

Construction of Boltzmann and McKean-Vlasov flows

(the sewing argument)

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In honor of Marta

1. Boltzmann type equations

$$\begin{aligned} & \int_{\mathbb{R}^d} \varphi(x) f_{s,t}(dx) \\ &= \int_{\mathbb{R}^d} \varphi(x) d\rho(x) + \int_{\mathbb{R}^d} \langle \nabla \varphi(x), b(x, f_{s,t}) \rangle dx \\ &+ \int_s^t \int_{\mathbb{R}^d \times \mathbb{R}^d} f_{s,t}(dx) f_{s,r}(dv) \int_E \left(\varphi(x + c(v, z, x, f_{s,t})) - \varphi(x) \right) \gamma(v, z, x) \mu(dz) dr. \end{aligned} \tag{1}$$

We look for a family of probability measures $f_{s,t}(dv)$, $s \leq t$ which verify the above weak integro differential equation.

- a. If the coefficients does not depend on $f_{s,t}$ this is a Boltzmann type equation.
- b. If $f_{s,r}(dv)$ is a given (known) $g_{s,r}(dv)$ and the coeffieints depend on $f_{s,t}$ this is a McKean Vlaso type equation

2. Stochastic equation

We consider the stochastic equation

$$\begin{aligned} X_{s,t} = X &+ \int_s^t b(X_{s,r}, \mathcal{L}(X_{s,r}))dr \\ &+ \int_s^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(v, z, X_{s,r-}, \mathcal{L}(X_{s,r})) \mathbf{1}_{\{u \leq \gamma(v, z, X_{s,r-})\}} N_{\mathcal{L}(X_{s,r})}(dv, dz, du, dr). \end{aligned} \quad (2)$$
$$(3)$$

With $N_{\mathcal{L}(X_{s,r})}$ a Poisson point measure with compensator

$$\widehat{N}_{\mathcal{L}(X_{s,r})}(dv, dz, du, dr) = \mathcal{L}(X_{s,r})(dv) \mu(dz) du dr$$

This represents a "probabilistic representation" (Tanaka) for the solution of the weak equation :

$$f_{s,r}(dx) = \mathcal{L}(X_{s,r})(dx)$$

3. Flow solution (a new formulation of the problem)

$$\mathcal{P}_1 = \{\mu \text{ probability, } \int_{\mathbb{R}^d} |x| \mu(dx) < \infty\}$$
$$W_1(\mu, \nu) = \sup_{|\nabla f| \leq 1} \left| \int f d\mu - \int f d\nu \right|$$

Flows of endomorphisms

$$\mathcal{E} = \{\theta : \mathcal{P}_1 \rightarrow \mathcal{P}_1\},$$
$$d_*(\theta, \theta') = \sup_{\rho \in \mathcal{P}_1} \frac{W_1(\theta(\rho), \theta'(\rho))}{1 + \int_{\mathbb{R}^d} |x| \rho(dx)}$$

Then $(\mathcal{P}_1, W_1)$ and $(\mathcal{E}, d_*)$ are complete metric spaces.

One step Euler scheme : Given $\rho \in \mathcal{P}_1$ we take $X \sim \rho$ and construct

$$Y_{s,t}(\rho) = X + b(X, \rho)(t - s) + \int_s^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(v, z, X, \rho) \mathbf{1}_{\{u \leq \gamma(v, z, X)\}} N_\rho(dv, dz, du, dr)$$

Then define $\Theta_{s,t} : \mathcal{P}_1 \rightarrow \mathcal{P}_1$ by

$$\rho \rightarrow \Theta_{s,t}(\rho) = \mathcal{L}(Y_{s,t}(\rho)).$$

Theorem (flow solution) There exists a unique flow $\theta_{s,t} \in \mathcal{E}$, $s < t$, such that

$$\theta_{s,t} = \theta_{r,t} \circ \theta_{s,r} \quad s < r < t \quad (\text{flow property})$$

and

$$d_*(\theta_{s,t}, \Theta_{s,t}) \leq C(t - s)^2.$$

Moreover, $\theta_{s,t}(\rho)$ solves the weak equation (1) and admits the stochastic representation (2). We call $\theta_{s,t}$ the "**flow solution**"

Remark 1 A similar point of view has been introduced by Devie in the framework of rough path equations.

