Space-time discretization schemes for the 2D Navier Stokes equations with additive noise

Annie Millet, SAMM, Université Paris 1
& LPSM
collaboration with
Hakima Bessaih, Florida International University

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The 2D Navier-Stokes equations on the torus

Time evolution of an **incompressible fluid** on $D=[0,L]^2$ with periodic b.c.

$$\begin{aligned} \partial_t u - \nu \Delta u + (u\cdot\nabla)u + \nabla\pi = 0 \quad &\text{in} \quad (0,T)\times D,\\ \text{div } u = 0 \quad &\text{in} \quad (0,T)\times D,\\ u(0)=u_0\in [H^1(D)]^2, \quad \text{div } u_0 = 0 \quad &\text{in } D, \end{aligned}$$

where $u=u(t,x)$ is the **velocity** field, $\pi(t,x)$ is the scalar **pressure**, $\nu>0$ is the viscosity, $\text{div}\, u = \sum_i \partial_i u^i$, $\quad ((u.\nabla)v)^i = \sum_j u^j \partial_j v^i$,

Let $\mathbb{L}^p := [L^p(D)]^2,\ p\in[1,\infty], \quad H=\{u\in\mathbb{L}^2,\ \text{div } u = 0 \text{ in } D\}$,
$V=[H^1(D)]^2\cap H$
$\Pi_H : \mathbb{L}_2 \to H$ Leray projection
$A=-\Pi_H\Delta$ is a positive unbounded operator on H
$\text{Dom}\,(A) = \mathbb{W}^{2,2}\cap H$.

The bilinear operator

- $B : V \times V \to V'$ continuous defined by $B(u,v) = (u.\nabla)v$,

$$\big\langle B(u_1,u_2),u_3\big\rangle = \sum_{i,j=1}^{2} \int_D u_1^j(x)\,\partial_j u_2^i(x)\,u_3^i(x)\,dx$$

$$\langle B(u_1,\,u_2),\,u_3\rangle = -\,\langle B(u_1,u_3)\,,\,u_2\rangle, \quad \forall u_i \in V.$$

- D is the **torus**,

$$\langle B(u,u),Au\rangle = 0, \quad \forall u \in \mathrm{Dom}\,(A).$$

The stochastic perturbation

• Let $\{\lambda_j\}_{j\geq 1}$ be **eigenvalues** of A with $0 < \lambda_1 \leq \lambda_2 < \ldots$,
$\{\zeta_j \in \mathbb{W}^{2,2} \cap H\}_{j\geq 1}$ be the **eigenfunctions**; suppose $\{\zeta_j\}_{j\geq 1}$ is a **CONS of** H. $\lambda_j \sim j^2$, and $\lambda_j \to +\infty$ as $j \to +\infty$.
• Q is a **symmetric bounded operator in** H with $Q\zeta_j = q_j \zeta_j$, $j = 1, 2, \ldots$, $q_j > 0$ and $\sum_j q_j < \infty$; then Q is **trace class**. Set

$$W(t) = \sum_{j\geq 1} \sqrt{q_j}\, \beta_j(t)\, \zeta_j,\ t \geq 0,$$

where $\{\beta_j(t)\}_{j\geq 1}$ are independent 1D Brownian motions
Assume the **stronger condition**

$$K_0 := \sum_{j\geq 1} \lambda_j\, q_j < \infty.$$

The stochastic NS equations with additive noise

Normalize the viscosity $\nu = 1$. Let (u, π) be the solution to

$$\partial_t u - \Delta u + (u \cdot \nabla) u + \nabla \pi = dW, \quad \mathrm{div}\ u = 0 \quad \text{in} \quad (0, T) \times D,$$

Suppose u_0 is $\mathcal{F}_0$-measurable and $\mathbb{E}\Big[\exp\big(\gamma_0 |A^{\frac12} u_0|^2_{\mathbb{L}^2}\big)\Big] < \infty$ for $\gamma_0 > 0$.
Dealing with the **solution** / its **time discretization**, **project** on divergence free fields (Leray projection); this **cancels the pressure** π.
This SPDE has a **unique "strong" solution** u, i.e.,

