The paradidactic: a theory of noospheres

Koji Otaki

Hokkaido University of Education, Japan

> Your readers have seen the sky with one moon in it any number of times, right? But I doubt they've seen a sky with two moons in it side by side. When you introduce things that most readers have never seen before into a piece of fiction, you have to describe them with as much precision and in as much detail as possible.
>
> Haruki Murakami, *1Q84*.

1. Introduction: dualizing the didactic

This paper—which is personal to some extent[1], even though that is supposed to be prepared for a plenary talk in the 7th international conference on the ATD, i.e. the *anthropological theory of the didactic* (CITAD 7: June 19–23, 2022; Barcelona, Spain)—describes a specialized or applied theory of the ATD. So, I would skip the definitions and/or examples of many basic concepts of the ATD. Please have Chevallard with Bosch (2020) by your side, which is a glossary of the ATD[2], when you are not familiar with the ATD. The information for references related to such basics is only put in the cases that given notions are not included in the glossary. In the first appearances of traditional technical terms in the following sentences, they will be written by the bold italic style like ***ATD***. And then, newly created ones in the oncoming theory will be informed by the underlined italic style like <u>*ATD*</u>. In addition, I am going to use mathematical, formal notations without explanation, within ATD's tradition of the modeling of didactic reality. Please do not escape from them even though you do not like mathematics (even hate it). They are not as difficult as they look, and actually very useful for *de-subjectifying* or *re-objectifying* different portions of didactic reality.

Let me begin to get down to business with an extension of an analogy in the ATD, although historical epistemology tells us that the abuse of familiar images is a major category of obstacles to the progression of science. That is for emphasizing the <u>*dualization of didactic reality*</u> which the

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[1] I use in this paper the adjective "personal" and the adverb "personally" for meaning that a given thing or action is *under-institutionalized*.

[2] You can get it freely from the following URL, which is an initial portion of an ATD book: https://www.researchgate.net/publication/338109032_Working_with_the_Anthropological_Theory_of_the_Didactic_in_Mathematics_Education_A_Comprehensive_Casebook. Thanks for sharing it!

ATD—or more precisely its subtheory related to the phenomenon of ***didactic transposition***—established. I want to focus here on the notion of ***didactic continent*** (cf. Chevallard, 2019a). The didactic continent is a reality constituted with two kinds of entities, which are our ***object*** of study and our research. Such twofold nature seems to be represented by the dual meaning of the adjective "didactic", which can indicate ***the didactic*** (didactic reality) and ***didactics*** ("didactological" reality, so to speak), the same as other disciplinary fields like mathematics, epistemology, and so on[3]. But, please beware. This is *not* the duality which I want to discuss here.

The usage of the word "continent" is a metaphor based on the notion of *space*. Any didactician, whether sedentary or gyrovague, "lives" in the continent. Let me add here the aspect of *time* to this analogy. The didactic continent has a *couple of two time periods* like day and night. One is the period of the ***school***, and the other is of the ***noosphere***. For instance, sometimes we study *actually taught* mathematics in classrooms, and other times we study mathematics *to be taught* on textbooks. In my view, the finding out of the phenomenon of didactic transposition indeed has been one of the most remarkable events in the history of didactics, but more important point in the event has been to identify the existence of the ***institution*** of noosphere which should be studied in didactics. Nevertheless, the didactic investigation into the noosphere seems to be *relatively* restricted so far. Please note that the word "restricted" here should be comprehended together with a metatheoretical principle: *doing it does not mean knowing about it*. The rest of this paper gives a theorization of the noospheric aspect of the didactic continent, which metaphorically speaking is the night side where it is difficult to see things due to darkness. Of course, there may exist entities of which clear demarcation is inappropriate like twilight.

2. From noospheres to paradidactic systems

In this section, basic notions for didactic research on noospheres are given and explained. I think that some of them are pragmatically helpful for studying phenomena in noospheres, and the others are fundamentally useful for evoking some readers' epistemological ***break*** in didactics. (Please remember here that a break together with the emergence of didactics has been *to regard the teacher as its object of study*).

