Transport noise models from two-scale systems with additive noise in fluid dynamics.

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In honor of Marta Sanz-Solé.

Stochastic model for turbulent flows

Stochastic Navier-Stokes equation with transport noise on a domain $\mathcal{O} \subset \mathbb{R}^d$, $d = 2, 3$:

$$
\begin{cases}
du = (\nu\Delta u + ((u+\bar{u})\cdot\nabla u) + \nabla p)dt + \phi dW \circ \nabla u + \tilde{\phi} d\widetilde{W} \\
\quad = (\nu\Delta u + K_\phi u + ((u+\bar{u})\cdot\nabla u) + \nabla p))dt + \phi dW \cdot \nabla u + \tilde{\phi} d\widetilde{W}, \\
\mathrm{div}\, u = 0,
\end{cases}
$$

- $u = u(x,t)$, $x \in \mathcal{O}$, $t > 0$ is the velocity; $p = p(x,t)$ is the pressure; W and $\widetilde{W}$ are cylindrical Wiener process and ϕ, $\tilde{\phi}$ are the covariance operator (in other words, since these are smoothing operator, ϕdW, $\tilde{\phi} d\widetilde{W}$ are spatially smooth time white noises); $\bar{u}$ is the Ito-Stokes drift.
- These equations are supplemented with boundary conditions: for instance Dirichlet, periodic if $\mathcal{O} = \mathbb{T}^d$, or decaying at infinity if $\mathcal{O} = \mathbb{R}^d$.

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- R. Mikulevicius & B. Rozovsky, E. Mémin and co-authors (see also Z. Brzezniak, M. Capinski, , and F. Flandoli): start with the Lagrangian description of the fluid and add a perturbation.
- D. Holm and co-authors: derive stochastic fluid equation from variational principle, SALT model (Stochastic advection by Lie transport). A slightly different equation is obtained. This equation has the property to have similar conservation properties as in the deterministic case.

Stochastic model for turbulent flows

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- Write
$$dx = u(x(t), t) + \phi(x(t), t) dW.$$
The noise models the small unresolved scales.
- Do the classical derivation of fluid mechanics equation by looking at the evolution of a volume transported by the stochastic flow. Ito-Wentzel formula introduces the additional terms in the Navier-Stokes equations.

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The noise models the small unresolved scales.
- Do the classical derivation of fluid mechanics equation by looking at the evolution of a volume transported by the stochastic flow. Ito-Wentzel formula introduces the additional terms in the Navier-Stokes equations.
- The derivation is not rigorous because: 1. The deterministic derivation is not rigorous. 2. Some terms like the derivative of the white noise have to be discarded.
- Many other models (quasi-geostrophic equations, primitive equations ...) can be derived in this way.

Mathematical framework

$$\begin{cases} du = (\nu\Delta u + ((u+\bar{u})\cdot\nabla u) + \nabla p)dt + \phi dW \circ \nabla u + \tilde{\phi} d\widetilde{W} \\ \mathrm{div}\, u = 0. \end{cases}$$

- Introduce

$$H = \{u \in (L^2(\mathcal{O}))^d,\ \mathrm{div}\, u = 0,\ u\cdot n = 0 \text{ on } \partial\mathcal{O}\},$$

  P the Leray projector on H, the Stokes operator

$$A = \nu P\Delta \text{ on } D(A) = (H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}))^d \cap H$$

  and $b(u,v) = P(u\cdot\nabla v)$.
- Rewrite the equation as:

$$du = Au + b(u+\bar{u}, u)dt + b^o(\phi dW, u) + P\tilde{\phi} d\widetilde{W}.$$

- Well-posedness has been studied by various authors.

