Stochastic Gross Pitaevskii Equation and invariant measure

A. de Bouard

CMAP, Ecole Polytechnique, France
Joint work with A. Debussche (ENS Rennes, France) and R. Fukuizumi (Tohoku University, Japan)

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I - Introduction

Modeling

$x_1, \ldots, x_N$ particles in a trapping potential V, two-body interactions

$$\widehat{H} = \sum_{j=1}^{N} \Big(-\frac{\hbar^2}{2m}\Delta_{x_j} + V(x_j) \Big) + \sum_{1 \le j,k \le N} U(x_j - x_k)$$

Ground state : minimum of energy corresponding to $\widehat{H}$ for the wave function $\tilde{\psi}(x_1, \ldots, x_N)$.

For small T, thermal wavelength

$$\lambda_T = \frac{\hbar}{(2\pi m k_B T)^{1/2}}$$

larger than particle distance
$\rightsquigarrow$ take into account statistical properties of the particles

$\rightsquigarrow$ replace interaction potential by

$$U_{eff}(x) = \frac{4\pi\hbar^2 a}{m}\delta_0(x)$$

a : atomic diffusion length (positive or negative)

Boson gaz : (Hartree approximation)

$$\tilde{\psi}(t, x_1, \ldots, x_N) = \prod_{j=1}^{N} \psi(t, x_j)$$

Moreover, large number of particles $\rightsquigarrow$ rescaling

Gross- Pitaevskii (1961, superfluids)

$$i\hbar\partial_t\psi(t,x) = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi + \frac{4\pi\hbar^2 a}{m}|\psi|^2\psi := L_{GP}\psi$$

V : confining potential

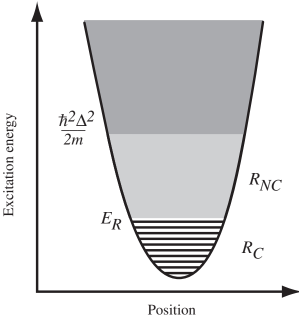
Non zero temperature

Aim : modeling of condensates close to critical T (phase transition) Weiler et al., Nature, 2008

⇝ need modeling of interactions with non condensed atoms, here assumed "thermalized"

Duine, Stoof, 2001 ;

Gardiner Davis, 2003



From Blackie et. al., Adv. Physics 2008

Stochastic projected GPE

ψ : wave function for the condensed atoms

$$L_{GP} = -\frac{\hbar^2}{2m}\Delta + V(x) + g|\psi(t,x)|^2$$

where m is the atomic mass, $g = \frac{4\pi\hbar^2 a}{m}$ and a the (positive) s-wave scattering length. Then

$$d\psi = \mathcal{P}_c\big[-\frac{i}{\hbar}L_{GP}\psi dt + \frac{\gamma}{k_B T}(\mu - L_{GP})\psi dt + dW_\gamma(t,x)\big]$$

where μ is the chemical potential, and $\mathcal{P}_c$ is a spectral cut-off (low energy modes)

$$\langle dW_\gamma^*(t,x)dW_\gamma(t',x')\rangle = 2\gamma\delta(t-t')\delta(x-x')dt$$

Additional terms : interaction thermal cloud–condensate

Infinite dimensional model

$$d\psi = (i-\gamma)\Big[A\psi - \mu\psi + |\psi|^2\psi\Big]dt + \sqrt{2\gamma}\,dW$$

$\psi(t,x)$ is the wave function ; $\gamma > 0$; W is a cylindrical Wiener process : $(h_n)_{n\in\mathbf{N}^d}$ real valued c.o.s. of $L^2(\mathbf{R}^d)$ s.t.

$$Ah_n := (-\Delta + |x|^2)h_n = \lambda_n h_n, \quad \lambda_n = 2|n| + d, \quad n \in \mathbf{N}^d$$

then W may be written as

$$W(t,x) = \sum_{n\in\mathbf{N}^d} \beta_n(t)h_n(x)$$

with $(\beta_n)_n$ sequence of independent complex valued BM.

$$H(\psi) = \frac{1}{2}|\nabla\psi|^2_{L^2} + \frac{1}{2}|x\psi|^2_{L^2} - \frac{\mu}{2}|\psi|^2_{L^2} + \frac{1}{4}|\psi|^4_{L^4}.$$