A formal definition of the word "noosphere"—roughly speaking, people who are interested in

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[3] In my understanding, in the case of Spanish, the former is expressed by *lo didáctico* with the neuter definite article and the male noun, while the latter is by *la didáctica* with the female definite article and noun. The case of French is not so different as the Spanish case except for that the male definite article is used for the former: *le didactique* and *la didacticque*. This kind of linguistic facts is *not* trivial for Japanese! We do not only lack the concept of grammatical gender, but also have weaker ideas about articles which usually disappear in statements.

education—is a possible stating point of this theoretical section, according to a logical manner. However, I want to begin with short historical description. It is because of my belief that an unfamiliar aspect of legacy of the *didactic transposition **theory*** can be emphasized by some historical facts, by which readers can be easy to find out some relationship between oncoming notions and your research. There have been two epochs in the emergence of the noosphere as a theme of didactics. The first has already been said in the introduction, that is, the "invention" of noospheres by Yves Chevallard (note that this usage of word *invention* is under a metatheoretical principle: *theories construct objects*). This impact is well-summarized in a ***work*** of the historical epistemology of didactics by Mariana Bosch and Josep Gascón, "twenty-five years of the didactic transposition" (Bosch & Gascón, 2006). That has already been accepted as a quite arresting affair, an epistemological break in didactics. By contrast, the second epoch seemed to attract relatively small attention, that is, the invention of *paradidactic reality* by Carl Winsløw. In my view, that can also be an epistemological break in, or toward, didactics, but such a break is still hindered by some epistemological obstacles, especially the (epistemological) *illusion of transparency*. The adjective ***paradidactic*** has been used in Winsløw (2012) for studying schoolteachers' didactic activities out of their own lessons, that is, designing and analyzing them. The relationship between the idea of paradidactic reality and the concept of noosphere will be gradually revealed in the rest of this section. Please note only at this point that the relationship is similar to that of the idea of didactic reality and the concept of school. Moreover, I want to add four very short remarks. The first is that Gascón (2003) had already used the same adjective with another meaning in the historical epistemology of didactics. Secondly, the "concept in action"—a notion in the *theory of conceptual fields* by Gérard Vergnaud—of paradidactic activity had already existed in Miyakawa & Winsløw (2009) at least which is comparative study of French *didactic engineering* and Japanese *lesson study*—where the ATD is also implicit, but a typical transpositive analysis of proportionality can be seen as well. Thirdly, the adjective "paradidactic" seems to be an application of the relatively lesser-known typology of "mathematical" objects in the didactic transposition theory: ***mathematical***, ***paramathematical***, and ***protomathematical*** (Chevallard, 1985/1991). Finally, the previous findings by the ATD includes *implicitly* paradidactic phenomena which are usually not explicitly identified as paradidactic ones: e.g., ***thematic confinement*** (cf. Barbé, Bosch, Espinoza, & Gascón, 2005); ***pedagogical generalism*** (cf. Florensa, Bosch, Cuadros, & Gascón, 2018); and the ***paradigm of questioning the world***. The reason why they are paradidactic will be gradually clarified from now on. As you know, the last one has the most influential for our research. The discovery of it leads us a completely new kind of didactics, which I personally call *non-Brousseaunian didactics*. Within aforesaid research milieu, Winsløw has opened up a new path of research on noospheres. The productive history after that can easily be

followed through chasing works by himself and his colleagues. However, in my view, although such great progression by the idea of paradidactic reality, the paradidactic perspective has not been able to fully develop its theoretical and epistemological potential yet. The rest of this paper is a small contribution for giving it a supportive push forward.

I want to enter from now on into theorization of noospheres. Let us start with a basic model of fundamental systems to be studied in didactics, that is, ***didactic systems*** which are social systems of a particular type through which some ***instance***—i.e. ***person***, institution, or (institutional) ***position***—teaches something to some instance. The notion of didactic system has quite broader meaning within the framework of the ATD, but it is firstly enough that you imagine ordinary lessons at school, for introducing the model denoted by $S(X, Y, \heartsuit)$. In the notation, X is a group of students, Y is a group of teachers, and $\heartsuit$ is a ***didactic stake***. In the case of ordinary lessons, their didactic systems are described as $S(X, y, w)$, where Y is a singleton and $\heartsuit$ is a *small **work** w* like a mathematical notion, technique, or theorem, which is also called a ***piece of knowledge*** (the antonym of it is the ***body of knowledge***) or metaphorically a ***monument*** (cf. Bosch & Gascón, 2006; Chevallard, 2015). Once we recognize lessons as didactic systems of a particular type, and thereby become able to identify other types of them. For example, doing homework is modeled by $S(x, \emptyset, w)$, and the opportunity where a student x_1 teaches w to another student x_2 is $S(x_2, x_1, w)$. In other cases, didactic stakes $\heartsuit$ are not small works w, but large works $\mathcal{W}$ usually called theories, books $\mathcal{B}$ as masterworks, great authors themselves $\mathcal{G}$, even ***questions*** q, whether the study of q is ***finalized*** or ***unfinalized*** (cf. Bosch, 2019; Chevallard, 2022a).