- u is an adapted V-valued process,
- $\mathbb{P}$ a.s. we have $u(.,.,\omega) \in C([0,T];V) \cap L^2(0,T;\mathrm{Dom}(A))$,
- $\mathbb{P}$ a.s.

$$\big(u(t), \phi\big) + \int_0^t \big(A^{\frac12} u(s), A^{\frac12}\phi\big) ds + \int_0^t \big\langle [u(s)\cdot\nabla] u(s), \phi \big\rangle ds$$
$$= \big(u_0, \phi\big) + \big(W(t), \phi\big), \quad \forall t \in [0,T],\ \forall \phi \in V.$$

Moment estimates / exponential moments of the solution

For every $p \in [2, +\infty)$, we have for $C > 0$ (additive / multiplicative)

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|u(t)\|_V^p + \int_0^T |Au(s)|_{\mathbb{L}^2}^2 \big(1 + \|u(s)\|_V^{p-2}\big)\, ds \Big) \le C\big[1 + \mathbb{E}(\|u_0\|_V^p)\big].$$

Theorem (Bessaih-M. ("additive"))

(i) Let $u_0 \in V$*; then for* $0 < \alpha < \tilde\alpha_0 = \frac{1}{\mathrm{Tr}(Q)}$*, there exists* $C(\alpha) > 0$ *s.t.*

$$\mathbb{E}\Big[\exp\Big(\alpha \sup_{t\in[0,T]} |A^{\frac12} u(t)|_{\mathbb{L}^2}^2 \Big) \Big] = C(\alpha) < \infty. \qquad (1)$$

(ii) Let u_0 *be* $\mathcal{F}_0$ *measurable with* $\mathbb{E}\big(e^{\gamma_0 |A^{\frac12} u_0|_{\mathbb{L}^2}^2}\big) < \infty$*. Then for* $0 < \alpha < \tilde\alpha_0 \frac{\gamma_0}{\gamma_0 + \tilde\alpha_0}$*, there exists* $C(\alpha) > 0$ *s t. (1) holds*

Remark: u_0 deterministic is formally $\gamma_0 = +\infty$.

Similar result proven by M. Hairer & J. Mattingly (2006)

Time regularity results ($u_0 \in V$)

Valid for an **additive** or a **multiplicative** perturbation.

• **In** $\mathbb{L}^2$ (Carelli-Prohl)

Given $q \geq 1$, there exists $C > 0$ s.t. for $s, t \in [0, T]$

$$\mathbb{E}\big(|u(t) - u(s)|_{\mathbb{L}^2}^{2q}\big) \leq C\ \big(1 + \mathbb{E}(\|u_0\|_V^{4q}\big)\ |t - s|^q.$$

• **in** V (Bessaih-M.)

Fix an integer $N \geq 1$ and for $j = 0, 1, ..., N$, set $t_j := j\frac{T}{N}$. Then given $\eta \in (0, 1)$, for every $q \in [1, +\infty)$ for some constant C

$$\mathbb{E}\Big(\Big|\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\big[\|u(s) - u(t_{j-1})\|_V^2 + \|u(s) - u(t_j)\|_V^2\big]\,ds\Big|^q\Big) \leq C\Big(\frac{T}{N}\Big)^{\eta q}.$$

Similar results can be deduced for the $\mathbb{L}^4$-norm (Gagliardo-Nirenberg)

Moments of time increments in V similar to the $\mathbb{L}^2$-one proven

- by Carelli-Prohl (far away from 0) for $u_0 \in V$;
- by Breit-Dogson for $u_0 \in \mathbb{H}^2 \cap H$.

