Solution properties of an incompressible Stochastic Euler system

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- Framework and main results
- Connection with geometric mechanics
- Assumptions
- Main steps of the proofs
- Final remarks

This is joint work with Oana Lang, Franco Flandoli and Darryl Holm and is based on the papers:

◦ O Lang, D Crisan, Well-posedness for a stochastic 2D Euler equation with transport noise, Stochastics and Partial Differential Equations: Analysis and Computations, 1-48, 2022.
◦ D. Crisan, F Flandoli, DD Holm, Solution properties of a 3D stochastic Euler fluid equation, Journal of Nonlinear Science 29 (3), 813-870, 2019.
◦ D Crisan, DD Holm, JM Leahy, T Nilssen, Solution properties of the incompressible Euler system with rough path advection, arXiv preprint arXiv:2104.14933.

- v is the fluid velocity, $\omega = \text{curl}\, v$ is the vorticity
- We work on a 3D torus denoted by $\mathbb{T}^3$.
- $\omega_t \in W^{2,2}\left(\mathbb{T}^3, \mathbb{R}^3\right)$
- ξ_k are divergence free given vector fields
- B_t^k are scalar independent Brownian motions
- $\mathcal{L}_A C := [A, C] = (A \cdot \nabla)C - (C \cdot \nabla)A, \quad A, C : \mathbb{R}^3 \mapsto \mathbb{R}^3$.

3D stochastic Euler equation

$$d\omega + [v, \omega]\, dt + \sum_{k=1}^{\infty} [\xi_k, \omega] \circ dB_t^k = 0, \qquad \mathrm{div}(\omega) = 0, \qquad \omega|_{t=0} = \omega_0 \tag{1}$$

- Equation (1) is stated in Stratonovich form; the corresponding Itô form is

$$d\omega + [v, \omega]\, dt + \sum_{k=1}^{\infty} [\xi_k, \omega]\, dB_t^k = \frac{1}{2} \sum_{k=1}^{\infty} [\xi_k, [\xi_k, \omega]]\ dt, \qquad \omega|_{t=0} = \omega_0. \tag{2}$$

Theorem (DC, Flandoli, Holm)

There exists a unique local solution in $\omega \in W^{2,2}\left(\mathbb{T}^3, \mathbb{R}^3\right)$ *of equation* (1).

Beale-Kato-Majda criterion for blow up:

Theorem (DC, Flandoli, Holm)

There exists a stopping time $\tau_{\max} : \Xi \to [0, \infty]$ *and a process* ω *such that:*

i) (ω is a solution) ω *has trajectories in* $C\left([0, \tau_{\max}); W^{2,2}\left(\mathbb{T}^3; \mathbb{R}^3\right)\right)$ *and equation* (1) *holds and* $\tau_{\max}$ *is the largest s.t. with this property; and*

ii) (Beale-Kato-Majda criterion) If $\tau_{\max} < \infty$, *then*

$$\int_0^{\tau_{\max}} \|\omega(t)\|_\infty \, dt = +\infty$$

and, in particular, $\limsup_{t \uparrow \tau_{\max}} \|\omega(t)\|_\infty = +\infty$.

Remarks.

• The BKM criterion can be used as a criterion for testing whether a given numerical simulation has shown finite-time blow up.

• Following Gibbon 2008, the classical Beale-Kato-Majda theorem implies that algebraic singularities of the type

$$\|\omega\|_\infty \geq (t^* - t)^{-p},$$

must have $p \geq 1$.

• We show that a corresponding BKM result also applies for the stochastic Euler fluid equations; hence, the same criterion applies here.

• The L^∞ condition in the BKM theorem was reduced to L^p, for finite p, at the price of imposing constraints on the direction of vorticity. We hope to obtain a similar result for the stochastic equation in future work.

- We work on a 2D torus denoted by $\mathbb{T}^2$.
- $\omega_0 \in W^{k,2}\left(\mathbb{T}^2, \mathbb{R}^2\right)$

2D stochastic Euler equation

$$d\omega_t + u_t \cdot \nabla\omega_t dt + \sum_{i=1}^{\infty}(\xi_i \cdot \nabla\omega_t) \circ dB_t^i = 0 \tag{3}$$

Theorem (DC, Oana Lang)

There exists a unique global solution of equation (3) in the space $C\left([0,\infty); \mathcal{W}^{k,2}(\mathbb{T}^2)\right)$. In particular, if $k \geq 4$ the solution is classical. Moreover if $\tilde{\omega} = \{\tilde{\omega}_t, t \in [0,\infty)\}$ is another solution of the equation, then

$$\mathbb{E}[e^{-CA_t}||\omega_t - \tilde{\omega}_t||_{k,2}^2] \leq ||\omega_0 - \tilde{\omega}_0||_{k,2}^2 \tag{4}$$

In (4) A is the process defined as $A_t := \int_0^t \|\omega_s\|_{k,2} ds$, for any $t \geq 0$. In particular the solution of (3) is unique.

