Semimartingales with jumps, weak Dirichlet processes and path-dependent martingale problems.

Francesco Russo, ENSTA Paris, Institut Polytechnique de Paris

Conference on Stochastic Analysis and Stochastic Partial Differential Equations

in honor of **Prof. Marta Sanz-Solé**

May 30th-June 3rd 2022, CRM Barcelona

Joint work with

Elena Bandini (Università Alma Mater, Bologna)

Marta

- High level Mathematics.
- Scientific and university politics at the European level.
- Humanity, Spontaneity, Unpretentiousness.

My first impact. Saint-Flour school.

Multiparameter processes, Malliavin calculus.

Outline

1. Basic elements of stochastic calculus via regularizations.
2. Weak Dirichlet processes and semimartingales.
3. Motivations and examples.
4. Stochastic calculus driven by discontinuous weak Dirichlet process.
5. BSDEs: the identification problem.
6. General (possibly path-dependent) martingale problems with jumps.

Basic References

- E. Bandini and F. Russo

  Weak Dirichlet processes and generalized martingale problems.

  Preprint 2022.
  https://hal.archives-ouvertes.fr/hal-03660061/

- E. Bandini and F. Russo

  Special weak Dirichlet processes and BSDEs driven by a random measure.

  Bernoulli, vol. 24(4A), pp. 2569-2609, 2018.

- E. Bandini and F. Russo

  Weak Dirichlet processes with jumps.

  Stochastic Processes and their applications, vol. 12, pp. 4139–4189, 2017.

- E. Bandini and F. Russo

  The identification problem for BSDEs driven by possibly non quasi-left-continuous random measures.

  Stochastics & Dynamics, vol. 20 (16), 2020.

Other available preprints and publications.

http://uma.ensta.fr/∼russo/

1 Basic elements of stochastic calculus via regularization

1.1 The covariation.

Russo and Vallois [1995]

We set

$$[X, Y]_s = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^s \left(X_{(r+\varepsilon)\wedge s} - X_r\right) \left(Y_{(r+\varepsilon)\wedge s} - Y_r\right) \mathrm{d}r \quad \text{u.c.p.}$$

Remark 1 *Let X, Y (càdlàg) semimartingales.*

1. *$[X, Y]$ is the usual (square) bracket.*
2. *If X is such that $[X, X]$ exists then X is said to be a finite quadratic variation process.*

Remark 2

- *One defines also various stochastic integrals (via regularization).*

  Russo and Vallois [1991, 1993]

- *The calculus is* pathwise *in the spirit, still* probabilistic *in the facts.*

- *In this talk we avoid the complications due to multidimensional aspects.*

- **Higher order irregularity:** *Errami and Russo [2003], Gradinaru et al. [2003], Gradinaru et al. [2005], Kruk and Russo [2010], Nourdin.*

- **Connections with rough paths integrals (Gubinelli, Friz and coauthors)***: see Ohashi et al. [2021].*

2 Weak Dirichlet processes

2.1 History and notion

In the case of continuous processes.

M. ERRAMI AND F. RUSSO (2003) *n-covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes.* Stoch. Process. Their Appl.

F. GOZZI AND F. RUSSO (2006) *Weak Dirichlet processes with a stochastic control perspective.* Stoch. Process. Their Appl.

$C^{0,1}$-chain rule of Itô type, when X continuous with finite quadratic variation.

Path-dependent framework

Di Girolami and Russo [2012], Leão et al. [2018], Bouchard et al. [2021].

In the case of càdlàg processes.

F. COQUET, A. JAKUBOWSKI, J. MEMIN, L. SLOMINSKI (2006)
Natural decomposition of processes and weak Dirichlet processes.
Lecture Notes in Mathematics. They have introduced the

notion of what we call **Special weak Dirichlet processes** without mentioning it.

Definition 3 *Bandini and Russo [2017].*
We consider a "usual" filtration $\mathbb{F}$, which will be often omitted, when self-explanatory.

A process X is said to be a weak Dirichlet process (resp. a (special weak Dirichlet process) if it admits the following.

1. $X = M + A$,
2. M local martingale,
3. A adapted (resp. predictable) and martingale orthogonal, i.e., $[A, N] = 0$ for every continuous local martingale N.
4. $A_0 = 0$.

Remark 4 **Basic examples of martingale orthogonal processes.**

- *A purely discontinuous local martingale.*
- *A càdlàg bounded variation process.*

Remark 5

1. *The decomposition of a special weak Dirichlet process is unique.*
2. *Generalization of the notion of special semimartingale.*
3. *Generalization of the notion of Dirichlet process* $([A, A] = 0)$.
4. *The notion of Dirichlet process not adapted in the jump case. Indeed* $[A, A] = 0$ *implies that* A *is continuous.*
5. *The decomposition of weak Dirichlet is generally not unique. As for càdlàg semimartingales one has uniqueness only after cutting big jumps.*

2.2 A new unique decomposition

This decomposition is also new and useful for semimartingales.

Remark 6 *Any local martingale M can be uniquely decomposed as the sum of a continuous local martingale M^c and a purely discontinuous local martingale M^d such that $M_0^d = 0$, see Theorem 4.18, Chapter I, in Jacod and Shiryaev [2003].*

A decomposition for weak Dirichlet processes is not unique, but the result below proposes a particularly natural one, which is unique.