II - Local/global existence of solutions

$$d\psi = (i-\gamma)\Big[A\psi - \mu\psi + |\psi|^2\psi\Big]dt + \sqrt{2\gamma}\,dW$$

Let Z be the stochastic convolution

$$\begin{aligned} Z(t) &= \sqrt{2\gamma}\int_{-\infty}^{t} e^{(i-\gamma)(t-s)A}\,dW(s) \\ &= \sum_{n\in\mathbf{N}^d} \sqrt{2\gamma}\int_{-\infty}^{t} e^{(i-\gamma)(t-s)\lambda_n}\,d\beta_n(s)h_n \\ &= \sum_{n\in\mathbf{N}^d} \frac{g_n(t)}{\sqrt{\lambda_n}}h_n \end{aligned}$$

with $(g_n)_n$ i.i.d. $\mathcal{N}(0, 1_{\mathbf{C}})$; Z is the stationary solution of the linear equation (without interaction term) ; then if $\psi = v + Z$, v solves

$$\partial_t v = (i-\gamma)[Av - \mu(v+Z) + |v+Z|^2(v+Z)]$$

Case $d = 1$

Proposition (Burq, Thomann, Tzvetkov, 2013) : For any $p > 2$, $Z \in L^p(\mathbf{R})$ a.s. More precisely,

$$\mathbf{E}(|Z(t)|_{L^p}^p) \leq C_p, \quad \text{for all } p > 2.$$

Remark :

- Use of $|h_k|_{L_x^p} \leq C_p \lambda_k^{-\theta(p)/12}$ for all $p > 2$, with $\theta(p) > 0$
- May actually prove for $p \geq 4$: for any $s < \frac{1}{6}$, and $\alpha < \frac{1}{12} - \frac{s}{2}$,

$$Z \in C^\alpha([0, T]; W^{s,p}(\mathbf{R}));$$

where

$$W^{s,p}(\mathbf{R}) = \{v,\ A^{s/2} v \in L^p(\mathbf{R})\}.$$

Consequence : Let $p \geq 3$, and $v(0) \in L^p$, then there is a unique local solution a.s. in $C([0, T^*); L^p(\mathbf{R}))$.

Estimates on the semi-group $e^{(i-\gamma)At}$ thanks to Mehler transform and estimates on the GL semi-group Ginibre, Velo, 1997

Case $d = 2$

Proposition : $Z \in W^{-s,q}$ a.s., for any $q \geq 2$ and $sq > 2$, but $\mathbf{P}(Z \in L^q) = 0$ for any q.

Consequence :

- need renormalization (Wick products) based on the family $(h_n)_n$ to define polynomial terms in Z (theory is well known for the torus $\mathbf{T}^2$, Nualart, Da Prato-Tubaro, ...) $\rightsquigarrow$ need generalization adapted to the family $(h_n)_n$ of hermite functions
- need to define the products $v : Z^k :$ and $|v|^2 Z$...
- need refined estimates on the corresponding linear semi-group $e^{(i-\gamma)tA}$

Case $d = 2$, renormalization

- Albeverio-Röckner, Da Prato-Tubaro, Gatarek-Goldys, $\sim$'90 : stochastic quantization (weak solutions)
- Da Prato-Debussche (2002-2003) : strong solutions : 2-D stochastic Navier Stokes, Φ_2^4
- Mourrat-Weber (2017) : global well-posedness for Φ_2^4 on the plane
- Tsatsoulis-Weber (2018) : spectral gap for Φ_2^4, irreducibility,...
- Trenberth (2019), Matsuda (2020) : stochastic complex GL on 2-D torus, strong Feller property
- Hoshino (2018) : stochastic complex GL on 3-D torus

Case $d = 2$, Wick products

Itô-Wiener decomposition : $L^2(\Omega, \mathcal{G}, \mathbf{P}) = \bigoplus_{k=0}^{+\infty} \mathcal{H}_k$, were $\mathcal{G}$ is generated by ξ, Gaussian white noise on $L^2(\mathbf{R}^2)$ and

$$\mathcal{H}_k = \mathrm{span}\{H_k((\xi, h_n)_{L^2}),\ n \in \mathbf{N}\},$$

where H_k Hermite polynomial of degree k.

