Probabilistic representation of the derivative of the killed diffusion semigroup

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Stopped processes

Let X be a one dimensional uniformly elliptic diffusion starting at $x > L$. Consider $\tau := \inf\{t > 0; X_t = L\}$.
Question: Is it possible to give meaning to $\partial_x X_{T\wedge\tau}$? (First consider the case $\tau > T$ for boundary type effects)
Some preliminary information:

- Airault, Malliavin and Ren (1999) prove that Malliavin derivatives of order $1/2$ of stopping times do not exist.

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Some preliminary information:

- Airault, Malliavin and Ren (1999) prove that Malliavin derivatives of order $1/2$ of stopping times do not exist.
- The killed semigroup $P_t f(x) = \mathbb{E}\left[f(X_T)1_{(\tau>T)}\right]$ with $f(L) = 0$ is smooth
- A discretized IBP formula for $\partial_x P_t f(x)$ was obtained in joint work with N. Frikha (Paris-Diderot) and L. Li (Australia).

- What to expect:
  - We need to change the notion of derivative: (You expect $\partial_x \mathbb{E}[f(X_T)] = \mathbb{E}[f'(X_T)\mathcal{E}_T]$)

$$\partial_x \mathbb{E}\left[f(X_T)1_{(\tau>T)}\right] = \mathbb{E}\left[f'(Y_T)\mathcal{E}_T\right]$$

- What to expect:
  - The derivative of the killed process is a pair $(Y, \mathcal{E})$ such that for any $f \in \mathcal{C}^1_b$ with $f(L) = 0$, we have

$$\partial_x \mathbb{E}\left[f(X_T)1_{(\tau > T)}\right] = \mathbb{E}\left[f'(Y_T)\mathcal{E}_T\right]$$

  - In fact, it is not difficult to expect that Y has to be a reflected process. Densities of stopped (reflected) BM with no-drift !! :

$$g_t(y - x) \underset{Y_t}{\overset{X_t}{-/+}} g_t(y + x - 2L), \quad y \geq L$$

$$g_t(y) = \frac{e^{-\frac{y^2}{2\sigma^2 t}}}{\sqrt{2\pi\sigma}}$$

When applying ∂_x to the above expression there is a change in sign!

1-dim with no drift: the symmetric case

$$\begin{aligned} X_t &= x + \int_0^t \sigma(X_s)dW_s, \\ Y_t &= x + \int_0^t \sigma(Y_s)dW_s + B_t, \\ \mathcal{E}_t &= 1 + \int_0^t \sigma'(Y_s)\mathcal{E}_s dW_s. \end{aligned}$$

Some ideas about the proof:

- Use approximations, do the calculations (hope for the best) and use at the end Mémin, Jakubowski, Pagès (or the contemporary version for sde's in Kurtz, Protter). We prefer this argument because it is easier to generalize.
- It is essential to consider $1_{(\tau > T)}$ in $\mathbb{E}\left[f(X_T)1_{(\tau > T)}\right]$ as a change of measure (see Gobet) that goes from stopping into reflection.

some details

Consider the EM approximation X_i^n and $U_i \sim U(0,1)$, iid. τ^n is the st. time for the continuously interpolated EM

$$\mathbb{E}\left[f(X_T^n)1_{(\tau^n>T)}\right] = \mathbb{E}\left[f(X_n^n)M_n^n\right]$$

$$M_j^n = \prod_{i=1}^{j} 1_{(U_i>p_i)}$$

$$p_i := \exp\left(-2\frac{(X_i^n - L)(X_{i-1}^n - L)}{\sigma^2(X_{i-1}^n)(t_i - t_{i-1})}\right)$$

Now differentiate the MChains and the “change of measure" M^n, rearrange and take limits

$$\partial_x \mathbb{E}\left[f\left(X_n^n\right)M_n^n\right] = \mathbb{E}\left[f'\left(X_n^n\right)E_n^n \bar{M}_n^n\right]$$

Here $\bar{M}_j^n = \prod_{i=1}^{j}(1 + 1_{(U_i \le p_i)})$ gives the reflecting process. The limit is

$$\partial_x \mathbb{E}[f(X_T)1_{(\tau>T)}] = \mathbb{E}\left[f'(Y_T)\exp\left(\int_0^t \sigma'(Y_s)dW_s - \frac{1}{2}\int_0^t \sigma'(Y_s)^2 ds\right)\right].$$

The drift case: Non-symmetric case

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The drift case: Non-symmetric case

• Most people, would say that the general case is finished after Girsanov and smooth transformations in multi-dim. ☺ Define $h_i = \frac{b_i}{\sigma_i} 1_{(U_i \le p_i)}$

$$\partial_x \mathbb{E}\left[f\left(X_n^n\right) M_n^n\right] = \mathbb{E}\left[f'\left(X_n^n\right) E_n^n \bar{M}_n^n\right] + \sum_{i=1}^{n} \mathbb{E}\left[f_i\left(X_i^n\right) E_{i-1}^n h_i \bar{M}_i^n\right].$$

Solve the following linear equation for $P_t f(x) = \mathbb{E}[f(X_T) 1_{(\tau > t)}]$

$$\partial_x P_T f(x) = \mathbb{E}[f'(Y_T)\hat{\mathcal{E}}_T] + \frac{b}{a}(L)\mathbb{E}\left[\int_0^T \partial_x P_{T-s} f(L)\hat{\mathcal{E}}_s dB_s\right]$$

$$\hat{\mathcal{E}}_t = 1 + \int_0^t \hat{\mathcal{E}}_s\left(b'(Y_s)ds + \sigma'(Y_s)dW_s + \frac{b}{\sigma^2}(L)\, dB_s\right).$$

Actually, Girsanov for the killed process with drift and its approximation is natural while for reflected processes with drift is not.
Here the proof is more involved as it requires the analysis of convergence of the honest time ρ_t.

Two theorems in 1-dim

Theorem

Let $f \in C_b^1$ then $\partial_x P_t f(x) = \mathbb{E}[f'(Y_T)\mathcal{E}_T]$ where

$$\mathcal{E}_t = 1 + \int_0^t \mathcal{E}_s \left(b'(Y_s)ds + \sigma'(Y_s)dW_s + 2\frac{b}{\sigma^2}(L)\, dB_s \right).$$

Theorem

Let $f : [L, \infty) \to \mathbb{R}$ be a measurable and bounded function such that $f(L) = 0$. Then for $\tau(s) := \inf\{u > s : Y_s = L\}$, we have

$$T\partial_x P_T f(x) = \mathbb{E}\left[f(Y_T) \int_0^T 1_{(\tau(s)>T)} \mathcal{E}_s \sigma^{-1}(Y_s) dW_s \right]$$

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