A Young approach to hyperbolic Anderson model

Aurélien DEYA (Nancy, France)

Barcelona, 2022

In honor of Marta Sanz-Solé

Thank you Marta!

Joint work with X. Chen, J. Song and S. Tindel

Sujet : Félicitations
Date : Tue, 19 Oct 2010 11:18:13 +0200 (CEST)
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Cher Aurélien,

J'imagine qu'en ce moment vous avez déjà le grade de Docteur. Mes chaleureuses félicitations!

Je suis desolée de ne pas avoir pu assister à la soutenance.

(...)

Avez-vous réflechi à une approche trajectorielle similaire pour l'équation des ondes? Je ne connais pas d'autres travaux que celui de Lluís et Samy.

[Did you think about a similar pathwise approach for the wave equation? The only related work I know is the one by Lluís and Samy.]

J'espère que nous pourrons nous rencontrer bientôt.
Meilleures salutations,
Marta

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Outline

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Hyperbolic Anderson model

General objective: we would like to develop a **pathwise** approach for the so-called hyperbolic Anderson model on $\mathbb{R}_+ \times \mathbb{R}^d$:

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + u\,\dot{B}(t,x), \quad t \in [0,T],\ x \in \mathbb{R}^d,$$

where:

- smooth initial conditions φ, ψ
- $\dot{B}$ a **space-time fractional noise** of index $H=(H_0,\ldots,H_d)\in(0,1)^{d+1}$

$$\mathbb{E}[\dot{B}(s,x)\dot{B}(t,y)] = |s-t|^{2H_0-2}\prod_{i=1}^{d}|x_i-y_i|^{2H_i-2}.$$

Remark 1. "Pathwise"= highlight regularity conditions on $\dot{B}$ so that we can interpret and solve the equation with deterministic arguments.

Remark 2. Here, we will rely on an interpretation of the stochastic wave integral as a limit of Riemann sums, that is in the Young sense.

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Remark 3. The range of our results is quite limited so far, regarding both the space dimension ($d \in \{1,2\}$) and the coefficients H_i.

Remark 4. We hope that these developments can be a first step toward more sophisticated expansion procedures.

Outline

Previously: white noise situation

Consider the Itô wave equation:

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + f(u) + \sigma(u)\,\dot{B}(t,x), \quad t \in [0,T],\ x \in \mathbb{R}^d,$$

where $\dot{B}$ is **white in time**, fractional in space.

Interpretation: Theory of martingale measures and extension

Properties of the solution: stochastic tools (Malliavin calculus,...)

Extensive literature. "Completely random" selection of works:

- **Marta Sanz-Solé** and A. Millet: A stochastic wave equation in two space dimension: smoothness of the law. *Ann. Probab.* (1999).
- **Marta Sanz-Solé** and A. Millet: Approximation and support theorem for a wave equation in two space dimensions. *Bernoulli* (2000).

- **Marta Sanz-Solé** and L. Quer-Sardanyons: Absolute continuity of the law of the solution to the 3-d stochastic wave equation. *JFA* (2004).
- **Marta Sanz-Solé** and L. Quer-Sardanyons: A stochastic wave equation in dimension 3: smoothness of the law. *Bernoulli* (2004).
- **Marta Sanz-Solé**: Properties of the density for a three-dimensional stochastic wave equation. *JFA* (2008).

- **Marta Sanz-Solé** and R. Dalang: Regularity of the sample paths of a class of second-order spde's. *JFA* (2005).
- **Marta Sanz-Solé** and R. Dalang: Hölder-Sobolev regularity of the solution to the stochastic wave equation in dimension 3. *Mem. AMS* (2009).
- **Marta Sanz-Solé** and R. Dalang: Criteria for hitting probabilities with applications to systems of stochastic wave equations. *Bernoulli* (2010).

• **Marta Sanz-Solé** and A. Süss: The stochastic wave equation in high dim: Malliavin differentiability and absolute continuity. *EJP* (2013).

• **Marta Sanz-Solé** and L. Quer-Sardanyons: Space semi-discretisations for a stochastic wave equation. *Potential Anal.* (2006).

• **Marta Sanz-Solé** and F. Delgado-Vences: Approximation of a stoch. wave equation in dimension 3. *Bernoulli* (2014).

• **Marta Sanz-Solé** and F. Delgado-Vences: Approximation of a stoch. wave equation in dimension 3: the non-stationary case. *Bernoulli* (2016).

• **Marta Sanz-Solé** and A. Millet: Global solutions to stochastic wave equations with superlinear coefficients. *SPA* (2021).

Previously: fractional situation, Skorohod approach

Consider the Skorohod wave equation:

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + u \diamond \dot{B}(t,x), \quad t \in [0,T],\ x \in \mathbb{R}^d,\ d \in \{1,2,3\},$$

where

$\dot{B}$ is a **space-time fractional noise** of index $H=(H_0,...,H_d)\in(0,1)^{d+1}$.