Proposition 7 *Let X be a càdlàg $\mathbb{F}$-weak Dirichlet process. Then there is a unique continuous $\mathbb{F}$-local martingale X^c and a unique $\mathbb{F}$-martingale orthogonal process A vanishing at zero, such that*

$$X = X^c + A. \tag{1}$$

Proof. *Existence.* Since X is an $\mathbb{F}$-weak Dirichlet process, it is a process of the type $X = M + \Gamma$, with M an $\mathbb{F}$-local martingale and Γ an $\mathbb{F}$-martingale orthogonal process vanishing at zero. Recalling Remark 6, it follows that X admits the decomposition

$$X = M^c + M^d + \Gamma, \tag{2}$$

that provides (1) by setting $A := M^d + \Gamma$ and $X^c := M^c$.

Uniqueness. Assume that X admits the two decompositions

$$X = M^1 + A^1, \quad X = M^2 + A^2$$

with M^1, M^2 continuous $\mathbb{F}$-local martingales and A^1, A^2 $\mathbb{F}$-martingale orthogonal processes vanishing at zero. So we have $0 = M^1 - M^2 + A^1 - A^2$. Taking the covariation of previous equality with $M^1 - M^2$, we get $[M^1 - M^2, M^1 - M^2] \equiv 0$. Since $M^1 - M^2$ is a continuous martingale vanishing at zero we finally obtain $M^1 = M^2$ and so $A^1 = A^2$. $\blacksquare$

3 Motivations and examples

- Irregular Markov processes solutions of SDEs with distributional drift with jumps.
- Solutions of (even continuous) path-dependent SDEs with distributional drift, see Ohashi et al. [2020b].
- (Path-dependent) Bessel processes, see Ohashi et al. [2020a].

- Applications in verification theorems in stochastic control, see Gozzi and Russo [2006b], in the continuous framework.
- Identification problem in BSDEs driven by random measure, Bandini and Russo [2018].

4 Stochastic calculus for discontinuous weak Dirichlet processes

4.1 Some preliminary notations

Let X be a càdlàg process.

- μ_X will denominate its **jump measure**)

$$\mu^X(dt\,dx) = \sum_{0<s\leq T} \mathbb{I}_{\{\Delta X_s \neq 0\}}\, \delta_{(s,\Delta X_s)}(dt\,dx). \tag{3}$$

- We denote by $\nu^X = \nu^{X,\mathbb{P}}$ the compensator of μ^X, see Jacod and Shiryaev [2003] (Theorem 1.8, Chapter II). The dependence on $\mathbb{P}$ will be omitted when self-explanatory.
- For a random field $(W_t(x))$ one denotes

$$\hat{W}_t = \int_{\mathbb{R}} W_t(x)\,\nu^X(\{t\} \times dx), \quad \tilde{W}_t = \int_{\mathbb{R}} W_t(x)\,\mu^X(\{t\} \times dx) - \hat{W}_t$$

if previous integral make sense.

- $\mathcal{A}^+$ (resp $\mathcal{A}^+_{\text{loc}}$) the collection of all adapted integrable increasing (resp. adapted locally integrable) processes.
- $\tilde{\mathcal{P}}$: predictable random fields on $\Omega \times [0, T] \times \mathbb{R}$.

- For every $q \in [1, \infty[$, we also introduce the linear spaces

$$\mathcal{G}^q(\mu^X) = \Big\{W \in \tilde{\mathcal{P}} :\ \forall s \geq 0 \int_{\mathbb{R}} |W(s,x)|\, \nu^X(\{s\} \times \mathbb{R}) < \infty, \\ \Big[\sum_{s \leq \cdot} |\tilde{W}_s|^2\Big]^{q/2} \in \mathcal{A}^+\Big\},$$

$$\mathcal{G}^q_{\mathrm{loc}}(\mu^X) = \Big\{W \in \tilde{\mathcal{P}} :\ \forall s \geq 0 \int_{\mathbb{R}} |W(s,x)|\, \nu^X(\{s\} \times \mathbb{R}) < \infty, \\ \Big[\sum_{s \leq \cdot} |\tilde{W}_s|^2\Big]^{q/2} \in \mathcal{A}^+_{\mathrm{loc}}\Big\}.$$

- We also introduce the norm $||W||_{\mathcal{L}^2(\mu^X)} := \mathbb{E}[|W|^2 \star \nu_T]$ and the space $\mathcal{L}^2(\mu^X) := \{W \in \tilde{\mathcal{P}} : \; ||W||_{\mathcal{L}^2(\mu^X)} < \infty\}$. We have

$$\mathcal{L}^2_{\mathrm{loc}} \subset \mathcal{G}^2_{\mathrm{loc}} \subset \mathcal{G}^1_{\mathrm{loc}}.$$

- Given a random measure ν We define

$$W \star \nu = \int_{[0,\cdot]\times\mathbb{R}} W_s(x)\, \nu(ds \times dx).$$

- If $W \in \mathcal{G}^1_{\mathrm{loc}}$ then the purely discontinuous local martingale $W \star (\mu_X - \nu_X)$ is well-defined.
- If $|W| \star \mu_X \in \mathcal{A}_{\mathrm{loc}}$ (locally integrable) then $|W| \star \nu \in \mathcal{A}^+_{\mathrm{loc}}$ and
$$W \star (\mu_X - \nu_X) = W \star \mu_X - W \star \nu_X.$$
- If $W \in \mathcal{G}^2_{\mathrm{loc}}$ then $W \star (\mu_X - \nu_X)$ is a square integrable local martingale and it admits an oblique bracket.

4.2 Truncation functions

Similar to the case of a semimartingale.

