^d, \end{aligned}$$

with measurable

$$\begin{aligned} &b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \\ &\sigma : [0, T] \times \mathbb{R}^d \to (d \times d) - \text{ real matrices;} \end{aligned}$$

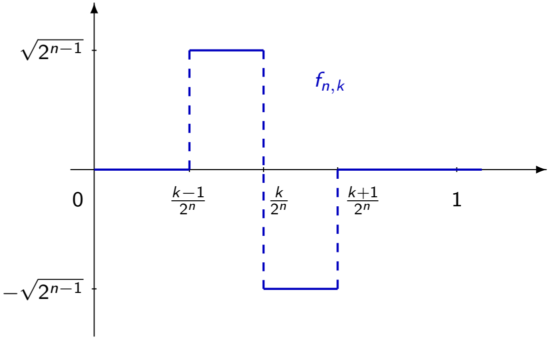
$W(t)$, $t \geq 0$, **Brownian motion** on $\mathbb{R}^d$,
i.e.

$$X(t)(\omega) = x + \int_0^t b(t, X(t)(\omega))dt + \left(\int_0^t \sigma(t, X(t))dW(t) \right)(\omega), \quad t \in [0, T].$$

Recall: Brownian Motion

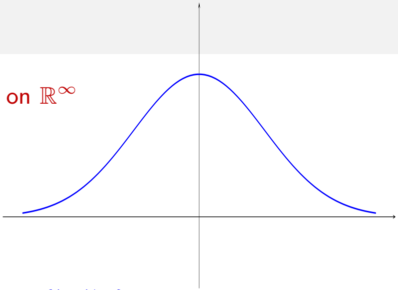
[Lévy–Wiener–Ciesielski]
First ingredient: Haarbasis of $L^2([0,1], \mathrm{d}t)$:

$f_{0,0} :\equiv 1$, and for $n \in \mathbb{N}$, $0 < k < 2^n$, k odd,



Observe: $(f_{n,k})_{\substack{0<k<2^n,\ k \text{ odd},\\ n\in\mathbb{N}}}$ is ONB of $L^2([0,1], \mathrm{d}t)$

Recall: Brownian Motion



Second ingredient: Standard normal distribution on $\mathbb{R}^\infty$

Standard normal (Gauss) distribution γ on $\mathbb{R}^1$:

$$\gamma(\mathrm{d}x) := \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \underbrace{\mathrm{d}x}_{\substack{\text{Lebesgue meas.}\\ \text{on } \mathbb{R}^1}}$$

Set $\gamma_{n,k} := \gamma$ and

$$\mathbb{P} := \bigotimes_{\substack{0<k<2^n,\\ k \text{ odd},\\ n\in\mathbb{N}}} \gamma_{n,k} \quad \text{product measure on } \mathbb{R}^\infty \; (= \mathbb{R}^{\{(n,k)|\dots\}} =: \Omega)$$

Define $\xi_{n,k} : \mathbb{R}^{\{(n,k)|n\in\mathbb{N},\, 0<k<2^n,\, k \text{ odd}\}\cup\{(0,0)\}} \to \mathbb{R}$ (projection) and for $t \in [0,1]$ the Brownian motion $W(t)(\omega)$ by

$$W(t)(\omega) := \sum_{(n,k)} \left(\xi_{n,k}(\omega) \int_0^t f_{n,k}(s)\, \mathrm{d}s \right) \text{(converges uniformly in } t \in [0,1]$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$).