A multiscale approach

Inspired by works by A. Majda, P. Kramer, I. Timorfeyev, E. Van den Eijden, F. Flandoli proposed to study a multiscale fluid equation:

$$\begin{cases} \partial_t u = Au + b(u+v,u), \\ dv = (Av + \frac{1}{\varepsilon}Cv)dt + b(u+v,v)dt + \frac{1}{\varepsilon}\phi dW, \end{cases}$$

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- A natural choice is $C = I$, this represents a friction.
- Formally, when $\varepsilon \to 0$, we get $v = (-C)^{-1}\phi dW$ and the Navier-Stokes equation with transport noise:

$$du = Au + b(u,u) + b^o((-C)^{-1}\phi dW, u).$$

This misses the Ito-Stokes drift.

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- $$du = Au + b(u,u) + b^o((-C)^{-1}\phi dW, u).$$
- F. Flandoli and U. Pappalettera studied the 2d case without viscosity: $A = 0$. They use the vorticity form of the equation and characteristic method. The solutions of the characteristics of the fast equations "converge" to a white noise and they obtain a sort of Wong-Zakai result.
- The same result would be obtain without the nonlinear term in the fast equation.

Remark: Another scaling

- The scaling

$$\begin{cases} \partial_t u = Au + b(u+v,u), \\ dv = (Av + \frac{1}{\varepsilon}Cv)dt + b(u+v,v))dt + \frac{1}{\varepsilon^{1/2}}\phi dW, \end{cases}$$

  is also very natural. This is the averaging regime.
- The limit does not contain any trace of the small scales or of the noise: $\partial_t u = Au + b(u,u)$

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  is also very natural. This is the averaging regime.
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- A non trivial limit would be obtained from:

$$\begin{cases} \partial_t u = Au + b(u+v,u), \\ dv = \frac{1}{\varepsilon}(Av + b(u+v,v))dt + \frac{1}{\varepsilon^{1/2}}\phi dW, \end{cases}$$

$$\rightsquigarrow \partial_t u = Au + b(u,u) + \int_H b(v,u)d\nu_u(v).$$

Generators

Consider first the simpler case:

$$\begin{cases} \partial_t u = Au + b(u+v, u), \\ dv = \frac{1}{\varepsilon} Cvdt + \frac{1}{\varepsilon} \phi dW, \end{cases}$$

- v is of order $\varepsilon^{-1/2} \rightsquigarrow w = \varepsilon^{1/2} v$ and

$$\begin{cases} \partial_t u = Au + b(u, u) + \frac{1}{\varepsilon^{1/2}} b(w, u), \\ dw = \frac{1}{\varepsilon} Cwdt + \frac{1}{\varepsilon^{1/2}} \phi dW, \end{cases}$$

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- The generator:

$$\mathcal{L}_\varepsilon \varphi(u,w) = \langle Au + b(u,u), D_u\varphi(u,w)\rangle + \frac{1}{\varepsilon^{1/2}} \langle b(w,u), D_u\varphi(u,w)\rangle$$

$$+\frac{1}{\varepsilon}\langle Cw, D_w\varphi(u,w)\rangle + \frac{1}{2\varepsilon} Tr(\phi^2 D^2_{ww}\varphi(u,w)).$$

Perturbed test function method

$$\begin{cases} \partial_t u = Au + b(u,u) + \frac{1}{\varepsilon^{1/2}} b(w,u), \\ dw = \frac{1}{\varepsilon} Cwdt + \frac{1}{\varepsilon^{1/2}} \phi dW, \end{cases}$$

- The generator:

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where

$$\mathcal{L}_w \varphi(u,w) = \langle Cw, D_w \varphi(u,w) \rangle + \frac{1}{2} Tr(\phi^2 D^2_{ww} \varphi(u,w)).$$

- Use correctors: $\varphi_\varepsilon(u,w) = \varphi(u) + \varepsilon^{1/2} \varphi_1(u,w) + \varepsilon \varphi_2(u,w)$
$\rightsquigarrow \varphi_1(u,w) = (D_u\varphi(u), b((-C)^{-1}w, u)).$