$\mathcal{K}$: the space of truncation function. Typical example of $k \in \mathcal{K}$ is

$$k(x) = x 1_{\{x \le a\}},$$

for some $a \in \mathbb{R}$.

4.3 Some typical assumptions

Let X be a càdlàg process, $v : [0, T] \times \mathbb{R} \to \mathbb{R}$ locally bounded.

Assumption 1 (Square-jumps)

$$\sum_{s \leq \cdot} |\Delta X_s|^2 < \infty \quad \text{a.s.} \tag{4}$$

Notice that condition (4) is equivalent to ask that $(1 \wedge |x|^2) \star \mu^X \in \mathcal{A}^+_{\text{loc}}$ where μ^X is the jump measure related to X defined in (3).

Assumption 2 (Basic-G1)

$v(t, X_t)$ is a càdlàg process, and for every $t \in [0, T]$,

$$\Delta v(t, X_t) = v(t, X_t) - v(t, X_{t-}); \quad (5)$$

$$\exists\, k \in \mathcal{K} \ \text{ s.t. } \quad (v(s, X_{s-} + x) - v(s, X_{s-}))\, \frac{k(x)}{x} \in \mathcal{G}^1_{\mathsf{loc}}(\mu^X). \quad (6)$$

Remark 8 *1. If v is continuous, then the pair (v, X) obviously fulfills* (5)*.*

2. If $v \in C^{0,1}([0,T] \times \mathbb{R})$ then Assumption (Basic-G1) is fulfilled.

3. Nevertheless Assumption (Basic-G1) is verified in many other situations, for instance if $(v(t, X_t)$ has bounded variation (vanishing continuous local martingale component).

4.4 Characteristics of a weak Dirichlet process.

Subtraction of large jumps.

Let X be an $\mathbb{F}$-weak Dirichlet process with jump measure μ^X satisfying Assumption (Square-jumps). Given $k \in \mathcal{K}$, by Corollary 20

$$X^k = X - \sum_{s \le \cdot} [\Delta X_s - k(\Delta X_s)]$$

is an $\mathbb{F}$-special weak Dirichlet process with unique decomposition

$$X^k = X^c + k \star (\mu^X - \nu^X) + B^{k,X}, \tag{7}$$

where

- X^c is the unique continuous $\mathbb{F}$-local martingale part of X introduced in Proposition 7; we set $C^X := \langle X^c \rangle$.
- $B^{k,X}$ is a predictable and $\mathbb{F}$-martingale orthogonal process.

$(B^{k,X}, C^X, \nu^X)$: Characteristics of the weak Dirichlet process X.

Remark 9 *When X is a semimartingale, $B^{k,X}$ is a bounded variation process, so in particular $\mathbb{F}$-martingale orthogonal.*

4.5 Characteristics and Itô formula for semimartingales.

Classical framework. X is a semimartingale with characteristics $(B^{k,X}, C^X, \nu^X)$ and $v \in C^{1,2}([0,T] \times \mathbb{R})$.

Proposition 10 (**Itô formula.**) *Let* $f : [0,T] \times \mathbb{R}$ *be a bounded function of class* $C^{1,2}$, X *a càdlàg semimartingale. Then*

$$
\begin{aligned}
f(t, X_t) &= f(0, X_0) + \int_0^t \partial_s f(s, X_s) ds + \frac{1}{2} \int_0^t \partial_{xx} f(s, X_s)\, dC_s^X \\
&+ \int_0^t \partial_x f(s, X_s)\, dB_s^{k,X} \\
&+ \int_{]0,t] \times \mathbb{R}} (f(s, X_{s-} + x) - f(s, X_{s-}) \\
&+ k(x)\, \partial_x f(s, X_{s-}))\, \nu^X(ds\, dx), \quad t \in [0,\, T].
\end{aligned}
$$

Underlying idea. To find substitution tools when f is not smooth and X is not a semimartingale.

4.6 A first fundamental chain rule

Theorem 11 (First-chain-rule) *Let X be a càdlàg and $\mathbb{F}$-adapted process satisfying Assumption (Square-jumps).*

Let $v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a locally bounded function such that (v, X) satisfies Assumption (Basic-G1).

Let $Y_t = v(t, X_t)$ be an $\mathbb{F}$-weak Dirichlet process. Then, for every $k \in \mathcal{K}$, one can write the decomposition

$$Y = Y^c + M^{k,d} + \Gamma^k(v) + (v(s, X_{s-} + x) - v(s, X_{s-}))\frac{(x - k(x))}{x} \star \mu^X, \tag{8}$$

with

$$M^{k,d} := (v(s, X_{s-} + x) - v(s, X_{s-}))\frac{k(x)}{x} \star (\mu^X - \nu^X) \qquad (9)$$

and $\Gamma^k(v)$ a predictable and $\mathbb{F}$-martingale orthogonal process.

Remark 12 *(i) Sufficient conditions for* Y *to be weak Dirichlet are given in Theorem (Stability-weak-Dir).*

(ii) Notice that, $\Gamma^k(Id) = B^{k,X}$*.*

Taking $k(x) = x\mathbb{I}_{\{|x|\le a\}}$ in Theorem (First-chain-rule) we get the following result.

Corollary 13 *Let X be a càdlàg $\mathbb{F}$-adapted process satisfying Assumption (Square-jumps). Let $v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a locally bounded function, such that (v, X) satisfies Assumption (Basic-G1). Assume moreover that, for some $a \in \mathbb{R}_+$,*

$$|\Delta X_t| \le a, \quad \forall t \in \mathbb{R}_+. \tag{10}$$

Then, if $Y_t = v(t, X_t)$ is an $\mathbb{F}$-weak Dirichlet process, then it is an $\mathbb{F}$-special weak Dirichlet process.