The fully implicit time scheme

Set $\mathrm{u}^0 = u_0$. Fix $N \geq 1$ let $t_l = lT/N$.
induction step For $l = 1, \cdots, N$, find pairs $(\mathrm{u}^l, \pi^l) \in V \times L^2_{per}$ such that $\mathbb{P}$ a.s. for all $\phi \in \mathbb{W}^{1,2}_{per}$ and $\psi \in L^2_{per}$,

$$\big(\mathrm{u}^l - \mathrm{u}^{l-1}, \phi\big) + \frac{T}{N}\Big[\big(\nabla \mathrm{u}^l, \nabla\phi\big) + \big\langle([\mathrm{u}^l \cdot \nabla]\mathrm{u}^l), \phi\big\rangle\Big] - \frac{T}{N}\big(\pi^l, \mathrm{div}\ \phi\big)$$
$$= \big(\Delta_l W, \phi\big), \quad \text{and} \quad \big(\mathrm{div}\ \mathrm{u}^l, \psi\big) = 0,$$

where $\Delta_l W = W(t_l) - W(t_{l-1})$.
project on divergence free fields; rewrite

$$\big(\mathrm{u}^l - \mathrm{u}^{l-1}, \phi\big) + \frac{T}{N}\Big[\big(A^{\frac{1}{2}}\mathrm{u}^l, A^{\frac{1}{2}}\phi\big) + \big\langle(\mathrm{u}^l \cdot \nabla \mathrm{u}^l), \phi\big\rangle\Big] = \big(\Delta_l W, \phi\big), \quad \forall \phi \in V.$$

Moment estimates of u^l - Carelli-Prohl (2012)

Unique solution $\{\mathrm{u}^l\}_{l=0}^N$; $\mathrm{u}^l \in L^2(\Omega; V)$ a.s. is $\mathcal{F}_{t_l}$-measurable.
Given $q \in [2, \infty)$ s.t. $\mathbb{E}(\|u_0\|_V^{2q}) < \infty$.

$$\sup_N \mathbb{E}\Big(\max_{0\le l\le N} \|\mathrm{u}^l\|_V^{2q} + \frac{T}{N} \sum_{l=1}^N \|\mathrm{u}^l\|_V^{2q-2} \, |A\mathrm{u}^l|_{\mathbb{L}^2}^2 \Big) < \infty.$$

Strong rate of convergence of the Euler time scheme

For $k = 0, ..., N$ set $e_k := u(t_k) - \mathrm{u}^k$.

Theorem (Bessaih-M.)

(i) Let $u_0 \in V$ *and suppose that* $T\,\mathrm{Tr}\,(Q) < \frac{2}{\bar{C}^2}$ *(*$\bar{C}$ *from the* $\mathbb{L}^4$ *Gagliardo Nirenberg inequality).*
Then for $\eta \in (0,1)$ *there exists* $C > 0$ *s.t. for "large"* N

$$\mathbb{E}\Big(\max_{1\leq j\leq N} |e_j|_{\mathbb{L}^2}^2 + \frac{T}{N} \sum_{1\leq j\leq N} |A^{\frac{1}{2}} e_j|_{\mathbb{L}^2}^2 \Big) \leq C \Big(\frac{T}{N} \Big)^{\eta}. \qquad (2)$$

(ii) If u_0 *is* $\mathcal{F}_0$*-measurable with* $\mathbb{E}\big(e^{\gamma_0 |A^{1/2} u_0|_{\mathbb{L}^2}^2}\big) < \infty$*. Suppose that for some* $\mu \in (0,1)$ *we have*
$T\,\mathrm{Tr}\,(Q) < \mu \frac{2}{\bar{C}^2}$ *and* $\gamma_0 \geq \frac{T\,\bar{C}^2}{2(1-\mu)}$.
Then for $\eta \in (0,1)$ *here exists* $C > 0$ *s.t. (2) holds for "large"* N *.*

If u_0 is deterministic, $\gamma_0 \to \infty$ and $\mu \to 1$: consistant results.