Heuristic motivation for this type of equations:

Let $F : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. Consider SODE

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t}(\omega) = F(t, X(t)(\omega), \underbrace{\dot{W}}_{\text{random path}}(t)(\omega)), \qquad t \in [0, T]$$

Worst case scenario:

$\dot{W}(t),\ t \in [0, T]$, all independent, (centered) with infinite variance, i.e.
$\mathbb{E}\left[\dot{W}^i(t)^2\right] = +\infty\ \forall i \in \mathbb{N}$, $\dot{W}^i(t) =$ i-th component of $\dot{W}(t)$.
Then by Taylor expansion (around 0) in the third variable up to first order:

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = F(t, X(t), 0) + D_3F(t, X(t), 0)\dot{W}(t)\ .$$

In integral form for $t \in [0, T]$

$$X(t) = X(0) + \int_0^t \underbrace{F(s, X(s), 0)}_{=:b(s,X(s))} \mathrm{d}s + \int_0^t \underbrace{D_3 F(s, X(s), 0)}_{=:\sigma(s,X(s))} \underbrace{\dot{W}(s)\mathrm{d}s}_{\mathrm{d}W(s)}$$

Assume: $\dot{W}(t),\ t \in [0, \infty)$, are Gaussian.

Then:

- $\dot{W}(t),\ t \in [0, \infty)$, "white noise", only exists as generalized function (= Schwartz distribution) in t.
- $W(t),\ t \in [0, \infty)$, Wiener process or Brownian motion on $\mathbb{R}^d$ (with $\dot{W}(t) := \frac{d}{dt} W(t)$ in the sense of generalized functions in t).

Likewise SPDE on $H \overset{(e.g.)}{=}$ separable Hilbert space

$$dX(t) = -A(t, X(t))dt + \sigma(t, X(t))dW(t), \quad t \in [0, T],$$

with measurable

$$A : [0, T] \times H \to H$$
$$\sigma : [0, T] \times H \to L(H, H) \text{ (:= all bounded linear operators on } H)$$

$W(t)$, $t \geq 0$, **Brownian motion** on H.

Let $F : [0, T] \times H \times H \to H$. Consider ODE

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = F(t, X(t), \underbrace{\dot{W}}_{\text{random path}}(t)), \qquad t \in [0, T]$$

Worst case scenario:

$\dot{W}(t)$, $t \in [0, T]$, all independent, (centered) with infinite variance, i.e. $\mathbb{E}\left[\dot{W}^i(t)^2\right] = +\infty \; \forall i \in \mathbb{N}$, $\dot{W}^i(t) =$ i-th component of $\dot{W}(t)$ w.r.t. ONB of H.
Then by Taylor expansion (around 0) in the third variable up to first order:

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = F(t, X(t), 0) + D_3F(t, X(t), 0)\dot{W}(t) .$$

In integral form for $t \in [0, T]$

$$X(t) = X(0) + \int_0^t \underbrace{F(s, X(s), 0)}_{=:-A(s,X(s))} \mathrm{d}s + \int_0^t \underbrace{D_3 F(s, X(s), 0)}_{=:\sigma(s,X(s))} \underbrace{\dot{W}(s)\mathrm{d}s}_{\mathrm{d}W(s)}$$

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3. Profiting from randomness

Recall: $W(t) : \Omega(:= \mathbb{R}^\infty) \to \mathbb{R}^d$,

Regularization by noise

Let $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ such that

$$\int_0^T \left(\int_{\mathbb{R}^d} |b(t,x)|^p_{\mathbb{R}^d} dx \right)^{\frac{q}{p}} dt < \infty$$

where $p \in [2, \infty)$, $q \in (2, \infty)$ with $\frac{d}{p} + \frac{2}{q} < 1$. Then ([Krylov/R.: Probab. Th. Rel. Fields 2005])

$$\begin{aligned} dX(t) &= b(t, X(t))dt + dW(t), \quad t \in [0, T] \\ X(0) &= x \in \mathbb{R}^d, \end{aligned} \qquad \text{(SDE)}$$

i.e.,

$$X(t)(\omega) = x + \int_0^t b(s, X(s)(\omega))ds + W(t)(\omega), \quad t \in [0, T], \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega$$

has a unique solution, where "unique" means:

$$X, \tilde{X} \text{ solve (SDE)} \Rightarrow \mathbb{P}\Big(\{\omega \in \Omega | X(t)(\omega) = \tilde{X}(t)(\omega) \, \forall t \in [0, T]\} \Big) = 1.$$

But without $W(t)$ **no** solution exists! **WHY?**

Analytic reason:

"Linearize" (SDE): Take solution $X(t,x)$, $t \in [0,T]$, of (SDE), and consider all $\varphi : \mathbb{R}^d \to \mathbb{R}$ smooth with compact support.
Then by Itô's formula (= chain rule)

$$d\varphi(X(t,x)) = (\nabla\varphi)(X(t,x))dX(t,x) + \frac{1}{2}\underbrace{(\Delta\varphi)}_{\substack{\text{appears, because } t \mapsto X(t,x) \text{ is \textbf{only} of} \\ \text{bounded quadratic variation} \\ \text{(\textbf{not} of bounded variation)} \\ \textbf{because of randomness}}}(X(t,x))dt$$

$$\underset{\text{(SDE)}}{=} (\nabla\varphi)(X(t,x)) \cdot b(t, X(t,x))dt + (\nabla\varphi)(X(t,x)) \cdot dW(t) + \frac{1}{2}(\Delta\varphi)(X(t,x))dt$$

i.e., $$\varphi(X(t,x)) = \varphi(x) + \int_0^t (\nabla\varphi)(X(s,x)) \cdot b(s, X(s,x))dt + \int_0^t \nabla\varphi(X(s,x)) \cdot dW(s) + \frac{1}{2}\int_0^t (\Delta\varphi)(X(s,x))ds$$

Now take $\int_\Omega \mathbb{P}(d\omega)$ to get

$$\int_\Omega \varphi(X(t,x)(\omega))\mathbb{P}(d\omega) = \varphi(x) + \int_0^t \int_\Omega (\nabla\varphi)(X(s,x)(\omega)) \cdot b(s, X(s,x)(\omega))\mathbb{P}(d\omega)ds$$
$$+ \underbrace{\int_\Omega \left(\int_0^t (\nabla\varphi)(X(s,x)) \cdot dW(s) \right)(\omega)\mathbb{P}(d\omega)}_{=0\ !}$$
$$+ \frac{1}{2}\int_0^t \int_\Omega (\Delta\varphi)(X(s,x)(\omega))\mathbb{P}(d\omega)ds$$

Then with $\mu_t^x := (X(t,x))^*\mathbb{P}$ (= push forward or image measure), $t \in [0,T]$ get

$$\int_{\mathbb{R}^d} \varphi(y)\mu_t^x(dy) = \varphi(x) + \int_0^t \int_{\mathbb{R}^d} b(s,y)\cdot\nabla\varphi(y)\mu_s^x(dy) + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \Delta\varphi(y)\mu_s^x(dy)ds.$$

Taking $\frac{d}{dt}$

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi(y)\mu_t^x(dy) = \int_{\mathbb{R}^d} \left(\frac{1}{2}\Delta + b(t,y)\cdot\nabla\right)\varphi(y)\mu_s^x(dy)$$

or (since "for all φ") get a linear PDE for "time marginal laws", μ_t^x, $t \in [0,T]$,

$$\frac{\partial}{\partial t}\mu_t^x = \left(\frac{1}{2}\Delta + b(t,\cdot)\cdot\nabla\right)^*\mu_t^x.$$ **Linear Fokker-Planck equation.**

Elliptic! Because of Δ, i.e. because of noise $W(t)$, $t \geq 0$.

So, have "profited from noise" to get well-posedness and understood where this comes from.

Remark 1

For critical case $\frac{d}{p} + \frac{2}{q} = 1,\ p, q \in (2, \infty)$, *see [**Guohuan Zhao**/R.: arXiv: 2103.05803].*

Remark 2

Have

$$SDE \Leftrightarrow FPE.$$

Have a **nonlinear** *analogue.*

$$\text{McKean Vlasov SDE} \rightleftarrows \text{nonlinear FPE}$$

*[**Barbu**/R.: Annals of Probability 2020]* *(Existence)*
*[**Barbu**/R.: arXiv: 2203.00122]* *(Uniqueness)*