Corollary 13 with $v \equiv \mathrm{Id}$ gives in particular the following result.

Corollary 14 *Let X be a càdlàg and $\mathbb{F}$-adapted process satisfying Assumption (Square-jumps) and* (10)*. Then, if X is an $\mathbb{F}$-weak Dirichlet process, it is an $\mathbb{F}$-special weak Dirichlet process.*

In fact it is also not difficult to prove the following.

Theorem 15 *Let X be càdlàg and $\mathbb{F}$-adapted process with bounded jumps satisfying Assumption (Square-jumps). Let $v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a locally bounded function such that (v, X) satisfies Assumption (Basic-G1). Set $Y_t = v(t, X_t)$, and assume that Y is an $\mathbb{F}$-weak Dirichlet process. Then Y is an $\mathbb{F}$-special weak Dirichlet process if and only if*

Assumption 3 (Cond-Special-Weak)

$$\exists\, a \in \mathbb{R}_+ \text{ s.t. } |v(s, X_{s-} + x) - v(s, X_{s-}|\, \mathbb{I}_{\{|x|>a\}} \star \mu^X \in \mathcal{A}^+_{\text{loc}}. \tag{11}$$

Remark 16 *If v is bounded and X is a càdlàg and $\mathbb{F}$-adapted, then Assumption (Cond-Special-Weak) is satisfied.*

Theorem 15 with $v \equiv \mathrm{Id}$ gives in particular the following characterization.

Corollary 17 *Let X be an $\mathbb{F}$-weak Dirichlet process satisfying Assumption (Square-jumps) Then X is an $\mathbb{F}$-special weak Dirichlet process if and only if*

$$\exists\, a \in \mathbb{R}_+ \text{ s.t. } x\, \mathbb{I}_{\{|x|>a\}} \star \mu^X \in \mathcal{A}^+_{\mathrm{loc}}. \tag{12}$$

Theorem 18 (First-chain-rule-special) *Let X be a càdlàg and $\mathbb{F}$-adapted process satisfying Assumption (Square-jumps).*

Let $v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a locally bounded function such that (v, X) satisfies Hypothesis Basic-G1. Let $Y_t = v(t, X_t)$ be an $\mathbb{F}$-weak Dirichlet process.

Assume moreover that the pair (v, X) satisfies Assumption (Cond-Special-Weak). Then the $\mathbb{F}$-special weak Dirichlet process $Y_t = v(t, X_t)$ admits the unique decomposition

$$Y = Y^c + (v(s, X_{s-} + x) - v(s, X_{s-})) \star (\mu^X - \nu^X) + \Gamma, \tag{13}$$

with Γ a predictable and $\mathbb{F}$-martingale orthogonal process.

Remark 19 *Let X be a càdlàg and $\mathbb{F}$-adapted process satisfying Assumption (Square-jumps). Let $v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be such that (v, X) satisfies Assumption (Basic-G1). If v is moreover a bounded function, by Remark 16 Assumption (Cond-Special-Weak) is satisfied as well, and Theorem (First-chain-rule-special) holds true.*

Theorems (First-chain-rule) and (First-chain-rule-special) with $v \equiv \mathrm{Id}$ give in particular the following result.

Corollary 20 *Let X be an $\mathbb{F}$-weak Dirichlet process satisfying Assumption (Square-jumps). Let X^c be the continuous martingale part of X. Then, the following holds.*

(i) Let $k \in \mathcal{K}$. Then X can be decomposed as

$$X = X^c + k(x) \star (\mu^X - \nu^X) + \Gamma^k(Id) + (x - k(x)) \star \mu^X. \quad (14)$$

(ii) If (12) *holds, then the $\mathbb{F}$-special weak Dirichlet process X admits the decomposition*

$$X = X^c + x \star (\mu^X - \nu^X) + \Gamma \quad (15)$$

with $\Gamma := \Gamma^k(Id) + x\, \mathbb{I}_{\{|x|>1\}} \star \nu^X$.

4.7 A relaxed notion of finite quadratic variation

Definition 21 *A càdlàg process X is said to be a weakly finite quadratic variation process if there is $\varepsilon_0 > 0$ such that the laws of the random variables $[X, X]^{ucp}_{\varepsilon}(T)$, $0 < \varepsilon \leq \varepsilon_0$, are tight.*

Below, $\varepsilon > 0$ will mean $0 < \varepsilon \leq \varepsilon_0$ for some ε_0 small enough. For instance, a family $(Z_\varepsilon)_{\varepsilon > 0}$ of random variables will indicate a sequence $(Z_\varepsilon)_{0 < \varepsilon \leq \varepsilon_0}$ for some ε_0 small enough.

Remark 22 *A finite quadratic variation process is a weakly finite quadratic variation process. Indeed, if* $\int_0^{\cdot} \frac{(X_{(s+\varepsilon)\wedge \cdot} - X_s)^2}{\varepsilon} ds$ *converges u.c.p., the random variable* $[X, X]_{\varepsilon}^{ucp}(T)$ *converges in probability, and so it also converges in law.*

Remark 23 *We set, for* $\varepsilon > 0$*,*

$$Z_\varepsilon) := \int_0^T \frac{(X_{(s+\varepsilon)\wedge T} - X_s)^2}{\varepsilon} ds,$$

Suppose that either

(i) $\sup_{\varepsilon>0} Z_\varepsilon < \infty$ ***a.s.***

(ii) $\sup_{\varepsilon>0} \mathbb{E}[Z_\varepsilon] < \infty$*.*

Then the family of distribution of $(Z_\varepsilon)_{\varepsilon>0}$ *is tight, so* X *is a weakly finite quadratic variation process.*

Proposition 24 *If* X *is a weakly finite quadratic variation process, then Assumption (Square-jumps) holds true.*

4.8 $C^{0,1}$-chain rules for semimartingales and weak Dirichlet processes.