Sketch of the proof

- $\phi \in V$ and $e_k := u(t_k) - \mathrm{u}^k$

$$(e_j - e_{j-1}, \phi) + \int_{t_{j-1}}^{t_j} \Big[\big(A^{\frac{1}{2}} u(s) - A^{\frac{1}{2}} \mathrm{u}^j \,,\, A^{\frac{1}{2}} \phi\big) \\ + \big\langle B(u(s), u(s)) - B(\mathrm{u}^j, \mathrm{u}^j)\big) \,,\, \phi \big\rangle \Big] ds = 0.$$

- estimate of the bilinear term: some "bad" coefficient. For $\delta_1, \delta_2 > 0$,

$$\big(e_j - e_{j-1}\,,\, e_j\big) + \frac{T}{N} |A^{\frac{1}{2}} e_j|^2_{\mathbb{L}^2} \leq (\delta_1 + \delta_2) \frac{T}{N} |A^{\frac{1}{2}} e_j|^2_{\mathbb{L}^2} + \tilde{T}_j \\ + \Big(\gamma_2 + \frac{\bar{C}^2}{4\delta_1} |A^{\frac{1}{2}} u(t_j)|^2_{\mathbb{L}^2}\Big) \frac{T}{N} |e_j|^2_{\mathbb{L}^2},$$

Sketch of the proof - continued

- collect these upper estimates, choose $\delta_1 \sim 1$ and $\delta_2 \in (0, 1-\delta_1)$

$$\max_{1\le j\le k} \frac{1}{2}|e_j|^2_{\mathbb{L}^2} \le Z + \frac{T}{N}\, g \sum_{j=1}^{k-1} |e_j|^2_{\mathbb{L}^2},$$

where

$$\big\{\mathbb{E}(Z^q)\big\}^{1/q} \le C\big(\tfrac{T}{N}\big)^{\eta},\ \eta \in (0,1) \text{ (time regularity of } u);$$
$$g = \Big(\gamma_2 + \tfrac{\bar{C}^2}{2\delta_1} \sup_{s\in[0,T]} |A^{\frac{1}{2}} u(s)|^2_{\mathbb{L}^2}\Big).$$

- Use the discrete Gronwall lemma ω by ω (no stochastic integral in RHS)
- exponential moments of $\sup_{s\in[0,T]} |A^{\frac{1}{2}} u(s)|^2_{\mathbb{L}^2}$ valid for "small" coefficient

Convergence in probability - multiplicative noise

In the RHS replace $dW(t)$ by $G(u(t))dW(t)$
require linear growth and globally Lipschitz conditions on G
• **Localized** $L^2(\Omega)$ convergence
$\mathbb{E}\big(1_{\Omega_T(K)}\big\{ \sup_{n\le N} |e_n|^2_{\mathbb{L}^2} + \frac{T}{N}\sum_{1\le n\le N} |A^{\frac12} e_n|^2_{\mathbb{L}^2}\big\}\big) \le C_K (T/N)^\eta$
First localization : $\Omega_T(K) = \{\sup_n |A^{\frac12} u^n|^2_{\mathbb{L}^2} \le K\}$
• Choosing $K(N)$, deduce the **rate of convergence in probability**
$P\Big(\sup_{n\le N} |e_n|^2_{\mathbb{L}^2} + (T/N)\sum_{1\le n\le N} |A^{\frac12} e_n|^2_{\mathbb{L}^2} \ge C(T/N)^\eta\Big) \to 0.$

rate of cv in probability:
- Carelli & Prohl (2012) **multiplicative noise**, $\mathbb{E}(\|u_0\|^8_V) < \infty$;
time exponent $\eta \in (0, \frac12)$ (**rate** almost **1/4**) ;
- Breit & Dogson (2021) **multiplicative noise**, $u_0 \in \mathbb{H}^2$.
time exponent $\eta \in (0,1)$ (**rate** almost **1/2**) .