Perturbed test function method

$$\begin{cases} \partial_t u = Au + b(u,u) + \dfrac{1}{\varepsilon^{1/2}} b(w,u), \\ dw = \dfrac{1}{\varepsilon} Cwdt + \dfrac{1}{\varepsilon^{1/2}} \phi dW, \end{cases}$$

- $\varphi_\varepsilon(u,w) = \varphi(u) + \varepsilon^{1/2}\varphi_1(u,w) + \varepsilon\varphi_2(u,w)$

$$\begin{aligned} \mathcal{L}_\varepsilon \varphi_\varepsilon(u,w) &= \langle Au + b(u,u), D_u\varphi_\varepsilon(u,w)\rangle + \frac{1}{\varepsilon^{1/2}} \langle b(w,u), D_u\varphi_\varepsilon(u,w)\rangle \\ &+ \frac{1}{\varepsilon}\mathcal{L}_w\varphi_\varepsilon(u,w) \\ &= \langle Au + b(u,u), D_u\varphi_\varepsilon(u,w)\rangle \\ &+ \int_H \left\langle D_u\varphi(u), b((-C)^{-1}y, b(y,u))\right\rangle d\nu(y) \\ &+ \int_H \left\langle D^2_{uu}\varphi(u)\cdot b(y,u), b((-C)^{-1}y,u)\right\rangle d\nu(y) \\ &+ O(\varepsilon^{1/2}) \\ &= \mathcal{L}_0\varphi(u) + O(\varepsilon^{1/2}). \end{aligned}$$

$\rightsquigarrow du = Au + b(u,u) + b^o((-C)^{-1}\phi dW, u).$

The proof

- Take $\varphi(u) = (u, h)$ and apply Ito formula to $\varphi_\varepsilon(u, w)$:

$$
\begin{aligned}
\varphi(u_t^\epsilon) = \varphi(u_0) &+ \int_0^t \mathcal{L}_0 \varphi(u_s^\epsilon) ds + \int_0^t \langle b((-C)^{-1} Q^{1/2} dW_s, u_s^\epsilon), h\rangle \\
&+ \epsilon^{1/2} \left(\varphi_1(u_0, w_0) - \varphi_1(u_t^\epsilon, w_t^\epsilon)\right) + \epsilon \left(\varphi_2(u_0, 0) - \varphi_2(u_t^\epsilon, w_t^\epsilon)\right) \\
&+ \epsilon^{1/2} \int_0^t \Phi_1(u_s^\epsilon, w_s^\epsilon, w_s^\epsilon) ds + \epsilon \int_0^t \Phi_2(u_s^\epsilon, w_s^\epsilon) ds \\
&+ \epsilon^{1/2} \int_0^t \langle D_w \varphi_2(u_s^\epsilon, w_s^\epsilon), Q^{1/2} dW_s \rangle,
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_1(u, w, w) &= \langle Au + b(u, u), D_u \varphi_1(w)\rangle + \langle b(u, w), D_w \varphi_1(u)\rangle \\
&\quad + \langle b(w, u), D_u \varphi_2(w)\rangle, \\
\Phi_2(u, w) &= \langle Au + b(u, u), D_u \varphi_2(w)\rangle.
\end{aligned}
$$

- The control of the various remaining terms is delicate due to A and b. Recall that $\varphi_1(u, w) = (D_u \varphi(u), b((-C)^{-1} w, u))$.

The proof

- Take $\varphi(u) = (u, h)$ and apply Ito formula to $\varphi_\varepsilon(u, w)$:

$$\begin{aligned}\varphi(u_t^\epsilon) = \varphi(u_0) &+ \int_0^t \mathcal{L}_0\varphi(u_s^\epsilon)ds + \int_0^t \langle b((-C)^{-1}Q^{1/2}dW_s, u_s^\epsilon), h\rangle \\ &+ \epsilon^{1/2}(\varphi_1(u_0, w_0) - \varphi_1(u_t^\epsilon, w_t^\epsilon)) + \epsilon(\varphi_2(u_0, 0) - \varphi_2(u_t^\epsilon, w_t^\epsilon)) \\ &+ \epsilon^{1/2}\int_0^t \Phi_1(u_s^\epsilon, w_s^\epsilon, w_s^\epsilon)ds + \epsilon\int_0^t \Phi_2(u_s^\epsilon, w_s^\epsilon)ds \\ &+ \epsilon^{1/2}\int_0^t \langle D_w\varphi_2(u_s^\epsilon, w_s^\epsilon), Q^{1/2}dW_s\rangle,\end{aligned}$$