Below we give a significant generalization of Proposition 3.10 in Gozzi and Russo [2006a], where the result was proven when X is continuous and of finite quadratic variation.

When X is càdlàg, even in the case when X is a finite quadratic variation process, the result is new.

Theorem 25 (Stability-weak-Dir) *Let X be an $\mathbb{F}$-weak Dirichlet process with weakly finite quadratic variation. Let $v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a function of class $C^{0,1}$. Then $Y_t = v(t, X_t)$ is an $\mathbb{F}$-weak Dirichlet with continuous martingale component*

$$Y^c = Y_0 + \int_0^{\cdot} \partial_x v(s, X_s)\, dX_s^c. \tag{16}$$

Idea of the proof.

We aim at proving that, for every $\mathbb{F}$-continuous local martingale N,

$$[v(\cdot, X), N]_t = \int_0^t \partial_x v(s, X_s)\, d[X^c, N]_s, \quad t \in [0, T]. \tag{17}$$

As a matter of fact, this would imply that $A(v) := v(\cdot, X) - Y^c$ is martingale orthogonal, and therefore by additivity $v(\cdot, X)$ is a weak Dirichlet process. Then (16) would follow by the uniqueness of the continuous martingale part of Y.

A consequence of Theorem (First-chain-rule) and Theorem ((Stability-weak-Dir)) is the following.

Corollary 26 (Stab-Special) *Let X be a weakly finite quadratic variation weak Dirichlet process.*

Let v of class $C^{0,1}$ fulfilling Condition (Cond-Special-Weak).

Then $Y = v(\cdot, X)$ is a special weak Dirichlet process and

$$\begin{aligned} Y &= Y_0 + \int_0^{\cdot} \partial_x v(s, X_s) dX^c \\ &+ (v(s, X_{s-} + x) - v(s, X_{s-})) \star (\mu^X - \nu^X) + \Gamma, \end{aligned}$$

with Γ a predictable and $\mathbb{F}$-martingale orthogonal process.

5 BSDEs: the identification problem.

5.1 General mathematical context

Probabilistic tool for representing semilinear PDEs

PDE:

$$\begin{cases} \partial_s u(s,x) + L_s u(s,x) + f(s,x,u(s,x),\sigma\partial_x u(s,x)) = 0 \\ u(T,x) = g(x),\ s \in [0,T], x \in E = \mathbb{R}^d, \end{cases} \tag{18}$$

where L_t is the generator of a diffusion of the type

$$dX_t = \sigma(t,X_t)dW_t + b(t,X_t)ds, X_s = x. \tag{19}$$

BSDE: (19) is coupled with

$$Y_t = g(X_T) + \int_t^T f(r, X_r, Y_r, Z_r)dr - \int_t^T Z_r dW_r. \qquad (20)$$

The link is the following.

1. If u is a classical solution of (18) then

$$Y_t = u(t, X_t), Z_t = \sigma(t, X_t)\partial_x u(t, X_t)$$

provide a solution to (20) (Itô formula).

2. Viceversa if, given $(s, x) \in [0, T] \times E$ and $X^{s,x}$ is given by (19), $(X^{s,x}, Y^{s,x}, Z^{s,x})$ is a solution to (20), then $u(s, x) := Y_s^{s,x}$ is a *viscosity solution* to (18).

What about $v(s,x) := Z_s^{s,x}$?

- If u is of class $C^{0,1}$ then $v(s,x) = \sigma(s,x)\partial_x u(s,x)$.
- What happens in the case of a BSDE driven by continuous martingale (for instance a Brownian motion) and a random measure?

5.2 BSDEs driven by a random measure.

More specifically

BSDE driven by a compensated random measure and a continuous martingale N.

Let ζ be a non-decreasing, adapted and continuous process, and a predictable random measure λ on $\Omega \times [0,T] \times \mathbb{R}$. Given a BSDE driven by a random measure $\mu - \nu$ and a continuous martingale M of the type

$$Y_t = \xi + \int_{]t,T]} \tilde{g}(s, Y_{s-}, Z_s) d\zeta_s + \int_{]t,T]\times\mathbb{R}} \tilde{f}(s, Y_{s-}, U_s(e))\lambda(ds\, de)$$
$$- \int_{]t,T]} Z_s dM_s - \int_{]t,T]\times\mathbb{R}} U_s(e)(\mu - \nu)(ds\, de), \qquad (21)$$

its solution is a triple of processes $(Y, Z, U(\cdot))$.

5.3 The identification problem

Suppose that the component Y of the solution can be expressed as $Y_t = v(t, X_t)$ for some v and some adapted càdlàg process X, the identification problem consists in expressing Z and $U(\cdot)$ in terms of v.

(i) Being $Y_t = v(t, X_t)$ a solution to a BSDE, it is a special weak Dirichlet process, and therefore (v, X) satisfies Assumption (Cond-Special-Weak).

Then, if Assumption (Basic-G1) holds for (v, X), then Theorem (First-chain-rule-special) allows to identify $U(\cdot)$.