$L^2(\Omega)$ rate of cv - Multiplicative/Additive

• **with localization** for a **multiplicative** noise:
Different localization by $\tilde\Omega_T(K) = \{\sup_{s\leq T} |A^{\frac12} u(s)|^2_{\mathbb{L}^2} \leq K\}$
Bessaih-M. (2019) for the time scheme
Idea: Balance $\mathbb{E}(1_{\tilde\Omega_T(K(N))} \max_n |e_n|^2_{\mathbb{L}^2})$ and $\mathbb{E}(1_{\tilde\Omega_T(K(N))^c} \max_n |e_n|^2_{\mathbb{L}^2})$
use moments of $|u(t_n)|^2_{\mathbb{L}^2}$ and $|u^n|^2_{\mathbb{L}^2}$ / "small" probability of $\tilde\Omega_T(K(N))^c$
Case 1: "linear" multiplicative noise Logarithmic rate of convergence
When $\mathbb{E}(\|u_0\|_V^{2^q}) < \infty$ replace $\big(\frac{T}{N}\big)^\eta$ by $\big[\ln(N)\big]^{-(2^{q-1}-1)}$
Case 2: bounded diffusion coefficient G**: Polynomial** rate $\big(\frac{T}{N}\big)^\eta$
When $\eta < \eta_0$ where η_0 depends on the bound $\tilde K$ of G, $\mathrm{Tr}\,(Q)$ (for fixed viscosity). Use exponential moments
As $\tilde K \mathrm{Tr}\,(Q) \to 0$, $\eta_0 \to \frac12$ (**time rate approaches 1/4**).

• **Without localization** for an **additive noise**
Bessaih-M. arXiv 2021, (Stoch. Dyn.);
$\eta \in (0,1)$ based on a **discrete Gronwall lemma used for fixed "**ω.

Exponential moments (used for the full scheme)

Theorem (Bessaih-M.)

(i) Let $u_0 \in V$*. Then for* $0 < \alpha < \tilde{\alpha}_0 = \frac{1}{\mathrm{Tr}\,(\mathrm{Q})}$*, there exists* $C_1(\alpha) > 0$ *s.t. for large* N

$$\mathbb{E}\Big[\exp\Big(\alpha \max_{0\le l\le N} |A^{1/2}\mathrm{u}^l|^2_{\mathbb{L}^2}\Big)\Big] = C_1(\alpha) < \infty. \qquad (3)$$

For $\beta < \frac{\tilde{\alpha}_0}{2}$*, there exists* $C_2(\beta) > 0$ *s.t. for large* N

$$\mathbb{E}\Big[\exp\Big(\beta \max_{0\le n\le N} \Big[|A^{1/2}\mathrm{u}^n|^2_{\mathbb{L}^2} + \frac{T}{N}\sum_{l=1}^{N} |A\mathrm{u}^l|^2_{\mathbb{L}^2}\Big]\Big)\Big] = C_2(\beta) < \infty. \qquad (4)$$

(ii) Let u_0 *be random,* $\mathcal{F}_0$*-measurable with* $\mathbb{E}\big(e^{\gamma_0 |A^{1/2}u_0|^2_{\mathbb{L}^2}}\big) < \infty$*.*
Set $\tilde{\beta}_0 := \tilde{\alpha}_0 \, \frac{\gamma_0}{\gamma_0 + \tilde{\alpha}_0}$*. Then for* $0 < \alpha < \tilde{\beta}_0$ *there exists* $C_1(\alpha) > 0$ *s.t. (3) holds for large* N *.*
Set $\tilde{\beta}_1 := \tilde{\alpha}_0 \, \frac{\gamma_0}{2\gamma_0 + \tilde{\alpha}_0}$*; then for* $0 < \beta < \tilde{\beta}_1$ *there exists* $C_2(\beta) > 0$ *s.t. (4) holds for large* N*.*

Mixed finite elements

Keep the pressure π to have a stable pairing of u and π which satisfies the discrete LBB condition.