Now if

$$(S_N Z)(x) = \sum_{n \in \mathbf{N}^2, |n| \leq N} \frac{1}{\sqrt{\lambda_n}} g_n h_n(x)$$

and

$$\rho_N^2(x) = \sum_{n \in \mathbf{N}^2, |n| \leq N} \frac{h_n^2(x)}{\lambda_n},$$

we obtain

$$:(S_N z(x))^k := P_{\mathcal{H}_k}(S_N z)^k = \rho_N(x)^k \sqrt{k!} H_k\left(\frac{S_N z(x)}{\rho_N(x)}\right).$$

Example : $:(S_N z(x))^3 := (S_N z)^3(x) - 3\rho_N^2(x)(S_N z)(x)$.

Note that : ρ_N diverges (in any L^p space) as N goes to infinity

Nelson inequality (moment estimates of random variables in $\mathcal{H}_k$) allows to get

$$\mathbf{E}(| :(S_N z)^k : |^q_{W^{-s,q}}) \lesssim |A^{-s/2}(S_N K)^k(\cdot,\cdot)|^{\frac{q}{2}}_{L^{\frac{q}{2}}(\Delta)}$$

where

$$K(x,y) = \sum_n \frac{1}{\lambda_n} h_n(x) h_n(y)$$

is the kernel of A^{-1}. It turns out that for any $r \geq 2$, and any k, $K^k \in L^r_x W^{\alpha,2}_y$, for all $\alpha < 1 - 2/r$.

Prop : For any power k, the sequence $(:(S_N Z)^k:)_{N\in\mathbf{N}}$ is Cauchy in $L^q(\Omega; W^{-s,q}(\mathbf{R}^2))$, for $q > 2$, $s > 0$ and $sq > 2$.

Case $d = 2$: estimates on the semigroup

Aim : Run a fixed point argument on the mild equation

$$
\begin{aligned}
v(t) &= e^{(i-\gamma)tA}(\psi_0 - Z(0)) \\
&\qquad +(i-\gamma)\int_0^t e^{(i-\gamma)(t-\tau)A} : |v+Z|^2(v+Z) : d\tau
\end{aligned}
$$

$\rightsquigarrow$ need estimates on the semigroup $e^{-(i-\gamma)tA}$ for positive γ ; let for $T > 0$, $\beta, s > 0$, $p, q \geq 1$,

$$
\mathcal{E}_T = C([0,T]; W^{-s,q}) \cap L^r(0,T; W^{\beta,p})
$$

Note we need q large (s small), but constrained on p

Prop : Let $\gamma > 0$, $\beta, s > 0$, $q > p > 2$ and assume $\frac{1}{r} - \frac{\beta+s}{2} - (\frac{1}{p} - \frac{1}{q}) > 0$; then

$$
\left| e^{(i-\gamma)tA}\psi \right|_{\mathcal{E}_T} \leq C_T |\psi|_{W^{-s,q}}
$$

Case $d = 2$: local existence

Prop : Let $q > p > 2$, $0 < s < \beta < 2/p$, assume $\beta - s - (2/p - \beta) > 0$ and $s + 2/p - \beta < 2(1 - 1/q)$; then for any f, g, if $\alpha = s + 2(2/p - \beta)$

$$|hg^2|_{W^{-\alpha,q}} \leq C|h|_{W^{-s,q}}|g|^2_{W^{\beta,p}}$$

Moreover, for any f,

$$\Big|\int_0^t e^{(i-\gamma)(t-\tau)A} f(\tau)d\tau\Big|_{\mathcal{E}_T} \leq C_T T^\delta |f|_{L^{r/3}(0,T;W^{-\alpha,q})}$$

provided $1/r - (\beta + s)/2 - (1/p - 1/q) > 0$ and $\delta := 1 - (2/p - \beta) - 3/r > 0$.

Conclusion : Choosing q large enough, p close to 2, β close to $2/p$ and s sufficiently small, we get local existence of a unique solution in $\mathcal{E}_T$, for small T ; can improve the result : for $\psi_0 = v_0 + Z(0)$ with $v_0 \in L^q(\mathbf{R}^2)$, then get local existence with v in $C(0, T; L^q(\mathbf{R}^2))$.

Global existence (large dissipation)

$$\partial_t v = (i-\gamma)[Av + \Theta(v, (:Z^k:)_{1\le k\le 3})]$$

with $\Theta(v, (:Z^k:)_{1\le k\le 3}) =: |v+Z|(v+Z):$ and $\mu = 0$ for simplicity.