More precisely, this provides

$$U(e) \star (\mu - \nu) = (v(s, X_{s-} + x) - v(s, X_{s-})) \star (\mu^X - \nu^X), \quad a.s.$$

(ii) Suppose now for simplicity $\mu = \mu^X$, even though this can be generalized. This provides

$$H(x) \star (\mu^X - \nu^X) = 0, \quad a.s.,$$

with $H(x) := U(x) - (v(s, X_{s-} + x) - v(s, X_{s-}))$. If $H \in \mathcal{G}^2_{\mathrm{loc}}(\mu^X)$, then the predictable bracket of $H(e) \star (\mu^X - \nu^X)$ can be calculated.

This gives (see Proposition 2.8 in Bandini and Russo [2018]) that there is a predictable process (l_s) such that

$$H_s(x) = l_s \mathbb{I}_K(s) \quad d\mathbb{P}\, \nu^X(ds\, dx) \text{ a.e.}$$

where

$$K := \{(\omega, t) : \nu^X(\omega, \{t\} \times \mathbb{R}) = 1\}.$$

(iii) If $v \in C^{0,1}$ then one can show (by Corollary (Stab-Special)) identifying the continuous local martingale component of the BSDE) that

$$Z_s = \partial_x v(s, X_s) \left(\frac{d[X,M]}{d[M,M]} \right)_s,$$

$d[M] \times dP$ a.e.

(iv)

If M is a Brownian motion and $[X,X]_t = \int_0^t \sigma^2(s, X_s) ds$ then

$$Z_s = \partial_x v(s, X_s) \sigma(s, X_s), \; dsdPa.e.$$

6 General (possibly path-dependent) martingale problems with jumps.

6.1 Definition

$\mathbb{D}(0, T)$: càdlàg functions on $[0, T]$.

$\mathbb{D}_-(0, T)$: càglàd functions on $[0, T]$.

$B([0, T])$: bounded functions on $[0, T]$.

Given $\eta \in \mathbb{D}_-(0,\, T)$, (resp. $\zeta \in \mathbb{D}(0,\, T)$), we will use the notation

$$\eta^t(s) := \begin{cases} \eta(s) & \text{if } s < t, \\ \eta(t) & \text{if } s \geq t \end{cases}$$

and

$$\eta_s^- := \eta(s-), s \in [0, T].$$

We will make use of the following for a triplet $(\mathcal{D}, \Lambda, \gamma)$.

Assumption 4 *Let* $\mathcal{D} \subseteq C^{0,1}$, $\Lambda : \mathcal{D} \to \mathbb{D}_-([0,\, T]; B([0, T])$ *be a linear map, together with the map* $\gamma : [0, T] \times \mathbb{D}_-(0,\, T) \to \mathbb{R}$ *be such that, for every* $\eta \in \mathbb{D}_-(0,\, T)$, $\gamma(\cdot, \eta)$ *is of bounded variation.*

We also suppose the non-anticipating property holds for Λ *and* γ, *i.e, for every* $v \in \mathcal{D}$, $\eta \in \mathbb{D}(0,\, T), t \in [0, T]$,

$$(\Lambda v)(t, \eta) = (\Lambda v)(t, \eta^t),$$
$$\gamma(t, \eta) = \gamma(t, \eta^t).$$

where we have denoted

$$(\Lambda v)(t, \eta) := (\Lambda v)(\eta)(t).$$

Definition 27 *Fix* $N \in \mathbb{N}$*. Let*

$$\mathcal{D} := \mathcal{D}_{\mathcal{A}} \subseteq C^{0,1},$$

$\Lambda_i : \mathcal{D}_{\mathcal{A}} \to D_-(0,\, T); B([0, T])$ *and* $\gamma_i : [0,\, T] \times \mathbb{D}_-(0,\, T) \to \mathbb{R}$*,* $i = 1, ..., N$*, satisfying Assumption 4. For every* $v \in \mathcal{D}_{\mathcal{A}}$*,* $\eta \in \mathbb{D}_-(0,\, T)$*, we set*

$$(\mathcal{A}v)(ds, \eta) := \sum_{i=1}^{N} \Lambda_i v(s, \eta)\, \gamma_i(ds, \eta). \qquad (22)$$

Definition 28 *Let* $(\mathcal{A}v)(ds,\cdot)$ *as in Definition 27. A process* X *is said to solve the martingale problem (under a probability* $\mathbb{P}$*) with respect to* $\mathcal{A}$*,* $\mathcal{D}_A$ *and* x_0*, if the following conditions hold:*

(i) Assumption (Square-jumps) holds under $\mathbb{P}$*;*

(ii) for any $v \in \mathcal{D}_{\mathcal{A}}$ *and bounded, the process*

$$M_t^v := v(t, X_t) - v(0, x_0) - \int_0^t (\mathcal{A}v)(ds, X_s^-) \qquad (23)$$

is an $(\mathcal{F}_t^X)$*-local martingale under* $\mathbb{P}$*.*

When the problem is "Markovian" (non path-dependent) then

$$(\mathcal{A}u)(dt,\eta) = \partial_t u(t,\eta(t)) + \mathcal{L}u(t,\eta(t))$$

and

$$\mathcal{D}_{\mathcal{A}} = C^1([0,T];\mathcal{D}_{\mathcal{L}}),$$

where $\mathcal{L} : \mathcal{D}_{\mathcal{L}} \subset C(\mathbb{R}) \to C(\mathbb{R})$ and $\mathcal{D}_{\mathcal{L}}$ is equipped with the graph topology.