The notion of didactic system can possess a wide variety of its examples, based on the ***anthropological principle*** of the ATD (cf. Bosch, Chevallard, García, & Monhagan, 2019): (we recognize, at least methodologically) *everything is **possibly** didactic* because we humans can be regarded as *inherently didactic species*, that is, *Homo discens* (learning man) as well as *Homo docens* (teaching man). According to this principle, the teacher's activity for preparing and reflecting on her own lessons also is supposed to be comprehended as behavior in some type of didactic systems. Indeed, the teacher can *learn*—even *teach* to her colleagues—in there *something about teaching*. Didactic system of such a type are called ***paradidactic systems*** (Winsløw, 2012). Let me tentatively model the paradidactic system by $S(\dot{X}, \dot{Y}, \clubsuit)$, where $\dot{X}$ (X with dot above) and $\dot{Y}$ are <u>*teaching students*</u> and <u>*teaching teachers*</u> respectively, and $\clubsuit$ is a <u>*paradidactic stake*</u> (please remember that X and Y can be *mathematics* students and *mathematics* teachers respectively, and $\heartsuit$ is a *didactic* stake). At this point, this model is mere *formally constructed* without meaning. Starting now, I want to fill this form with contents. However, before that, note that this model is different from one in Winsløw (2012), which more focus on the so-called *milieu*, but this

difference is neither problematic nor contradictive. Generally speaking, models of systems to be studied are constructed and selected based on certain research questions and their theoretical frameworks.

Let us consider paradidactic systems $S(\dot{X}, \dot{Y}, \clubsuit)$ based on a typical example in the definition of Winsløw (2012), *lesson study* (*jugyō kenkyū*) which is Japanese schoolteachers' methodology for designing and reflecting on their own lessons. In a "simple" format of lesson study, $\dot{X}$ is a team $\{\dot{x}_1, \dot{x}_2, \dot{x}_3\}$ of teachers in a school attached with a university, and $\dot{Y}$ is a teacher educator $\dot{y}$ in the university. And then, $\clubsuit$ is a lesson which can be modeled as a didactic system $S(X, \dot{x}_1, w)$ where the didactic stake is a small work to be studied by X, and the teacher is a member $\dot{x}_1$ of $\{\dot{x}_1, \dot{x}_2, \dot{x}_3\}$. Such paradidactic system is denote as $S(\{\dot{x}_1, \dot{x}_2, \dot{x}_3\}, \dot{y}, S(X, \dot{x}_1, w))$. This should be easily schematized to an universal format $S(\dot{X}, \dot{Y}, S(X, Y, \heartsuit))$. Once such formalization is achieved, we can describe different types of paradidactic systems: $S(\dot{x}, \emptyset, S(X, \dot{x}, w))$; $S(\{\dot{x}_1, \dot{x}_2\}, \dot{y}, S(X, \dot{y}, w))$; $S(\dot{X}, \dot{Y}, S(X, y, q))$; and so on. I introduce here a particular type of paradidactic systems by prospective teachers, which is modeled as $S(\{\dot{x}_1, \dot{x}_2, \dot{x}_3, ...\}, \dot{y}, S(\{\dot{x}_2, \dot{x}_3, ...\}, \dot{x}_1, w))$. In there, under supervision by a teacher educator $\dot{y}$, the prospective teachers $\{\dot{x}_1, \dot{x}_2, \dot{x}_3, ...\}$ design a lesson, fictively realize it by themselves acting as students, and discuss the lesson after that. In Japan, the lesson of such type is called the lesson simulation (*mogi jugyō*).