Remark 2 The construction of the flow solution is done as limit of the Euler schemes :

$$\Theta_{s,t}^{\mathcal{P}} = \Theta_{s_{n-1},s_n} \circ \dots \circ \Theta_{s_0,s_1} \quad \mathcal{P} = \{s = s_0 < \dots < s_n = t\}$$

Then

$$d_*(\theta_{s,t}, \Theta_{s,t}^{\mathcal{P}}) \leq C(t - s) \times \max_{i=1,n} (s_{i+1} - s_i).$$

Remark 3 : Uniqueness of the solution of the weak equation is a difficult problem - for the flow solution we have uniqueness easily : there exists a unique solution constructed as **limit of Euler schemes** (but this produces just "one possible solution" of the weak equation).

Remark 3 : Stability

$$W_1(\theta_{s,t}(\rho), \theta_{s,t}(\xi)) \leq C(t - s)W_1(\rho, \xi).$$

Sewing Lemma (Feyel De la Pradele, et independemment Gubinelli 2004) Take V abstract set and consider the endomorphisms

$$\mathcal{E}(V) = \{\Theta : V \rightarrow V\} \quad \text{endomorphisms}$$

$$d_* = \quad \text{metric on } \mathcal{E}(V) \text{ such that } (\mathcal{E}(V), d_*) \text{ is complete.}$$

In our case : $V = \mathcal{P}_1$ and $\Theta_{s,t}$ is the one step Euler scheme. We construct Euler schemes

$$\Theta_{s,t}^{\mathcal{P}} = \Theta_{s_{n-1},s_n} \circ \dots \circ \Theta_{s_0,s_1} \quad \mathcal{P} = \{s = s_0 < \dots < s_n = t\}$$

We consider $\Theta_{s,t} \in \mathcal{E}(V)$, $s < t$ which has the following two properties :

$$\textbf{Lipschitz} \quad d_*(\Theta_{s,t}^{\mathcal{P}} \circ U, \Theta_{s,t}^{\mathcal{P}} \circ U') \leq C d_*(U, U') \quad \forall U, U' \in \mathcal{E}(V).$$

And **SEWING** property

$$d_*(\Theta_{s,t}, \Theta_{r,t} \circ \Theta_{s,r}) \leq C(t-s)^{1+\varepsilon} \quad \forall s < r < t.$$

Sewing Lemma Under the above hypothesis there exists a unique $\theta_{s,t} \in \mathcal{E}(V)$, $s < t$, which has the flow property $\theta_{s,t} = \theta_{r,t} \circ \theta_{s,r}$ and such that $d(\Theta_{s,t}, \theta_{s,t}) \leq C(t-s)^{1+\varepsilon}$.

Proof of the main result : $X \sim \rho$

$$Y_{s,t}(\rho) = X + b(X, \rho)(t - s) \\ + \int_s^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(v, z, X, \rho) \mathbf{1}_{\{u \leq \gamma(v, z, X)\}} N_\rho(dv, dz, du, dr)$$

Then define $\Theta_{s,t} : \mathcal{P}_1 \rightarrow \mathcal{P}_1$ by $\rho \rightarrow \Theta_{s,t}(\rho) = \mathcal{L}(Y_{s,t}(\rho))$.

Lipschitz Take $\Pi(\rho, \bar{\rho})$ optimal coupling of ρ and $\bar{\rho}$ and $(X, \bar{X}) \sim \Pi(\rho, \bar{\rho})$. Then

$$W_1(\Theta_{s,t}(\rho), \Theta_{s,t}(\bar{\rho})) \leq E |Y_{s,t}(\rho) - Y_{s,t}(\bar{\rho})| \\ \leq E |X - \bar{X}| + C \int_s^t E |X - \bar{X}| + W_1(\rho, \bar{\rho}) dr \\ \leq W_1(\rho, \bar{\rho}) \times (1 + C(t - s)).$$

We concatenate

$$W_1(\Theta_{s,t}^{\mathcal{P}}(\rho), \Theta_{s,t}^{\mathcal{P}}(\rho)) \leq W_1(\rho, \bar{\rho}) \times e^{C(t-s)}$$

Sewing Property One takes $s < r < t$ and writes

$$\begin{aligned} Y_{s,t}(X) &= \textcolor{red}{X} + \int_s^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(v, z, \textcolor{red}{X}) \mathbf{1}_{\{u \leq \gamma(v, z, \textcolor{red}{X})\}} N_{\textcolor{red}{\rho}}(dv, dz, du, dr) \\ &= Z + \int_r^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(v, z, \textcolor{red}{X}) \mathbf{1}_{\{u \leq \gamma(v, z, \textcolor{red}{X})\}} N_{\textcolor{red}{\rho}}(dv, dz, du, dr) \end{aligned}$$

with

$$Z = \textcolor{red}{X} + \int_s^r \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(v, z, \textcolor{red}{X}) \mathbf{1}_{\{u \leq \gamma(v, z, \textcolor{red}{X})\}} N_{\textcolor{red}{\rho}}(dv, dz, du, dr)$$