Remark 29 *Solutions of martingale problems: typical examples of weak Dirichlet processes.*

6.2 Examples

6.2.1 Semimartingales

Let B^k be an predictable process with bounded variation on the canonical space, with finite variation on finite intervals, and $B_0^k = 0$, C be an adapted continuous process of finite variation with $C_0 = 0$, and ν be a predictable random measure on $\mathbb{R}_+ \times \mathbb{R}$ with decomposition $\nu(\omega, ds\, dx) = \phi_s(\omega, dx)\chi(\omega, ds)$.

Let X be an adapted càdlàg real semimartingale with characteristics (B^k, C, ν).

Then $(X, \mathbb{P})$ is a solution of the martingale problem in Definition 28 with respect to $\mathcal{A}$, $\mathcal{D}_{\mathcal{A}}$ and x_0, with $\mathcal{D}_{\mathcal{A}} = C_b^{1,2}$ and

$$
\begin{aligned}
(\mathcal{A}f)(ds, \eta) \quad &:= \quad \partial_s f(s, \eta_s) ds + \frac{1}{2} \partial^2_{xx} f(s, \eta_s)\, d(C \circ \eta)_s \\
&+ \quad \partial_x f(s, \eta_s)\, d(B^k \circ \eta)_s \\
&+ \quad \int_{\mathbb{R}} (f(s, \eta_s + x) - f(s, \eta_s) \\
&- \quad k(x)\, \partial_x f(s, \eta_s))\, \phi_s(\eta, dx) d\chi_s(\eta), \quad \eta \in \mathbb{D}_-(0,\, T).
\end{aligned}
$$

6.2.2 Weak Dirichlet processes derived from semimartingales

Let X be a weak Dirichlet process with characteristics (B^k, C, ν).

Assume that there exists $h \in C^{0,1}$, $h(t, \cdot)$ bijective and $h(t, X_t)$ is a semimartingale with characteristics $(\bar{B}^k, \bar{C}, \bar{\nu})$.

Then $(X, \mathbb{P})$ is a solution of the martingale problem in Definition 28 with respect to $\mathcal{A}$, $\mathcal{D}_{\mathcal{A}}$ and x_0, with

$$\mathcal{D}_{\mathcal{A}} = \{f \in C_b^{0,1} : \; f \circ h^{-1} \in C_b^{1,2}\}$$

and, for every $\gamma \in \mathbb{D}_-(0,\, T)$,

$$
\begin{aligned}
(\mathcal{A}f)(ds,\gamma) &= \partial_s(f\circ h^{-1})(s,h(s,\gamma_s))ds \\
&+ \frac{1}{2}\partial^2_{xx}(f\circ h^{-1})(s,h(s,\gamma_s))\,(\partial_x h(s,\gamma_s))^2\,d(C\circ\gamma)_s \\
&+ \partial_x(f\circ h^{-1})(s,h(s,\gamma_s))\,d(\bar{B}^k\circ h(\cdot,\gamma))_s \\
&+ (f(s,\gamma_s+x)-f(s,\gamma_s) \\
&- \frac{k(h(s,\gamma_s+x)-h(s,\gamma_s)}{\partial_x h(s,\gamma_s)}\,\partial_x f(s,\gamma_s))\,\phi_s(\gamma,dx)d\chi_s(\gamma
\end{aligned}
$$

6.2.3 Discontinuous Markov processes with distributional drift

We have studied existence and uniqueness for a martingale problem with distributional drift in a discontinuous Markovian framework.

Let $k \in \mathcal{K}$ be a continuous function. Let $\beta = \beta^k : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ be continuous functions, with σ not vanishing at zero. We consider formally the PDE operator of the type

$$L\psi = \frac{1}{2}\sigma^2\psi'' + \beta'\psi' \tag{24}$$

in the sense introduced by Flandoli et al. [2000, 2004]. In particular we suppose the following.

Assumption 5 *1. There is a function*

$$\Sigma(x) := 2 \int_0^x \frac{\beta'}{\sigma^2}(y)dy \tag{25}$$

in some sense, generally by smoothing.

2. The function Σ in (25) is upper and lower bounded, and belongs to C^{α}_{loc}.

Let $\mathcal{D}_L$ be the set of $f \in C^1$ such that there is $\phi \in C^1$ with $f' = e^{-\Sigma}\phi$. For any $f \in \mathcal{D}_L$, we set

$$Lf = \frac{\sigma^2}{2}(e^{\Sigma} f')' e^{-\Sigma}.$$

This defines without ambiguity $L : \mathcal{D}_L \subset C^1 \to C^0$. We also define

$$\mathcal{D}_{\mathcal{L}} := \mathcal{D}_L \cap C^{1+\alpha}_{\text{loc}} \cap C^0_b, \tag{26}$$

equipped with the graph topology of L, the natural topology of $C^{1+\alpha}_{\text{loc}}$ and the uniform convergence.

Then we consider a transition kernel $Q(\cdot, dx)$ from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, with $Q(y, \{0\}) = 0$, satisfying the following condition.

Assumption 6 *The map* $y \mapsto \int (1 \wedge |x|^{1+\alpha})\, Q(y, dx)$ *is continuous in the total variation topology.*

For every $f \in \mathcal{D}_{\mathcal{L}}$, we finally introduce the operator

$$\mathcal{L}f(y) := Lf(y) + \int_{\mathbb{R}\backslash 0} (f(y+x) - f(y) - k(x)\, f'(y)\,)Q(y,dx), \tag{27}$$

that we assume to take values in C^0.