The formalization of the model of paradidactic systems also implies the possibility of generalization of the *profession* of possible "stakeholders" in there. As a preparation for explaining that, I want to define the notion of a noosphere. In the ATD, the word "noosphere" basically means *didactic* noosphere, even though it can be more general, that is, people who think about some "job", i.e. *métier* in French, like cooking, defensing in court, researching, and so on (cf. Chevallard, 2013). Then, roughly speaking again, a noosphere is people who think about *teaching*. A more precise definition can be created in the fundamental subtheory of the ATD, the ***theory of instantial relations***. In the theory, knowledge of an instance $\hat{i}$ about an object o is defined as the ***relation*** of $\hat{i}$ to o, which is denoted by $R(\hat{i}, o)$. The word "relation" has very broad referents in the ATD, and thereby indicates the bundle of all the *properties* or *predicates* of o from the $\hat{i}$'s point of view. Within this framework, given a ***school system*** Σ—where ***school establishments*** σ belong, i.e., $\sigma \in \Sigma$ (cf. Chevallard, 2019b)—, the noosphere $\dot{N}$ on Σ is defined by the following way: $\dot{N} \overset{\text{def}}{=} \{x \mid R(x, \Sigma) \neq \emptyset\}$. This definition of noospheres implies that any noosphere is dependent on its targeted school Σ, that is to say, $\dot{N}$ should be rigorously written like $\dot{N}_\Sigma$. In addition, I want to emphasize that this definition intentionally omits the variable of ***society*** (in a very broader sense including ***civilization*** and ***humankind***) where Σ is located, denoted by $\check{S}_\Sigma$. From this point of view, the definition can be elaborated with: $\dot{N}_\Sigma \overset{\text{def}}{=} \{x \in \check{S}_\Sigma \mid R(x, \Sigma) \neq \emptyset\}$. However, for

the sake of brevity, I will basically make the marks of "Σ" of "$\dot{N}_\Sigma$" and "$\check{S}_\Sigma$" implicit.

A *noospheric profession* $\dot{P}$ is any *subinstitution* of a given noosphere defined together with a ternary relation denoted by $\mathfrak{S}(x, \dot{N}, \dot{p})$, which indicates that a person x is **subjected** to a determined noosphere $\dot{N}$ in a certain position $\dot{p}$ in $\dot{N}$: $\dot{P} \overset{\text{def}}{=} \{x \mid \mathfrak{S}(x, \dot{N}, \dot{p})\}$. And then, we can instantly generalize this definition to an *institutional profession P* around an institutional position (I, p): $P \overset{\text{def}}{=} \{x \mid \mathfrak{S}(x, I, p)\}$. A representative of noospheric professions $\dot{P}$ is the *profession of schoolteacher* denoted by $\dot{P}_t$, which is the set of persons subjected to $\dot{N}$ in the *position of schoolteacher* denoted by $\dot{p}_t$, and usually called the *teaching profession*. Let me emphasize here that in this paper the profession of schoolteacher is different from the *position of teacher* denoted by p_t together with the *profession of teacher* denoted by P_t (= Y), which indicates a *school profession*. (Please remember the **Humpty Dumpty principle**; cf. Bosch et al., 2019.) For instance, a person x who occupies the *position of schooler* $\dot{p}_s$ in $\dot{N}$, and thereby subjected to the *profession of schooler* $\dot{P}_s$ $\subset \dot{N}$, can belong tentatively to the teacher profession P_t in a **school class**, i.e. x teaches something to her friends, even though x is mainly supposed to act as in the *student profession* P_s (= X). Another remarkable noospheric profession is of the *curriculum-developer* denoted by $\dot{P}_c$. In addition, we can identify the institution of *scholar of knowledge to be taught* as a noospheric profession. In the case of mathematics education, the major of such a *noospheric* profession denoted by $\dot{P}_{\check{s}}$ comes from the *social* profession of mathematician. For the sake of simplicity, from now on, the word "profession" basically means the *noospheric* profession, even though an institutional profession in this theory is a very general notion which can be defined around any institutional position.