4. Taming uncertainty

Let us go back to our SPDE on $H =$ separable Hilbert space, e.g. here $H := L^2(\mathcal{O})$, $\mathcal{O} \subset \mathbb{R}^d$, open, bounded

$$dX(t) = -A(t, X(t))dt + \underbrace{\sigma(t, X(t))}_{\underset{\text{Taylor}}{\approx} \sigma(t,0)+D_2\sigma(t,0)X(t)} dW(t). \qquad (SPDE)$$

Assume that $W(t)$ has the following representation

$$W(t) := \sum_{j=1}^{\infty} \mu_j e_j(\cdot) W^j(t),$$

where W^j, $j \in \mathbb{N}$, independent Brownian motions in $\mathbb{R}$, $\{e_j | j \in \mathbb{N}\}$ ONB of $H = L^2(\mathcal{O})$, $\sum_{j=1}^{\infty} \mu_j^2 \, (1 + \|e_j\|_\infty) < \infty$, and for simplicity assume

$$\sigma(t, X(t))dW(t) = \underbrace{X(t)}_{\in L^2(\mathcal{O})} \cdot d\underbrace{W(t)}_{\in L^\infty(\mathcal{O})} .$$

Assume have $V \subset H \subset V'$ Gelfand triple (e.g. $H^1 \subset L^2 \subset (H^1)'$)

$$A : [0, T] \times V \longrightarrow V',$$

$$\sigma : [0, T] \times H \longrightarrow L(H, H),$$

$W(t)$, $t \geq 0$, Brownian motion on H.

Examples

Fix $\mathcal{O} \subset \mathbb{R}^d$, open, bounded, $\partial\mathcal{O}$ smooth.

(1) **Stochastic porous media equations**
Let $H := H^{-1}(\mathcal{O})$.

$$\mathrm{d}X(t) \underbrace{-\Delta(\psi(X(t)))}_{A(t,X(t))} \mathrm{d}t = X(t)\mathrm{d}W(t), \quad t \in [0, T] .$$

- $\psi : \mathbb{R} \to \mathbb{R}$ continuous, $\psi(0) = 0$
- $(\psi(r) - \psi(r^{'}))(r - r^{'}) \geq 0 \quad (\Leftrightarrow\ \psi$ increasing)
- $r\psi(r) \geq c_1|r|^p - c_2 \qquad ; c_1, c_2 \in (0, \infty), p \in (1, \infty)$
- $|\psi(r)| \leq c_3|r|^{p-1} + c_4 \quad ; c_3, c_4 \in (0, \infty)$

E.g.: $\psi(r) = r|r|^{p-1}$, $\quad p > 1$,
i.e. stochastic classical porous media equation.

(2) **Stochastic nonlinear parabolic equations**
Let $H := L^2(\mathcal{O})$.

$$\mathrm{d}X(t) \underbrace{- \operatorname{div}(a(\nabla X(t)))}_{A(t,X(t))} \mathrm{d}t = X(t)\mathrm{d}W(t) , \quad t \in [0, T] ,$$

- $a : \mathbb{R}^d \to \mathbb{R}^d$, continuous, $a(0) = 0$
- $\langle a(r) - a(r'), r - r' \rangle_{\mathbb{R}^d} \geq 0$ "a increasing"
- $\langle r, a(r) \rangle_{\mathbb{R}^d} \geq c_1 \|r\|^p_{\mathbb{R}^d} - c_2$; $c_1, c_2 \in (0, \infty)$, $p \in (1, \infty)$
- $\|a(r)\|_{\mathbb{R}^d} \leq c_3 \|r\|^{p-1}_{\mathbb{R}^d} + c_4$; $c_3, c_4 \in (0, \infty)$

E.g.: $a(r) = r\|r\|^{p-1}_{\mathbb{R}^d}$, $p > 1$,
i.e. stochastic parabolic p-Laplace equation.