$\longrightarrow$ Chaos expansion procedures: series of works (2012+) by Balan, Song, Chen, D., Tindel...

Previously: additive noise, pathwise approach

Pathwise approach to wave equations with additive noise:

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + u^p + \dot{B}, \quad t \in [0,T],\ x \in \mathbb{T}^d,\ d \geq 1$$

where

$\dot{B}$ is a **space-time fractional noise** of index $H=(H_0,\dots,H_d)\in(0,1)^{d+1}$.

$\longrightarrow$ "Da Prato-Debussche trick" and extensions: recent series of works (2017+) by Gubinelli, Koch, Oh, D., Tolomeo...

Previously: pathwise approach to hAm

• *[L. Quer-Sardanyons and S. Tindel (2007): The 1-d stochastic wave equation driven by a fractional Brownian sheet.]*

$$\partial_t^2 u = \Delta u + u\,\dot{W}, \quad t \in [0,T],\ x \in \mathbb{R},$$

where $\dot{W}$ is a "rotated" fractional noise on $\mathbb{R} \times \mathbb{R}$.

• *[Z. Brzézniak and N. Rana (2020): Low regularity solutions to the stochastic geometric wave equation driven by a fractional Brownian sheet.]*

$$\partial_t^2 u = \Delta u + u\,\dot{W}, \qquad t \in [0,T],\ x \in \mathbb{R},$$

with u takes values in a manifold, and $\dot{W}$ is a "rotated" fractional noise.

• *[R. Balan (2022): Stratonovich solution for the wave equation.]*

$$\partial_t^2 u = \Delta u + u\,\dot{W}, \quad t \in [0,T],\ x \in \mathbb{R}^d,\ d \in \{1,2\},$$

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1. Introduction
   - The model
   - Previous related works
2. **About kernels regularization effect**
3. Young wave integral
   - Main pathwise results
   - Application
   - Pespectives

Let us go back to our setting: $\dot{B}$ is a space-time fractional noise,

$$\partial_t^2 u = \Delta u + u\,\dot{B}, \quad t \in [0,T],\ x \in \mathbb{R}^d .$$

Recall that $\dot{B} = \partial_t \partial_x B$, with B a space-time fractional field.

We can thus write the equation in the mild form as

$$\boxed{u_t = (\partial_t \mathcal{G})_t \varphi + \mathcal{G}_t \psi + \int_0^t \mathcal{G}_{t-r}\big(u_r\, d(\partial_x B)_r\big)},$$

where $\mathcal{G}$ is the wave kernel, i.e. $\mathcal{F}(\mathcal{G}_t)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}$, $\xi \in \mathbb{R}^d$.

Objective: find (almost sure) regularity conditions on $\partial_x B$, as a path with values in a space of negative-order distributions, so that we can interpret

$$\int_0^t \mathcal{G}_{t-r}\big(u_r\, d(\partial_x B)_r\big)$$

and solve the equation with deterministic arguments, in a suitable space of functions $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$.

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About kernels regularization effect

There have been many developments about the pathwise approach to the stochastic **heat** equation (regularity structures,...)

Pathwise approach to the stochastic **wave** equation is more delicate, due to the weaker regularizing effect of the kernel.

Heat setting (G = heat kernel in $\mathbb{R}^d$):

For any $\alpha \in \mathbb{R}$ and any test-function $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}$,

$$\left\| G *_{t,x} f \right\|_{\mathcal{B}^{\alpha+2}} \lesssim \left\| f \right\|_{\mathcal{B}^{\alpha}}$$

where $\mathcal{B}^{\alpha}$ is a space of *space-time* distributions of regularity α.

$\rightarrow$ One of the starting points in the theory of regularity structures

There is no such property for the **wave** kernel $\mathcal{G}$

$\rightarrow$ The theory of regularity structures does not apply

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Classical "waves" Strichartz estimates ($\mathcal{G}$ = wave kernel in $\mathbb{R}^d$):
For all $\alpha \in \mathbb{R}$, $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$,

$$\mathcal{N}\big[\mathcal{G} *_{t,x} f; L^\infty([0,T]; \mathcal{H}^{\alpha+1}(\mathbb{R}^d))\big] \lesssim \mathcal{N}[f; L^2([0,T]; \mathcal{H}^\alpha(\mathbb{R}^d))] \, ,$$

where $\mathcal{H}^\alpha$ is the standard Sobolev space of order α.