Let me here define *the intendedly paradidactic*. The intendedly paradidactic $\mathbb{I}$ is defined as $\{(\hat{i}, \pi, w, \hat{j}) \mid \mathfrak{T}(\hat{i}, \pi, w, \hat{j})\}$, where the quaternary relation $\mathfrak{T}(\hat{i}, \pi, w, \hat{j})$—I call it *transposing relation*—means that an instance $\hat{i}$ does some **gesture** π with the intention of helping an instance $\hat{j}$ to teach some work w to someone—such a gesture is a *intendedly paradidactic gesture*. Moreover, we can also find gestures *without* such intention, but they have potential for helping the teaching of something. We call such gestures the *unintendedly paradidactic gesture*. I define here *the paradidactic* $\mathbb{P}$ by refining the definition of the intendedly paradidactic, together with a quaternary relation $\mathfrak{T}'(\hat{i}, \pi', w, \hat{j})$, an instance $\hat{i}$ does some gesture π' with the *possibility* of helping an instance $\hat{j}$ to teach some work w: $\mathbb{P} \overset{\text{def}}{=} \{(\hat{i}, \pi', w, \hat{j}) \mid \mathfrak{T}'(\hat{i}, \pi', w, \hat{j})\}$. Any quadruplet $(\hat{i}, \pi', w, \hat{j})$ is called a *paradidactic fact* denoted by $\mathbb{p}$. This definition of the paradidactic implies that the object of study of paradidactic research is the *possibly* paradidactic.

A crucial entity which bridges the distance between didactic reality and paradidactic reality is the

cognitive nucleus denoted by $\tilde{n}$. In the framework of the ATD, it is defined as the quintuple of ($\hat{\imath}$, o, I, p, $\hat{v}$), by which *systemic objects* in a given *cognitive universe* can be indicated. In there, $\hat{\imath}$ is a studying instance, o is an object studied by $\hat{\imath}$, (I, p) is an institutional position targeted by $\hat{\imath}$, and $\hat{v}$ is an evaluating instance for $R(\hat{\imath}, o)$. Any cognitive nucleus $\tilde{n}$ is divided into two subsystems, that is the **cognitive base** $\bar{n} = (\hat{\imath}, o)$ and the **cognitive reference frame** $\underline{n} = (I, p, \hat{v})$. I want to emphasize here that such cognitive nuclei usually means <u>school nuclei</u> denoted by $\tilde{n}_\Sigma$, even though in principle it has very general meaning as is usual with in the ATD. In the case of $\tilde{n}_\Sigma$, its **nuclear fusion**—$\bar{n}_\Sigma \frown \underline{n}_\Sigma \to \tilde{n}_\Sigma$—is the entrance into school, and its **nuclear fission**—$\tilde{n}_\Sigma \to \bar{n}_\Sigma \frown \underline{n}_\Sigma$—is the graduation or giving up (cf. Chevallard, 2020). This kind of analysis implies that there are <u>noospheric nuclei</u> denoted by $\tilde{n}_N$. For clarifying that, I slightly replace the four letters $\hat{\imath}$, (I, p), and $\hat{v}$ with $\hat{x}$, P_s, and $\dot{P}$ respectively, and thereby make explicit that $\hat{\imath}$ is a studying instance in a school profession of student P_s of o, and $\dot{P}$ is a noospheric profession of evaluator which **judges** the **conformity** of $R(\hat{x}, o)$ and $R_\Sigma(p_s, o)$: $\tilde{n}_\Sigma \stackrel{\text{def}}{=} (\hat{x}, o, P_s, \dot{P})$. I define the noospheric nucleus by another quadruple: $\tilde{n}_N \stackrel{\text{def}}{=} (\hat{y}, \tilde{n}_\Sigma, \dot{P}_d, \check{P})$. In there, $\hat{y}$ is a *teaching instance*, $\tilde{n}_\Sigma$ is a school nucleus operated by $\hat{y}$, $\dot{P}_d$ is a schoolteacher profession which submits $\hat{y}$, and $\check{P}$ is a production institution of knowledge of $\tilde{n}_\Sigma$ like the teaching profession itself, mathematicians, didacticians, and so on. In the quadruple of $\tilde{n}_N$, the part of $(\hat{y}, \tilde{n}_\Sigma)$ is the <u>noospheric base</u> $\bar{n}_N$, and the rest of $(\dot{P}_t, \check{P})$ is the <u>noospheric reference frame</u> $\underline{n}_N$. Note that on the one hand the school nucleus has a noospheric object of the evaluating instance $\dot{P}$, and on the other hand the noospheric nucleus includes a school object of the teaching instance $\hat{y}$. Such hybridity is based on the dialectical interplay between didactic reality and paradidactic reality. However, the noospheric nucleus indeed is an entity in the cognitive universe of the noosphere $\dot{N}$, $\Omega(\dot{N})$, and the school nucleus belongs to the cognitive universe of the school Σ, $\Omega(\Sigma)$.