By the transformation

$$Y(t)(\omega) := e^{-W(t)(\omega)} X(t)(\omega), \quad t \in [0, T],$$

and Itô's formula (SPDE) transforms into random PDE

$$\frac{dy}{dt} + e^{-W(t)(\omega)} A(t, e^{W(t)(\omega)} y(t)) + \mu y(t) = 0 \text{ for } t - \text{a.e. } t \in (0, T) \qquad \text{(RPDE)}$$
$$y(0) = x,$$

and vice versa, where

$$\mu(\xi) := \frac{1}{2} \sum_{j=1}^{\infty} \mu_j^2 e_j^2(\xi), \ \ \xi \in \mathcal{O}.$$

Now consider the following (huge!) spaces of random paths:

$\mathcal{V}$:= all measurable maps $y : [0, T] \times \Omega \to V$,
such that

$$|y|_{\mathcal{V}} := \left(\int_\Omega \int_0^T \left| e^{W(t)(\omega)} y(t)(\omega) \right|_V^p dt\, \mathbb{P}(d\omega) \right)^{\frac{1}{p}} < \infty;$$

$\mathcal{H}$:= all measurable maps $y : [0, T] \times \Omega \to H$,
such that

$$|y|_{\mathcal{H}} := \left(\int_\Omega \int_0^T \left| e^{W(t)(\omega)} y(t)(\omega) \right|_H^2 dt\mathbb{P}(d\omega) \right)^{\frac{1}{2}} < \infty;$$

$\mathcal{V}'$:= all measurable maps $y : [0, T] \times \Omega \to V'$,
such that

$$|y|_{\mathcal{V}'} := \left(\int_\Omega \int_0^T \left| e^{W(t)(\omega)} y(t)(\omega) \right|_{V'}^{p'} dt\mathbb{P}(d\omega) \right)^{\frac{1}{p'}} < \infty.$$

Now define $\mathcal{A} : \mathcal{V} \longrightarrow \mathcal{V}'$ and $\mathcal{B} : D(\mathcal{B}) \subset \mathcal{V} \longrightarrow \mathcal{V}'$ by

$$
\begin{aligned}
(\mathcal{A}y)(t) &= e^{-W(t)}A(t)(e^{W(t)}y(t)), \quad t \in (0,T),\, y \in \mathcal{V}, \\
(\mathcal{B}y)(t) &= \frac{dy}{dt}(t) + \mu y(t), \quad t \in (0,T),\; y \in D(\mathcal{B}), \\
D(\mathcal{B}) &= \Big\{ y \in \mathcal{V} : y \in AC([0,T];V') \cap C([0,T];H),\; \mathbb{P}\text{-a.s.}, \\
&\qquad\qquad \frac{dy}{dt} \in \mathcal{V}',\; y(0) = x \Big\}.
\end{aligned}
$$

Here, $AC([0,T];V')$ is the space of all absolutely continuous V'-valued functions on $[0,T]$. Then

$$\mathcal{A} + \mathcal{B} : \mathcal{D}(\mathcal{B}) \subset \mathcal{V} \to \mathcal{V}'$$

and (RPDE) can be rewritten as

$$(\mathcal{A} + \mathcal{B})(y) = 0.$$

This means our solutions to (SPDE) resp. (RPDE) are zeros of the map

$$\mathcal{A} + \mathcal{B} : \mathcal{D}(\mathcal{B}) \subset \mathcal{V} \to \mathcal{V}'.$$

So, to have existence of solutions, it is more than enough to prove that this map is **onto**. And for uniqueness that it is one-to-one.

This can in fact be done (see [Barbu/R.: JEMS 2015]) by using the theory of (maximal) monotone operators on a Gelfant triple. The proof of the monotonicity of the (time) operator $\mathcal{B}$ depends on an infinite dimensional Itô formula!

So, well-posedness for such SPDEs resp. existence and uniqueness of the corresponding stochastic dynamics (+ properties...) achieved through "taming uncertainty" by mathematical tools from analysis.