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- The control of the various remaining terms is delicate due to A and b. Recall that $\varphi_1(u, w) = (D_u\varphi(u), b((-C)^{-1}w, u))$. We cannot take $C = I$, except when $d = 2$ on $\mathbb{T}^2$.

The proof

- Take the test function $\varphi(u) = (u, h)$:

$$\varphi(u_t^\epsilon) = \varphi(u_0) + \int_0^t \mathcal{L}_0\varphi(u_s^\epsilon)ds + \int_0^t \langle b((-C)^{-1}Q^{1/2}dW_s, u_s^\epsilon), h\rangle \tag{1}$$

$$+ \epsilon^{1/2}(\varphi_1(u_0, w_0) - \varphi_1(u_t^\epsilon, w_t^\epsilon)) + \epsilon(\varphi_2(u_0, 0) - \varphi_2(u_t^\epsilon, w_t^\epsilon)$$

$$+ \epsilon^{1/2}\int_0^t \Phi_1(u_s^\epsilon, w_s^\epsilon, w_s^\epsilon)ds + \epsilon\int_0^t \Phi_2(u_s^\epsilon, w_s^\epsilon)ds$$

$$+ \epsilon^{1/2}\int_0^t \langle D_w\varphi_2(u_s^\epsilon, w_s^\epsilon), Q^{1/2}dW_s\rangle,$$

- Prove tightness of $(u_\varepsilon)_{\varepsilon>0}$ and take the limit for a subsequence above.
- A limit point is a weak solution of the stochastic Navier-Stokes equation.

The proof

- Take the test function $\varphi(u) = (u, h)$:

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$$+ \epsilon^{1/2}(\varphi_1(u_0, w_0) - \varphi_1(u_t^\epsilon, w_t^\epsilon)) + \epsilon(\varphi_2(u_0, 0) - \varphi_2(u_t^\epsilon, w_t^\epsilon))$$

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- Prove tightness of $(u_\varepsilon)_{\varepsilon>0}$ and take the limit for a subsequence above.
- A limit point is a weak solution of the stochastic Navier-Stokes equation.
- If pathwise uniqueness holds for the limit problem, the whole sequence converge in probability.

The full problem

$$\begin{cases} \partial_t u = Au + b(u+v,u), \\ dv = \varepsilon^{-1}(\varepsilon Av + Cv)dt + b(u+v,v)dt + \varepsilon^{-1}\phi dW. \end{cases}$$

- Split $v = r + \varepsilon^{1/2}w$:

$$\begin{cases} \partial_t u = Au + b(u+\varepsilon^{-1/2}w + r, u), \\ dw = \varepsilon^{-1}(\varepsilon Aw + Cw)dt + \varepsilon^{-1/2}\phi dW, \\ \partial_t r = \varepsilon^{-1}(\varepsilon Ar + Cr)dt + b(u+\varepsilon^{-1/2}w + r, \varepsilon^{-1/2}w + r). \end{cases}$$

- An averaging phenomenon appears for r, we expect that it converges to

$$\bar{r} = (-C)^{-1}\int_H b(w,w)d\nu(w).$$

- This is a Ito-Stokes drift.