The choice for this equation comes from geometric mechanics. Holm (2015) introduced a new set of stochastic PDEs modelling the motion of an either compressible, or incompressible fluid resulting from a stochastically constrained variational principle $\delta S = 0$, with action, S, given by

$$S(u,p,q) = \int \left(\ell(u,q)dt + \langle p \, , \, dq + \mathcal{L}_{dx_t} q \, \rangle_V \right) , \tag{5}$$

- $\ell(u,q)$ unperturbed deterministic fluid Lagrangian, written as a functional of velocity vector field u and advected quantities q.
- $\langle \, p \, , \, q \, \rangle_V := \int < p(x), q(x,t) > dx$, $q \in V$ and their dual elements $p \in V^*$.
- $\mathcal{L}_{dx_t} q$ is the Lie derivative of the advected quantity $q \in V$, along a vector field dx_t

$$dx_t(x) = u(x,t)\, dt - \sum_i \xi_i(x) \circ dW_i(t) \, . \tag{6}$$

- DD Holm, Variational principles for stochastic fluid dynamics, Proc. R. Soc, 2015.
- O Street, DC, Semi-martingale driven variational principles, Proc. R. Soc, 2021.
- DC DD Holm, JM Leahy, T Nilssen, Variational principles for fluid dynamics on rough paths, Advances in Mathematics 404, 2022.

The SPDEs resulting from the SVP $\delta S = 0$ has the form

$$d\,\frac{\delta \ell}{\delta u} + \mathcal{L}_{dx_t}\frac{\delta \ell}{\delta u} - \frac{\delta \ell}{\delta q} \diamond q\, dt = 0\,, \quad \text{and} \quad dq + \mathcal{L}_{dx_t} q = 0\,, \tag{7}$$

The diamond operation $(\diamond) : T^*V \to \mathfrak{X}^*$ is defined for a vector space V with $(q, p) \in T^*V$ and vector field $\xi \in \mathfrak{X}$ is given by $\langle p \diamond q\,,\, \xi \rangle_{\mathfrak{X}} := \langle p\,,\, -\mathcal{L}_\xi q \rangle_V$ for the pairings $\langle\,\cdot\,,\,\cdot\,\rangle_V : T^*V \times TV \to \mathbb{R}$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{X}} : \mathfrak{X}^* \times \mathfrak{X} \to \mathbb{R}$ with $p \diamond q \in \mathfrak{X}^*$.

If we choose in equation (3) the Lagrangian

$$l(u) = \frac{1}{2}\|u\|^2_{L^2} = \frac{1}{2}\int |u|^2 dx$$

that is, the kinetic energy of the incompressible fluid, constrained to only allow divergence free velocity vector fields, independent of the advected variable q, and compute the curl of right hand side we obtain the stochastic Euler equation.

• the methodology incorporates physically meaningful stochastic perturbations in fluid dynamics equations.

Kelvin's circulation theorem (1869): In a barotropic ideal fluid with conservative body forces, the circulation around a closed curve (which encloses the same fluid elements) moving with the fluid remains constant with time.

Theorem (Kelvin circulation theorem for the SALT Euler model)

$$d\oint_{C(dx_t)} \boldsymbol{u}\cdot d\boldsymbol{x} = 0, \tag{8}$$

where

$$dx_t = u(x,t)\,dt - \sum_i \xi_i(x)\circ dW_i(t).$$

Assumptions on $\{\xi_k\}$ and basic bounds on Lie derivatives

- the vector fields ξ_k are sufficiently smooth
- $\sum_{k=1}^{\infty} ||\mathcal{L}_{\xi_k} f||_{L^2}^2 + ||\mathcal{L}_{\xi_k}^2 f||_{L^2}^2 \leq C \, \|f\|_{W^{2,2}}^2$, $\sum_{k=1}^{\infty} \|\xi_k\|_{W^{3,2}}^2 < \infty$.