$\mathcal{T}_h$ quasi-uniform triangulation $[0,L]^2$; triangles of **maximal diameter** $h>0$. Set $\bar{D} := \cup_{K\in\mathcal{T}_h}\bar{K}$,

$\mathbb{P}_1(K)=[P_1(K)]^2$: **polynomial** vector fields on K of degree at most 1

$$\begin{aligned}\mathbb{H}_h &:= \{\mathrm{U}\in C^0(\bar{D})\cap\mathbb{W}^{1,2}_{per}(D) : \mathrm{U}\in\mathbb{P}_1(K), \quad \forall K\in\mathcal{T}_h\},\\ L_h &:= \{\Pi\in L^2_{per}(D) : \Pi\in P_1(K), \quad \forall K\in\mathcal{T}_h\},\end{aligned}$$

with discrete **LBB-condition** $\sup_{\Phi\in\mathbb{H}_h}\frac{(\mathrm{div}\Phi\,,\Pi)}{|A^{1/2}\Phi|_{\mathbb{L}^2}}\geq C\,|\Pi|_{\mathbb{L}^2}$ for $\Pi\in L_h$.

Let $\mathbb{V}_h :=\{\Phi\in\mathbb{H}_h \,:\, (\mathrm{div}\,\Phi,\Lambda)=0,\ \forall\Lambda\in L_h\}$ (**in general** $\mathbb{V}_h\not\subset V$)

$\mathrm{Q}^0_h:\mathbb{L}^2\to\mathbb{V}_h$ **projection** defined by $(z-\mathrm{Q}^0_h,\Phi)=0, \quad \forall\Phi\in\mathbb{V}_h$.

$$|\mathrm{z}-\mathrm{Q}^0_h\mathrm{z}|_{\mathbb{L}^2}+h|A^{\frac12}(\mathrm{z}-\mathrm{Q}^0_h\mathrm{z})|_{\mathbb{L}^2}\leq\ C\,h^2\,|A\mathrm{z}|_{\mathbb{L}^2}, \quad \forall \mathrm{z}\in V\cap\mathbb{W}^{2,2}(D),$$

$$|\mathrm{z}-\mathrm{Q}^0_h\mathrm{z}|_{\mathbb{L}^2}\leq C\,h\,|A^{\frac12}\mathrm{z}|_{\mathbb{L}^2}, \quad \forall\mathrm{z}\in V.$$

Space-time discretization

Let $\tau := \frac{T}{N}$ denote the constant **time mesh**.
Let $U_0 \in \mathbb{H}_h$ a.s., $\mathcal{F}_0$ measurable, s.t. for some $q \in [1, +\infty)$
$\mathbb{E}(|u_0 - U^0|^{2q}_{\mathbb{L}^2}) \leq C(q)\, h^{2q}$ and $\mathbb{E}(|A^{\frac{1}{2}} U^0|^{2q}_{\mathbb{L}^2}) \leq C(q)$ (e.g. $U^0 = Q^0_h u_0$)
Due to the nonlinear effect of **discretely divergence-free velocity**

$$\tilde{b}(U_1, U_2, U_3) := ([U_1 \cdot \nabla] U_2 \,,\, U_3) + \frac{1}{2}\big([\mathrm{div} U_1]\, U_2 \,, U_3\big), \ \forall U_1, U_2, U_3 \in \mathbb{W}^{1,2}$$

once more antisymmetry $\tilde{b}(U_1, U_2, U_3) = -\tilde{b}(U_1, U_3, U_2)$
Induction step: For $l = 1, \cdots, N$, find a pair $(U^l, \Pi^l) \in \mathbb{H}_h \times L_h$, $\mathcal{F}_{t_l}$-measurable s.t. for $(\Phi, \Lambda) \in \mathbb{H}_h \times L_h$

$$\begin{aligned}(U^l - U^{l-1}, \Phi) + \tau\, (A^{\frac{1}{2}} U^l, A^{\frac{1}{2}} \Phi) + \tau\big([U^{l-1} \cdot \nabla] U^l, \Phi\big) + \frac{\tau}{2}\big([\mathrm{div}\, U^{l-1}]\, U^l, \Phi\big) \\ -\tau(\Pi^l, \mathrm{div}\, \Phi) = (\Delta_l W, \Phi), \\ (\mathrm{div}\, U^l, \Lambda) = 0.\end{aligned}$$