Theorem 30 *Under Assumptions 5 and 6, the martingale problem in Definition 28 with respect to* $\mathcal{D}_{\mathcal{A}}$*,* $\mathcal{A}$ *and* $x_0 \in \mathbb{R}$ *admits existence and uniqueness.*

6.2.4 Continuous path-dependent SDEs with distributional drift

In Ohashi et al. [2020b] one investigates the martingale problem related to a path-dependent SDE of the type

$$dX_t = \sigma(X_t)dW_+(\beta'(X_t) + \Gamma(t, X^t))dt, \tag{28}$$

where $\Gamma : [0, T] \times \mathbb{D}(0, T) \to \mathbb{R}$ is a bounded Borel functional (for σ and β we refer to the notations of previous example).

Consider the operator $L : \mathcal{D}_L \to C(\mathbb{R})$, with $\mathcal{D}_L$ as before.

Proposition 31 *Let $(X, \mathbb{P})$ be a solution to the martingale problem above. Then $(X, \mathbb{P})$ is a solution to the martingale problem in Definition 28 with respect to $\mathcal{D}_{\mathcal{A}} := C^1([0, T]; \mathcal{D}_L)$, $\mathcal{A}$ and x_0, with*

$$(\mathcal{A}v)(ds, \eta) = (\partial_s v(s, \eta(s) + Lv(s, \eta(s)) + \Gamma(s, \eta^s)\partial_x v(s, \eta(s))\, ds.$$

■

6.2.5 The PDMPs case

We omit the details.

References

E. Bandini and F. Russo. Weak Dirichlet processes with jumps. *Stochastic Process. Appl.*, 127(12):4139–4189, 2017. ISSN 0304-4149. doi: 10.1016/j.spa.2017.04.001. URL https://doi.org/10.1016/j.spa.2017.04.001.

E. Bandini and F. Russo. Special weak Dirichlet processes and BSDEs driven by a random measure. *Bernoulli*, 24(4A): 2569–2609, 2018.

B. Bouchard, G. Loeper, and X. Tan. A $c^{0,1}$-functional itô's formula and its applications in mathematical finance. 2021.

C. Di Girolami and F. Russo. Generalized covariation and extended Fukushima decompositions for Banach space valued processes. application to windows of Dirichlet processes. *Infinite Dimensional Analysis, Quantum Probability and Related Topics (IDA-QP).*, 15(2), 6 2012. doi: 10.1142/S0219025712500075.

M. Errami and F. Russo. n-covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes. *Stochastic Process. Appl.*, 104(2):259–299, 2003.

F. Flandoli, F. Russo, and J. Wolf. Some sdes with distributional drift part i: General calculus. *Osaka J. Maths*, 40(2):493–542, 2000.

F. Flandoli, F. Russo, and J. Wolf. Some SDEs with distributional drift. II. Lyons-Zheng structure, Itô's formula and semimartingale characterization. *Random Oper. Stochastic Equations*, 12(2):145–184, 2004.

F. Gozzi and F. Russo. Weak Dirichlet processes with a stochastic control perspective. *Stochastic Process. Appl.*, 116(11):1563–1583, 2006a.

F. Gozzi and F. Russo. Verification theorems for stochastic optimal control problems via a time dependent Fukushima-Dirichlet decomposition. *Stochastic Process. Appl.*, 116(11): 1530–1562, 2006b.

M. Gradinaru, F. Russo, and P. Vallois. Generalized covariations, local time and Stratonovich Itô's formula for fractional Brownian motion with Hurst index $H \geq \frac{1}{4}$. *Ann. Probab.*, 31 (4):1772–1820, 2003.

M. Gradinaru, Ivan Nourdin, F. Russo, and P. Vallois. m-order integrals and generalized Itô's formula: the case of a fractional Brownian motion with any Hurst index. *Ann. Inst. H. Poincaré Probab. Statist.*, 41(4):781–806, 2005.

J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2003. ISBN 3-540-43932-3. doi: 10.1007/978-3-662-05265-5.

I. Kruk and F. Russo. Malliavin-Skorohod calculus and Paley-Wiener integral for covariance singular processes. *ArXiv:1011.6478v1*, 2010.

D. Leão, A. Ohashi, and A. B. Simas. A weak version of path-dependent functional Itô calculus. *The Annals of Probability*, 46(6):3399–3441, 2018.

A. Ohashi and F. Russo. Rough paths, Skorohod integrals driven by covariance singular Gaussian processes. *In preparation*, 2022.

A. Ohashi, F. Russo, and A. Teixeira. SDEs for Bessel processes in low dimension and path-dependent extensions. 2020a. Preprint.

A. Ohashi, F. Russo, and A. Teixeira. On path-dependent SDEs involving distributional drifts. 2020b. Modern Stochastics: Theory and Applications, to appear. Preprint Hal-02465590.

A. Ohashi, Gomes A.O., F. Russo, and A. Teixeira. Rough paths and regularization. *Journal of Stochastic Analysis (JOSA).*, 2 (4):1–21, 12 2021. doi: 10.31390/josa.2.4.01.

F. Russo and P. Vallois. Intégrales progressive, rétrograde et symétrique de processus non adaptés. *C. R. Acad. Sci. Paris Sér. I Math.*, 312(8):615–618, 1991.

F. Russo and P. Vallois. Forward, backward and symmetric stochastic integration. *Probab. Theory Related Fields*, 97(3): 403–421, 1993.

F. Russo and P. Vallois. The generalized covariation process and Itô formula. *Stochastic Process. Appl.*, 59(1):81–104, 1995.

Congratulations Marta!