Lemma

$$\left\langle \mathcal{L}_{\xi_k}^2 f, f \right\rangle + \langle \mathcal{L}_{\xi_k} f, \mathcal{L}_{\xi_k} f \rangle \leq C_k^{(0)} \, \|f\|_{L^2}^2 \tag{9}$$

$$\left\langle D^\alpha \mathcal{L}_{\xi_k}^2 f, D^\alpha f \right\rangle + \langle D^\alpha \mathcal{L}_{\xi_k} f, D^\alpha \mathcal{L}_{\xi_k} f \rangle \leq C_k^{(2)} \, \|f\|_{W^{|\alpha|,2}}^2 \,, \tag{10}$$

- $\sum_{k=1}^{\infty} C_k^{(0)} < \infty, \qquad \sum_{k=1}^{\infty} C_k^{(2)} < \infty.$

Uniqueness of the solution (3D):

- Let $f_R : [0,\infty) \to [0,\infty)$, be a function with smooth compact support, equal to 1 on $[0,R]$ then there is a unique solution of the truncated equation

$$d\omega + f_R\left(\|\nabla v\|_\infty\right)[v,\omega]\,dt + \sum_{k=1}^{\infty}[\xi_k,\omega]\circ dB_t^k = 0,\ \ \omega|_{t=0} = \omega_0.$$

  The constant R is chosen such that $R > \|\nabla v_0\|_\infty$.
- Let $\omega_0 \in W_\sigma^{2,2}\left(\mathbb{T}^3,\mathbb{R}^3\right)$ and $(\tau_{\max},\omega)$ be a maximal solution of the stochastic 3D Euler equations (2) s.t. either $\tau_{\max} = \infty$ or $\limsup_{t\uparrow\tau_{\max}}\|\omega(t)\|_{W^{2,2}} = +\infty$. Moreover, let $(\tau,\tilde{\omega})$ be another maximal solutions of the same equation with the same initial condition. Then necessarily $\tau = \tau_{\max}$ and $\omega = \omega'$on $[0,\tau_{\max})$.

  Proof. From the local uniqueness we get that $\omega = \tilde{\omega}$ on $[0,\min(\tau,\tau_{\max}))$. By the explosion property, we cannot have $\tau_{\max} < \tau$ on any non-trivial set. Hence $\tau \le \tau_{\max}$. But then from the maximality property of $(\tau,\tilde{\omega})$ it follows that necessarily $\tau = \tau_{\max}$ and therefore $\omega = \tilde{\omega}$ on $[0,\tau_{\max})$.

Existence of a solution:

◦ 3D case

- we use a sequence of approximation equations for a highly regularized problem (use $\nu\Delta^5$) with (mild) solutions in $C\left([0,T]; W_\sigma^{2,2}\left(\mathbb{T}^3;\mathbb{R}^3\right)\right) \cap C\left([\delta,T]; W^{4,2}\left(\mathbb{T}^3;\mathbb{R}^3\right)\right)$ and estimates independent of $\nu > 0$.
- Sending $\nu \to 0$, one gets a global-in-time smooth solution of the truncated Euler equation which is also a local solution of the original equation
- This proves the existence of a local solution. The maximal solution is obtained by taking the truncation paramenter to ∞.
- The relative compactness is shown by means of a combination of the Gyöngy-Krylov convergence criterion and the Flandoli-Gatarek criterion for tightness.

◦ 2D case

- we introduce a sequence of approximation equations for a regularized problem (use $\nu\Delta$) with strong solutions $C\left([0,T]; W^{k,2}\left(\mathbb{T}^2;\mathbb{R}^2\right)\right) \cap L^2\left([0,T]; W^{k+1,2}\left(\mathbb{T}^2;\mathbb{R}^2\right)\right)$ and estimates independent of $\nu > 0$.
- Sending $\nu \to 0$ and $R \to \infty$ one gets a global solution of the equation.
- The relative compactness is shown by means of a combination of the Protter-Kurtz criterion convergence criterion and the Kurtz criterion for tightness.

- We have now a good understanding of the qualitative properties of solutions of SPDEs with realistic noise, relevant for modelling geophysical fluid dynamics models (incorporating temperature, gravity, rotation, buoyancy, bathymetry).
- Incorporating stochasticity transport into geophysical fluid dynamics equations allows quantification of model uncertainty.
- Local/global existence results are proved using a variety of methodology. Smoothness properties are currently being studied (analyticity, Gevrey spaces, etc) as well as SPDE with boundary conditions. McKean-Vlasov SPDEs. General existence and uniqueness criteria. RPDEs have also been considered.
- The next step in the programme is to calibrate the noise of these equations to data, first simulated and then real data.
- Finally we intend to develop data assimilation methodology using particle filters. Properly calibrated and modified, particle filters can be used to solve high dimensional data assimilation problems (state of the art: 10^6 degrees of freedom.
- For further details of the programme, see Stochastic transport in upper ocean dynamics (STUOD):

  https://www.imperial.ac.uk/ocean-dynamics-synergy/

Happy Birthday Marta !