$U^l \in \mathbb{V}_h$ with moment estimates similar to u^l (Carelli-Prohl)

Strong rate of convergence of the full scheme (Algo. 1)

Theorem

Let $\mathrm{U}^0 \in \mathbb{H}_h$ *be* $\mathcal{F}_0$*-measurable, s.t. for some "large"* q
$\mathbb{E}\big(\|\mathrm{U}^0\|^q_{\mathbb{W}^{1,2}}\big) < \infty$ *and* $\mathbb{E}(|u_0 - \mathrm{U}^0|^{2q}_{\mathbb{L}^2}) \leq C(q)\, h^{2q}$.
(i) $u_0 \in V$ **deterministic** *Suppose* $\mathrm{Tr}(Q) < \frac{4}{13\,[T\,\bar{C}^2 + 4\,\sigma^2]}$.
Given $\eta \in (0,1)$, *for "large"* N, $\tau = T/N$ *and* $h \in (0,1)$ *(inconditional)*

$$\mathbb{E}\Big(\max_{0\leq l\leq N} |u(t_l) - \mathrm{U}^l|^2_{\mathbb{L}^2} + \frac{T}{N} \sum_{l=1}^N \big| A^{\frac12}(u(t_l)) - A^{\frac12}\mathrm{U}^l\big|^2_{\mathbb{L}^2}\Big) \leq C\big[\tau^\eta + h^2\big]. \quad (5)$$

(ii) u_0 **is** $\mathcal{F}_0$**-measurable s.t.** $\mathbb{E}\big[\exp(\gamma_0\,|A^{1/2} u_0|^2_{\mathbb{L}^2})\big] < \infty$
Suppose that for some $\mu \in (0,1)$,
$\mathrm{Tr}(Q) < \mu \frac{4}{13\,\big(T\,\bar{C}^2 + 4\,\sigma^2\big)}$, and $\gamma_0 \geq \frac{13\,(T\,\bar{C}^2 + 2\sigma^2)}{4\,(1-\mu)}$.
Then for any $\eta \in (0,1)$ (5) *holds.*

• Proof: Use the strong convergence of the time Euler scheme ($\tau = T/N$)

$$\mathbb{E}\Big(\max_{0\leq l\leq N} |\mathrm{u}^l - \mathrm{U}^l|^2_{\mathbb{L}^2} + \tau \sum_{l=1}^{N} \big|A^{\frac12}\mathrm{u}^l - A^{\frac12}\mathrm{U}^l\big|^2_{\mathbb{L}^2}\Big) \leq C\big[\tau^\eta + h^2\big]$$

• If one uses **divergence-free FE** (such as Scott Vogelius)

$$(\mathrm{U}^l - \mathrm{U}^{l-1}, \Phi) + \tau\,(A^{\frac12}\mathrm{U}^l, A^{\frac12}\Phi) + \tau\big([\mathrm{U}^{l-1}\cdot\nabla]\mathrm{U}^l, \Phi\big) + \frac{\tau}{2}\big([\mathrm{div}\mathrm{U}^{l-1}]\,\mathrm{U}^l, \Phi\big) = \big(\Delta_l W, \Phi\big), \qquad \forall \Phi \in \mathbb{V}_h.$$

Same rate of convergence (which is "optimal") with **weaker conditions** (no need of the constant σ in the Sobolev embedding).
$\mathrm{Tr}\,(Q) < \frac{4}{5\bar{C}^2 T}$ if $u_0 \in V$ is deterministic.
$\mathrm{Tr}(Q) < \mu\,\frac{4}{5\bar{C}^2 T}$ and $\gamma_0 \geq \frac{5\bar{C}^2 T}{4\,(1-\mu)}$ for $\mu \in (0,1)$ if u_0 has exp. moments.