Prop : Let $\gamma > \gamma(q)$,with q as before, then

$$\frac{d}{dt}|v(t)|^q_{L^q} + \delta|v(t)|^q_{L^q} \le C\left(\sum_{k=1}^{3}|:Z_N^k:|^{\gamma_k}_{W^{-s,q}}\right)$$

for some positive $\delta = \delta(\gamma, q)$ with C depending only on γ, q.

Conclusion : Global existence en $C(R^+; L^q(\mathbf{R}^2))$ if $\gamma > \gamma(q)$ and $v_0 \in L^q$. Smoothing properties of the semi-group $\rightsquigarrow$ global existence in $C(\mathbf{R}^+; W^{-s,q})$ if $v_0 \in W^{-s,q}$. Same is true in 1-D (no need of Wick products).

Global existence (small dissipation)

Previous estimate still true in L^{q_0} for small γ, provided $q_0 > 2$ close to 2.

Strategy :

- Starting from $u_0 \in W^{-s,q}$, prove $u(t_0) \in L^{q_0}$, $q_0 > 2$ close to 2 as soon as $t_0 > 0$ $\rightsquigarrow$ bound in L^{q_0}
- Prove then that $u(t)$ bounded in $W^{\sigma,p}$, $\sigma > 0$ (small), $p > 2$ close to 2 (smoothing of the semi-group)
- boostrap argument (T. Matsuda) : upgrade regularity from $\sigma > 0$ to $\sigma < 1$ close to 1
- Finaly get bound in $W^{-s,q}$, large q by Sobolev embeddings

$\rightsquigarrow$ global existence in $W^{-s,q}$ for any dissipation

III - Existence of invariant measures

Gibbs measures

Constructive quantum field theory (N-body problem)
Simon, Lieb, ..., '60
Mean field limits for Gibbs measures Lewin, Pham, Rougerie, 2018

Leibowitz, Rose, Speer, 1988, Bourgain, 1994 : Gibbs measures and global existence for dispersive equations (Hamiltonian systems) ; lots of results since then Burq, Gerard, Tzvetkov, Colliander, Oh, Bourgain, Bulut,...

Here :

$$d\psi = J\nabla H(\psi) - \gamma \nabla H(\psi) + \sqrt{2\gamma}\, dW$$

with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $\mathbf{C}$ is identified with $\mathbf{R}^2$; note that $J\nabla H$ is a Hamiltonian operator, with

$$H(\psi) = \frac{1}{2}|\nabla\psi|_{L^2}^2 + \frac{1}{2}|x\psi|_{L^2}^2 - \mu|\psi|_{L^2}^2 + \frac{1}{4}|\psi|_{L^4}^4.$$

The generator $\mathcal{L}$ of the transition semi-group P_t associated with the previous equation has the form

$$\begin{aligned}\mathcal{L}(\Phi)(\psi) &= \gamma \operatorname{Tr} D^2\Phi(\psi) - \gamma\langle\nabla\Phi(\psi), \nabla H(\psi)\rangle \\ &\quad + \langle\nabla\Phi(\psi), J\nabla H(\psi)\rangle\end{aligned}$$

with $\langle\cdot,\cdot\rangle$ the inner product in $L^2(\mathbf{R}^d, \mathbf{C})$ i.e.

$$\langle u, v\rangle = \Re \int_{\mathbf{R}^d} u(x)\bar{v}(x)dx$$

Then formally, if

$$\nu(d\psi) = z^{-1}\, e^{-H(\psi)} d\psi,$$

for some normalizing coefficient z, one may compute for any bounded continuous function Φ on the state space E :

$$z \int_E (\mathcal{L}\Phi)(\psi)\nu(d\psi) = 0$$

so that $\mathcal{L}^*\nu = 0$, ν is (formally) invariant for P_t, even for $\gamma = 0$

Interpretation and support of ν : the case $d = 1$

Burq, Thoman, Tzvetkov, 2013 : rigorous definition of the Gibbs measure ν and invariance for the Hamiltonian flow ($d = 1$)

Note that

$$H(\psi) = \frac{1}{2}\langle \psi, A\psi \rangle - \frac{\mu}{2}|\psi|^2_{L^2} + \frac{1}{4}|\psi|^4_{L^4},$$

with, as before, $A\psi = -\Delta\psi + x^2\psi$ with eigenvalues $\lambda_n = 2n + 1$, and eigenfunctions h_n, and ih_n (Hermite functions) ;