Not sufficient to interpret the integral $\int_0^t \mathcal{G}_{t-r}\big(u_r \, d(\partial_x B)_r\big)$, because:

(i) We cannot handle the time-derivative $d(\partial_x B)_r$ in the scale $L^p([0,T], .)$

$\to$ The only information we can get is about the Hölder regularity of $t \mapsto (\partial_x B)_t$

(ii) $\partial_x B$ is expected to live in a *weighted* Sobolev space $\mathcal{H}^\alpha_w(\mathbb{R}^d)$

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Definition (Rychkov). For all $\mu \geq 0$ and $f : \mathbb{R}^d \to \mathbb{R}$, we set

$$\|f\|_{L^2_\mu} := \left(\int_{\mathbb{R}^d} |f(x)|^2 \, e^{-\mu|x|} \, dx \right)^{1/2} .$$

Then for every $\alpha \in \mathbb{R}$, we define the space $\mathcal{B}^\alpha_\mu$ as the completion of $\mathcal{D}(\mathbb{R}^d)$ with respect to the norm

$$\|f\|_{\mathcal{B}^\alpha_\mu} := \sup_{j \geq 0} \left(2^{j\alpha q} \big\|\varphi_j * f\big\|_{L^2_\mu} \right),$$

where $\varphi_j(x) = 2^{dj}\varphi(2^j x)$, for φ in a suitable subspace of $\mathcal{D}(\mathbb{R}^d)$.

Proposition (CDST). Let $d \in \{1, 2\}$, $T > 0$, and $0 \leq \mu \leq 1$. Then for all $\alpha \in \mathbb{R}$, $f : \mathbb{R}^d \to \mathbb{R}$ and $0 \leq s < t \leq T$,

$$\boxed{\|\{\mathcal{G}_t - \mathcal{G}_s\} *_x f\|_{\mathcal{B}^{\alpha+\kappa}_\mu} \lesssim |t-s|^{\rho_d - \kappa} \|f\|_{\mathcal{B}^\alpha_\mu}},$$

$$\text{where} \quad \rho_d := \begin{cases} 1 & \text{if } d = 1 \\ \frac{1}{2} & \text{if } d = 2 \end{cases} \quad \text{and} \quad \kappa \in [0, \rho_d].$$

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Proposition. Fix $d \in \{1,2\}$ and $T > 0$.

Assume that $\boxed{\partial_x B \in \mathcal{C}^\theta([0,T];\mathcal{B}_\mu^{-\alpha})}$ and pick $\boxed{u \in \mathcal{C}^\gamma([0,T];\mathcal{B}_{\mu_\cdot}^{\kappa})}$.

Under suitable conditions on $\theta, \alpha, \gamma, \kappa$, we can define

$$\boxed{\int_0^t \mathcal{G}_{t-r}\big(u_r\, d(\partial_x B)_r\big) := \lim_{n\to\infty} \sum_{k=0}^{m-1} \mathcal{G}_{t-t_k^n}\big(u_{t_k^n}\,\{\partial_x B_{t_{k+1}^n} - \partial_x B_{t_k^n}\}\big)}.$$

Moreover, this Young integral verifies

$$\Big\| \int_0^\cdot \mathcal{G}_{\cdot-r}(u_r\, d\dot{W}_r)\Big\|_{\mathcal{C}_T^\gamma \mathcal{B}_{\mu_\cdot}^\kappa} \lesssim \big\|\mathcal{G}_\cdot(u_0\, \partial_x B_T)\big\|_{\mathcal{C}_T^\gamma \mathcal{B}_{\mu_\cdot}^\kappa} + \|\partial_x B\|_{\mathcal{C}_T^\theta \mathcal{B}_\mu^{-\alpha}} \|u\|_{\mathcal{C}_T^\gamma \mathcal{B}_{\mu_\cdot}^\kappa}.$$

(C) The coefficients $\theta, \alpha, \kappa, \gamma$ all sit in the interval $[0,1]$, and we have

$$\gamma + \theta > 1\,, \quad \kappa > \alpha\,, \quad \kappa + \alpha + \gamma + (1-\rho_d) < \theta\,.$$

Theorem. In the above setting, the (Young) hyperbolic Anderson model

$$u_t = (\partial_t \mathcal{G})_t \varphi + \mathcal{G}_t \psi + \int_0^t \mathcal{G}_{t-r}\big(u_r\, d(\partial_x B)_r\big)$$

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Application

Proposition. Let B be a fractional field on $\mathbb{R}_+ \times \mathbb{R}^d$, with Hurst indexes $H_0, H_1, \ldots, H_d \in (0,1)^{d+1}$. Then, almost surely, we have

$$\partial_x B \in \mathcal{C}^{\theta}([0, T]; \mathcal{B}_{\mu}^{-\alpha})$$

for all $\mu > 0$,

$$\theta \in (0, H_0) \quad \text{and} \quad \alpha > d - H_+, \quad \text{where } H_+ := \sum_{i=1}^{d} H_i.$$

Corollary. Consider Hurst parameters $H_0, H_i \in (0,1)$ satisfying

$$\begin{cases} H_0 + H_1 > \frac{3}{2}, & \text{if } d = 1, \\ H_0 + H_1 + H_2 > \frac{11}{4}, & \text{if } d = 2. \end{cases}$$

Then the (Young) hyperbolic Anderson model admits a unique solution.

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Perspectives

• We hope that these developments can be seen a first step toward more sophisticated expansion procedures.

• We would like to compare this Young (“Stratonovich”) solution with the Skorohod solution, and possibly deduce new stochastic controls.

Thank you!

And thank you again Marta!