Other space-time rate of conv. in probability

multiplicative noise

• Carelli & Prohl (2012) $u_0 \in L^8(\Omega; V)$; diffusion coeff. controlled in $\mathbb{W}^{1,2}_{per}$.
coupling between space and time meshes
part of upper estimate $\tau^\eta h^{-\epsilon}$ for $\eta < \frac{1}{2}$ and $\mathbb{E}(\tau \sum_{n\leq N} |\nabla \pi^n|^2_{\mathbb{L}^2})$

• Breit & Dogson (2021) $u_0 \in L^8(\Omega; V)$ and is $\mathbb{H}^2$-valued
introduce a **decomposition** (kind of de Rham) to **suppress the pressure** at the end;
diffusion coefficient controlled in $\mathbb{W}^{1,2}_{per}$
coupling between time and space meshes : requires $L\tau \leq \big(- \epsilon \ln(\tau)\big)^{-1}$
upper estimate $\tau^{\eta-\epsilon} + h^2$ with $\eta < 1$.

$L^2(\Omega)$ cv - Multiplicative (Bessaih-M., 2021)

• **Multiplicative noise** $\mathbb{E}(\|u_0\|_V^{2^{q_0}}) < \infty$; **logarithmic** convergence
- **mixed FE**:
coupling $h^2\tau^{-1} \to 0$ if diffusion coefficient controlled in $\mathbb{W}^{1,2}_{per}$
estimate $\Big| \ln \big[\tau + h^2\tau^{-1}\big] \Big|^{-(2^{q_0-2}-\frac{1}{2})}$.
no coupling if diffusion coefficient controlled in V
estimate $\Big| \ln \big(\tau + h^2\big) \Big|^{-(2^{q_0-2}-\frac{1}{2})}$
- **divergence free FE**: **no coupling** between τ and h
$\Big| \ln \big(\tau + h^2\big) \Big|^{-(2^{q_0-1}-1)}$ for diffusion controlled in $\mathbb{W}^{1,2}_{per}$

• **bounded diffusion coefficient**
- **mixed FE**: coupling / **rate**
$\exp\Big(-\gamma \Big| \ln \big[\tau + h^2 k^{-1}\big] \Big|^{\frac{1}{2}}\Big)$ or $\exp\Big(-\gamma \Big| \ln \big[\tau + h^2\big] \Big|^{\frac{1}{2}}\Big)$.
- **divergence free FE**: no coupling / **rate** if bound in $\mathbb{W}^{1,2}_{per}$
$\big[(\tau + h^2)^{\gamma_1} + \tau^{\gamma}\big]$ for γ and γ_1 depending on strength of the noise.

Some further works

• Convergence **in probability** with **additive noise in 2D**
Breit & Prohl (2021 arXiv preprint) $u_0 \in \mathbb{H}^3 \cap H$, **additive noise** with control of noise in $V \cap \mathbb{W}^{3,2}$
$\max_n P(|u(t_n) - u^n|^2_{\mathbb{L}^2} + h\sum_{l=0}^N |\nabla[u(t_l) - u^l]|^2_{\mathbb{L}^2} \geq C\tau^\alpha) \to 0,\ \alpha < 2$
Similar result for space-time discretization with U^l and $\tau^\alpha + h^\alpha$ instead of τ^α ; **no coupling of meshes**.

• H. Bessaih & A.M. **Strong** - i.e. $L^2(\Omega)$ - convergence in **3D with "Brinkman-Forchheimer"** smoothing (add $a|u|^{2\alpha}u$ in LHS, $\alpha \geq 1$); some connection with porous media
$u_0 \in V$, **"optimal" rate** for the implicit time Euler scheme holds, **multiplicative** setting, diffusion coeff. controlled in V, **no constraint** on the strength of the noise

• **Work in progress**
- More involved hydrodynamical models (with H. Bessaih).
- Use some IMEX ideas for results better suited for implementation (with H. Bessaih, O. Landoulsi and V. Gitting)
- Anisotropic 3D models with smoothing term (with H. Bessaih).

Last but not least ...

Et desitjo tot el millor, Marta!

Treballar amb tu ha sigut genial!

Espero que continuarem fent recerca juntes durant els propers anys.