Assumptions

- $Tr(-C)^{-1}\phi^2 \ < \infty$.
- There exists $\Gamma \geq \gamma > 1/4$ such that for $s \in \mathbb{R}, \ \beta > 0$:

$$|x\|^2_{H^{s+\beta\gamma}} \lesssim \|(-C)^{\beta/2}x\|^2_{H^s} \lesssim \|x\|^2_{H^{s+\beta\Gamma}}.$$

- $\nu = \mathcal{N}(0, \frac{1}{2}(-C)^{-1}\phi^2)$. It is supported by H^{s_0} for some s_0 depending on d, Γ:

$$\int_H \|w\|^2_{H^{s_0}}\nu(dw) < \infty.$$

- C and ϕ commute.

Theorem

Let $u_0,\ v_0 \in H$ be given. For $\varepsilon > 0$ there exists a weak solution to:

$$\begin{cases} \partial_t u = Au + b(u+v,u), \\ dv = \varepsilon^{-1}(\varepsilon Av + Cv)dt + b(u+v,v)dt + \varepsilon^{-1}\phi dW. \end{cases}$$

with initial data $u_0,\ v_0$ which is uniformly bounded in

$$\left(L^\infty(\Omega, C([0,T],H) \cap L^2([0,T],H^1))\right) \times \left(L^2(\Omega, C([0,T],H) \cap L^2([0,T],H^1))\right)$$

The laws of $(u_\varepsilon)_{\varepsilon>0}$ are tight in $L^2(0,T,H) \cap C([0,T],H^{-\beta})$ for $\beta > 0$ and every limit point is a weak solution of

$$du = Au + b(u+\bar{r},u) + b^o((-C)^{-1}\phi dW, u).$$

For $d = 2$, the solutions are probabilistically strong and convergence holds in probability.
Moreover on the torus, if $u_0 \in (H^1(\mathbb{T}^2))^2$, we can take $C = I$,

- Similar ideas are used in a recent work of J. Garnier and L. Mertz for finite dimensional SDEs and in the context of the limit from stochastic Zakharov to stochastic Nonlinear Schrödinger equation by A. de Bouard, A.D. and G. Barrué.
- In finite dimension, it is possible to obtain an order of convergence. In our case, it requires a lot of smoothness.

- Similar ideas are used in a recent work of J. Garnier and L. Mertz for finite dimensional SDEs and in the context of the limit from stochastic Zakharov to stochastic Nonlinear Schrödinger equation by A. de Bouard, A.D. and G. Barrué.
- In finite dimension, it is possible to obtain an order of convergence. In our case, it requires a lot of smoothness.
- The model obtained by E. Mémin also contains an additive noise which we do not capture here. This would be captured by the system:

$$\begin{cases} \partial_t u = A(u+v) + b(u+v,u) + \tilde{\phi} d\widetilde{W}, \\ dv = \varepsilon^{-1}(\varepsilon A(u+v) + Cv)dt + b(u+v,v)dt + \varepsilon^{-1}\phi dW. \end{cases}$$

- Another difference is that the divergence of the velocity is not zero in Mémin's model.

Vanishing noise

Following ideas of F. Flandoli and D. Luo we consider:

$$\begin{cases} \partial_t u = Au + b(u+v,u), \\ dv = \varepsilon^{-1}(\varepsilon Av + Cv)dt + b(u+v,v)dt + \varepsilon^{-1}\phi_N dW. \end{cases}$$

and assume that the noise in the limit equation vanishes:

$$\lim_{n\to\infty} \sum_k \|\phi_n e_k\|^2_{H^{-1-2\gamma}} = 0.$$

- If we let $N \to \infty$ first and then $\varepsilon \to 0$, we obtain the deterministic Navier-Stokes equation. No trace of the high scale v at the limit.