Hence, we may formally write :

$$\begin{aligned} \nu(d\psi) &= z^{-1}\, e^{-H(\psi)} d\psi \\ &= z^{-1}\, e^{-\frac{1}{4}|\psi|^4_{L^4}} e^{-\frac{1}{2}\langle \psi, A\psi \rangle + \frac{\mu}{2}|\psi|^2_{L^2}} d\psi \end{aligned}$$

If $\mu < \lambda_0 = 1$, then the last term is a Gaussian measure, with support in $L^p(\mathbf{R})$, for any $p > 2$

Case $d = 2$

Remark : Gaussian measure has support in $W^{-s,q}$, $q \geq 2$, $sq > 2$; moreover, $\langle 1, : |\psi|^4 : \rangle$ not well defined for $\psi \in W^{-s,q} \rightsquigarrow$ no hope to use a duality argument (Da Prato, Debusshe, Φ_2^4, 2003)

However : Galerkin approximation of the purely dissipative equation ($\psi_N = S_N \psi$) :

$$d\psi = -\gamma \Big[A\psi + S_N(: |\psi_N|^2 \psi_N :) \Big] dt + \sqrt{2\gamma}\, \Pi_N dW$$

in $E_N = \mathrm{span}\{h_1, \cdots, h_N\}$, has a (unique) invariant Gibbs measure

$$\nu_N(d\psi) = \Gamma_N e^{-\mathcal{H}_N(\psi_N)} d\psi$$

with

$$\mathcal{H}_N(\psi) = \frac{1}{2} |\nabla \psi|^2_{L^2} + \frac{1}{2} |x\psi|^2_{L^2} + \frac{1}{4} \int_{\mathbf{R}^2} : |\psi|^4 : dx$$

Alternatively :

$$\begin{cases} \partial_t v &= -\gamma\Big[Av + S_N(:|v_N + Z_N|(v_N + Z_N):)\Big] \\ dZ &= -\gamma AZ\,dt + \sqrt{2\gamma}\,\Pi_N dW \end{cases}$$

has an invariant measure μ_N (Krylov-Bogolyubov) on $E_N \times E_N$ with

$$\int_{E_N} \varphi(x)\nu_N(dx) = \int_{E_N\times E_N} \varphi(u+z)\mu_N(du,dz)$$

thanks to uniqueness of ν_N.

Question : tightness of (ν_N) (or (μ_N)) ?

Unfortunately : L^q-estimate not valid for Galerkin approximations (due to S_N)

However : another estimate for v_N :

$$\frac{d}{dt}|v_N(t)|_{L^2}^2 + \frac{\gamma}{2}|A^{1/2}v_N(t)|_{L^2}^2 \leq C\sum_{k=1}^{3}| : S_N Z^k : |_{W^{-s,q}}^{m_k}$$

(interpolation, Sobolev embeddings...) Now, if v_N is stationary, then integrating in time between 0 and 1, and taking expectations implies

$$\mathbf{E}(|A^{1/2}v_N(t)|_{L^2}^2) \leq C.$$

Conclusion : Bound on v_N (indep. of N) in $W^{1,2} \subset L^q \subset W^{-s',q}$ for $s' > 0$. Since Z_N is bounded in $W^{-s',q}$ with $s'q > 2$, we deduce that (μ_N) is tight in $W^{-s,q}$, for $s > s'$.

Thm : Up to a subsequence, (ν_N) has a weak limit ν, which is an invariant measure for P_t, for any $\gamma > 0$.

Conclusion and open problems

Case $d = 1$:

- Strong Feller property ($\gamma > 0$) and irreducibilty of $P_t \rightsquigarrow$ ergodicity of ν
- Cvgence to equilibrium (exponential mixing) in $L^2(\nu)$ with rate $\gamma(\lambda_0 - \mu)$ (Poincaré inequality)

Case $d = 2$:

- Irreducibility not so clear : need information on the support of $\mathbf{P}_{\bar{Z}}$ with $\bar{Z} = (Z, :Z^2:, :Z^3:)$ in $(W_{-s,q})^3$.
- Uniqueness of ν ?
- Singularity of ν w.r. to Gaussian measure ?
- Invariance for $\gamma = 0$?

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Thank you Marta !