Notice that

$$Z \sim \Theta_{s,r}(\rho).$$

Then write

$$Y_{r,t}(Z) = Z + \int_r^t \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(v, z, \textcolor{red}{Z}) \mathbf{1}_{\{u \leq \gamma(v, z, \textcolor{red}{Z})\}} N_{\Theta_{s,r}(\rho)}(dv, dz, du, dr).$$

Then, standard computations give

$$\begin{aligned} E |Y_{s,t}(X) - Y_{r,t}(Z)| &\leq C \int_r^t E |X - Z| + W_1(\rho, \Theta_{s,r}(\rho)) \\ &\leq C(t - r)E |X - Z| \leq C(t - r)(r - s). \end{aligned}$$

Since

$$Y_{s,t}(X) \sim \Theta_{s,t}(\rho) \quad \text{and} \quad Y_{r,t}(Z) \sim \Theta_{r,t} \circ \Theta_{s,r}(\rho)$$

we get

$$\begin{aligned} W_1(\Theta_{s,t}(\rho), \Theta_{r,t} \circ \Theta_{s,r}(\rho)) &\leq E \left| Y_{s,t}(X) - Y_{r,t}(Z) \right| \\ &\leq C(t-r)(r-s) \leq C(t-s)^2. \end{aligned}$$

Remark Finite variation :

$$E \left| Y_{s,t}(X) - Y_{s,t+h}(Z) \right| \sim h \quad \rightarrow \quad h \times h = h^2$$

If a martingale term appears then

$$E \left| Y_{s,t}(X) - Y_{s,t+h}(Z) \right| \sim h^{1/2} \quad \rightarrow \quad h^{1/2} \times h^{1/2} = h$$

Particle system approximation. For $i = 1, \dots, N$

$$\begin{aligned}
 X_{k+1}^i &= X_k^i + b(X_k^i, \frac{1}{N} \sum_{i=1}^N \delta_{X_k^i})(s_{k+1} - s_k) \\
 &\quad + \int_{s_k}^{s_{k+1}} \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} c(v, z, X_k^i, \frac{1}{N} \sum_{i=1}^N \delta_{X_k^i}) \mathbf{1}_{\{u \leq \gamma(v, z, X_k^i)\}} N_k^i(dv, dz, du, dr)
 \end{aligned}$$

with

$$\widehat{N}_k^i(dv, dz, du, dr) = (\frac{1}{N} \sum_{i=1}^N \delta_{X_k^i}(dv)) \times \mu(dz) du dr$$

Theorem

$$\left| E\left(\frac{1}{N} \sum_{i=1}^N f(X_{nk}^i)\right) - \int f(x) \theta_{0,t}(\rho)(dx) \right| \leq \frac{C}{N^{1/d}} + \frac{C}{n}$$

Homogenous Boltzmann equation

$$V_{k+1}^i = V_k^i + \int_{s_k}^{s_{k+1}} \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} A(z)(V_k^i - v) \mathbf{1}_{\{u \leq |V_k^i - v|^\alpha\}} N_k^i(dv, dz, du, dr)$$

with

$$\widehat{N}_k^i(dv, dz, du, dr) = \mathcal{L}_{V_k^i}(dv) \times \mu(dz) du dr$$

Inhomogenous Boltzmann equation

$$X_{k+1}^i = X_k^i + V_k^i(s_{k+1} - s_k)$$

and

$$V_{k+1}^i = V_k^i + \int_{s_k}^{s_{k+1}} \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} A(z)(V_k^i - v) \mathbf{1}_{\{u \leq |V_k^i - v|^\alpha\}} \mathbf{1}_{\{|\textcolor{red}{X}_k^i - x| \leq \textcolor{red}{R}\}} N_k^i(d(v, x), dz, du, dr)$$

with

$$\widehat{N}_k^i(d(x, v), dz, du, dr) = (\mathcal{L}_{(X_k^i, V_k^i)}(d(x, v))) \times \mu(dz) du dr$$

Inhomogeneous equation with min field interaction

$$V_{k+1}^i = V_k^i + \int_{s_k}^{s_{k+1}} \int_{\mathbb{R}^d \times E \times \mathbb{R}_+} A(z)(V_k^i - v) \mathbf{1}_{\{u \leq |V_k^i - v|^\alpha\}} \mathbf{1}_{\{u \leq \frac{1}{N} \sum_{j=1}^N n_\varepsilon(X_k^i - X_k^j)\}} \\ N_k^i(d(v, x), dz, du, dr)$$