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- If we let $N \to \infty$ first and then $\varepsilon \to 0$, we obtain the deterministic Navier-Stokes equation. No trace of the high scale v at the limit.
- If first $\varepsilon \to 0$ and then $N \to \infty$, it may happen that the noise disappears but not the Stratonovitch corrections

$$\kappa_N(u) = \sum_{k\in\mathbb{N}} \frac{q_{N,k}}{2\lambda_k^2} b(e_k, b(e_k,u))$$

- It is easy to check that κ is symmetric and

$$\sum_{k\in\mathbb{N}} \frac{q_{N,k}}{2\lambda_k^2} \langle b(e_k, b(e_k,u)), u\rangle = -\sum_{k\in\mathbb{N}} \frac{q_{N,k}}{2\lambda_k^2} \|b(e_k,u)\|^2_H.$$

Vanishing noise

Theorem Let ϕ_N satisfy the assumptions of the previous result and κ_N is continuous from H^{σ_0} to H for some σ_0, and $\kappa_N(u) \to \kappa(u)$ in H as $N \to \infty$ for $u \in H^{\sigma_0}$. Finally, assume

$$\lim_{N\to\infty} \sum_{k\in\mathbb{N}} \|Q_N^{1/2} e_k\|^2_{H^{-1-2\gamma}} = 0.$$

Then $\{u^{\epsilon,N}\}_{\epsilon>0,N\in\mathbb{N}}$ are tight in $L^2([0,T],H) \cap C([0,T],H^{-\beta})$, and every weak accumulation point u as $\epsilon \to 0$ and $N \to \infty$ is a weak solution of

$$\partial_t u = Au + b(u,u) + \kappa(u).$$

If $u_0 \in H^1$ and $\langle \kappa(u), u\rangle \leq -\kappa_0 \|u\|^2_{H^1}$ for every $u \in H^1$ for some κ_0 sufficiently large (depending on $u_0,\ T$). Then u is the unique strong solution on $[0,T]$, and the whole sequence $\{u^{\epsilon,N}\}_{\epsilon>0,N\in\mathbb{N}}$ converges to u as $\epsilon \to 0$ and $N \to \infty$.

Vanishing noise

- For any $\alpha > 0$, it is possible to construct an example of noise such that
$$\kappa(u) = \alpha \Delta u$$
(from F. Flandoli and D. Luo)
- We can choose $\varepsilon = \varepsilon_N$. The same result holds under the condition:
$$\varepsilon_N^{\frac{\delta}{2(1-\gamma)}} Tr((-A)^{\theta_0/2}(-C)^{-1} Q_N^{1/2}) \to 0,$$
for some $\delta \in (0, \gamma - 1/4)$.
- With the noise constructed by Flandoli and Luo, this amounts to:
$$\varepsilon_N N^{\beta} \to 0$$
for some $\beta > 0$.

2D Surface Quasi-Geostrophic equations

In vorticity form on $\mathbb{T}^2$:

$$\begin{cases} d\xi_t^\epsilon = & -\left((-\nu\Delta)^{1/2}\xi_t^\epsilon + u_t^\epsilon\cdot\nabla\xi_t^\epsilon + \frac{1}{\epsilon^{1/2}}y_t^\epsilon\cdot\nabla\xi_t^\epsilon\right)dt, \\ d\eta_t^\epsilon = & \left(\frac{1}{\epsilon}C\eta_t^\epsilon + (-\nu\Delta)^{1/2}\eta_t^\epsilon - (u_t^\epsilon\cdot\nabla)\eta_t^\epsilon - \frac{1}{\epsilon^{1/2}}(y_t^\epsilon\cdot\nabla)\eta_t^\epsilon\right)dt \\ & +\frac{1}{\epsilon^{1/2}}dW_t, \\ u_t^\epsilon = & -\nabla^\perp(-\Delta)^{-1/2}\xi_t^\epsilon, \\ y_t^\epsilon = & -\nabla^\perp(-\Delta)^{-1/2}\eta_t^\epsilon, \end{cases}$$