Marta's CONTRIBUTIONS — a selection:

Hitting probabilities of anisotropic Gaussian random fields; stochastic wave equations with superlinear coefficients, hitting probablities of stochastic Poisson equations; support theorem in Hölder norm of a stochastic wave equation; hitting probabilities for nonliear systems of stochastic waves; absolute conitnuity for SPDEs; logarithmic asymptotics of the densities of SPDEs; Malliavin differentiability and absolute continuity; a Laplace principle for a stochastic wave equation; hitting probabilities with applications to systems of stochastic wave equations; fractional Poisson equation; Hölder-Sobolev regularity of the solution; mild solutions for a class of fractional SPDEs; properties of the density for a three-dimensional stochastic wave equation; approximation of rough parts of fractional Brownian motion; smoothness of the functional law; probability density for a hyperbolic SPDE; large deviations for rough paths of the fractional Brownian motion; regularity of the sample paths of a class of second-order SPDEs; Malliavin calculus; stochastic wave equation; absolute continuity of the law of the solution to the 3-dimensional stochastic wave equation; existence of density for the solution to the 3-dimensional stochastic wave equation; equivalence and Hölder-Sobolev regularity of solutions; positivity of the density for the stochastic wave equation in two spatial dimensions; Hölder continuity for the stochastic heat equation with spatially correlated noise; applications of Malliavin calculus to SPDEs; Hölder-Sobolev regularity; logarithmic estimates for the density of hypoelliptic two-parameter diffusions; stochastic Volterra equations in the plane; asymptotic behavior of the density in a parabolic SPDE;

path properties of a class of Gaussian processes with applications to SPDEs; support theorem for a wave equation; stochastic delay equations with hereditary drift; large deviations for stochastic Volterra equations in the plane; expansion of the density: a Wiener-chaos approach; logarithmic estimates for the density of an anticipating stochastic differential equation; Taylor expansion of the density in a stochastic heat equation; regularity of the law for a class of anticipating stochastic differential equations; points of positive density for the solution to a hyperbolic SPDE; small perturbations in a hyperbolic stochastic partial differential equation; existence and regularity of density; anticipating stochastic differential equations: regularity of the law; Varadhan estimates for the density; the law of the solution to a nonlinear hyperbolic SPDE; strong approximations for stochastic differential equations with boundary conditions; a Fubini theorem for generalized Stratonovich integrals; a nonlinear hyperbolic SPDE: approximations and support; approximation and support theorem; Green formulas in anticipating stochastic calculus; support theorem for diffusion processes; support of the solution to a hyperbolic SPDE; Hilbert-valued anticipating stochastic differential equations; Doob-Meyer decomposition and integrator properties of the Wong-Zakai anticipating integral; on the support of a Skorhod anticipating stochastic differential equation; moduls of continuity for stochastic flows; Skorohod integral; The Hu-Meyer formula for nondeterministic kernels; large deviations for a class of anticipating stochastic differential equations; un théorème de support pour une équation aux dérivées partielles stochastique hyperbolique; Itô formula for two-parameter martingales; une remarque sur la théorie des grandes déviations; small perturbations for quasilinear anticipating stochastic differential equations; Doob-Meyer decomposition for anticipating processes;

composition of large deviation principles and applications; nonadaptive stochastic calculus; on generalized multiple stochastic integrals and multiparameter anticipative calculus; application of Malliavin calculus; two-parameter continuous martingales; déviation stochastique de diffusions réfléchies; Planar semimartingales; smoothness of the solution; two-parameter continuous martingales and Itô's formula; time reversal for infinite-dimensional diffusions; integration by parts and time reversal for diffusion processes; two-parameter continuous martingales; Malliavin calculus for two-parameter Wiener functionals; Malliavin calculus for two-parameter processes; a singular stochastic integral equation; two-parameter strong martingales; conditional independence property in filtrations associated to shopping lines; a Markov property for two-parameter Gaussian processes; caractérisation des martingales à deux paramètres indépendantes du chemin; stochastic differential calculus for processes with n-dimensional parameter; processus de Wiener à deux paramètres; etc. ...

THANK YOU VERY MUCH, MARTA!

AND ALL THE BEST FOR MANY MORE HAPPY AND SCIENTIFICALLY PRODUCTIVE YEARS !