- $H = L_0^2(\mathbb{T}^2),\ Au = (-\nu\Delta)^{1/2},\ b(\xi,\eta) = \nabla^\perp(-\Delta)^{-1/2}\xi\cdot\nabla\eta.$
- We obtain convergence to:

$$d\xi_t = A\xi_t dt + b(\xi_t,\xi_t)dt + b((-C)^{-1}Q^{1/2}\circ dW_t,\xi_t) + b(r,\xi_t)dt$$

$$\text{with } r = \int(-C)^{-1}b(w,w)d\mu(w).$$

The primitive equations

$$
\begin{cases}
du_t^\epsilon = \nu\Delta u_t^\epsilon dt - (u_t^\epsilon \cdot \nabla_{\mathbf{x}})u_t^\epsilon dt - v_t^\epsilon \partial_z u_t^\epsilon dt \\
\qquad -\epsilon^{-1/2}(y_t^\epsilon \cdot \nabla_{\mathbf{x}})u_t^\epsilon dt - \epsilon^{-1/2} w_t^\epsilon \partial_z u_t^\epsilon dt + \nabla_{\mathbf{x}} p_t^\epsilon dt, \\
dy_t^\epsilon = \epsilon^{-1} C y_t^\epsilon dt + \nu\Delta y_t^\epsilon dt - (u_t^\epsilon \cdot \nabla_{\mathbf{x}})y_t^\epsilon dt - v_t^\epsilon \partial_z y_t^\epsilon dt \\
\qquad -\epsilon^{-1/2}(y_t^\epsilon \cdot \nabla_{\mathbf{x}})y_t^\epsilon dt - \epsilon^{-1/2} w_t^\epsilon \partial_z y_t^\epsilon dt + \epsilon^{-1/2} d\mathcal{W}_t + \nabla_{\mathbf{x}} q_t^\epsilon dt, \\
\partial_z p_t^\epsilon = 0, \ \partial_z q_t^\epsilon = 0, \\
\mathrm{div}_{\mathbf{x}} u_t^\epsilon + \partial_z v_t^\epsilon = 0, \ \mathrm{div}_{\mathbf{x}} y_t^\epsilon + \partial_z w_t^\epsilon = 0,
\end{cases}
$$

We define:

$$
H = \left\{ u \in [L^2(\mathbb{T}^d)]^{d-1} : \int_{\mathbb{T}^d} u(\mathbf{x}, z) d\mathbf{x} dz = 0, \int_0^1 \mathrm{div}_{\mathbf{x}} u(\mathbf{x}, z) dz = 0 \right\}.
$$

and with $v(\mathbf{x}, z) = -\int_0^z \mathrm{div}_{\mathbf{x}} u_t(\mathbf{x}, z') dz'$:

$$
(\Pi u)(\mathbf{x}, z) = u(\mathbf{x}, z) - \int_0^1 u(\mathbf{x}, z') dz,' \quad Au = \nu\Delta u,
$$

$$
b(u, u') = -\Pi(u \cdot \nabla_{\mathbf{x}})u' - \Pi v \partial_z u', \quad Q^{1/2} W = \Pi \mathcal{W},
$$

We prove convergence to

$$\begin{aligned} du_t &= Au_t dt + b(u_t, u_t)dt + b((-C)^{-1}Q^{1/2} \circ dW_t, u_t) + b(r, u_t)dt \\ &= \nu \Delta u_t dt - \Pi(u_t \cdot \nabla_{\mathbf{x}})u_t dt - \Pi v_t \partial_z u_t dt \\ &\quad - \Pi((-C)^{-1}Q^{1/2} \circ dW_t \cdot \nabla_{\mathbf{x}})u_t - \Pi dw_t \partial_z u_t dt \\ &\quad - \Pi(r \cdot \nabla_{\mathbf{x}})u_t dt - \Pi q \partial_z u_t dt, \end{aligned}$$

with the Ito-Stokes drift $r = \int (-C)^{-1} b(w, w) d\mu(w)$ and where v, w, q are defined implicitly by the incompressibility conditions

$$\operatorname{div}_{\mathbf{x}} u_t + \partial_z v_t = 0, \quad \operatorname{div}_{\mathbf{x}}(-C)^{-1}Q^{1/2}W_t + \partial_z w_t = 0, \quad \operatorname{div}_{\mathbf{x}} r + \partial_z q = 0.$$

We need stronger assumptions on C.