3. Paradidactic analysis

This section will be devoted to propose a kind of analysis for paradidactic research. For that, the current Japanese paradidactic reality is analyzed. Such analysis could be characterized as a didactic variation of the so-called *psychoanalysis*. I use here the word "psychoanalysis" being inspired by the application of it to epistemology by Gaston Bachelard rather than the original psychological usage of it. It is the *psychoanalysis of knowledge*, which analyzes relatively rational but implicit parts of institutional relations. Especially, the psychoanalysis of knowledge of noospheres focuses on *didactic knowledge*. Note that a crucial part of didactic knowledge is *disciplinary knowledge* at stake like mathematical knowledge. In my view, a major theme of

paradidactic analysis is the didactic transposition of disciplinary knowledge. That is, the **transpositive analysis** (cf. Gascón & Nicolás, 2019a) can be regarded as the _paradidactic analysis of first kind_. Then, I want to develop the _second kind_ which more focuses on didactic knowledge _in a narrow sense_.

Different epistemological properties or _dispositions_ could also be found out in the noospheric relation. Let us begin with a particular type of paradidactic systems $\dot{S} = S(\dot{x}, \dot{Y}, S(X, \dot{x}, \heartsuit))$, in which a teacher $\dot{x}$ ($= y$) prepares and reflects on her responsible didactic system $S(X, \dot{x}, \heartsuit)$ together with supervisors $\dot{Y}$ who are often a heterogeneous set of facilitators, colleagues, and "virtual" or "imaginary" charismas encountered on books and articles (For instance, we can virtually "meet" Jean Piaget through his books). Especially, I want to focus on the case of lesson study. In my personal observation of discussion in such systems, $\dot{x}$ and $\dot{Y}$ frequently refers to a criteria, which can be summarized as the following: _whether or not predetermined answers or keywords are said by students_? At first glance, this similar to the principle of "constructivism" (note that I use this term as a meaning shared by the TDS, i.e. _theory of didactic situations_, and the ATD, which is not limited to the American tradition of didactics). However, the word "said" does not means "constructed". Indeed, preservice teachers in their teaching practices tend to regard answers and keywords to be said by students as "taboo words" for the teachers. As results, the teachers give hints to their students and the students take hints for the teacher—that is exactly the so-called _Topaze effect_[4] (cf. Brousseau, 1997). The Topaze effect is usually related to _didactic contracts_. This _answering taboo_ certainly affects a clause in the ordinary didactic contract: answering questions is not a role of the teacher position. Let us proceed this analysis to the basis of this taboo. An important point is that said things are perceptual or **ostensive** (cf. Arzarello, Bosch, Gascón, & Sabena, 2008). As all of us deeply understand that, it is quite difficult to observe the construction of knowledge. Then, teachers prefer to rely on ostensive evidences about students' learning which are easy to be gathered. This could be a paradidactic variation of the so-called _positivism_, which can especially be compatible with modern bureaucratized school systems.

The aforementioned dispositions are specifically related to the profession of schoolteacher. However, the notion of paradidactic system is not restricted to it. I want to add in our scope the profession of curriculum developer. In that case, various documents of noospheric discourses are

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[4] The Topaze effect was identified by Guy Brousseau in the setting of the TDS, together with an example from the teaching of French in the very initial scene of the _Topaze_. Let me give here a possible theorization of this phenomenon within the framework of the ATD. This is a type of changes of the nature of didactic systems $S(X, Y, \heartsuit)$, in which the stake changes from an originally intended work w to the didactic system itself. I want to express such a process with a looped arrow "$\hookrightarrow$", which indicates "to become", by: $S(X, Y, w) \hookrightarrow S(X, Y, S(X, Y, w))$. This means that the Topaze effect is the phenomenon of _paradidactization_ of didactic systems.

more convenient for analysis. When looking at them, we can find out in there similar epistemological habitus which Bachelard called them prescientific. A representative disposition of that is the *substantialism*, a symptom of which is the easy and instant *fabrication of illusory categories of objects*. In my view, this symptom can more frequently be observed in the discourse on humanity rather than in natural (pre)science which is Bachelard's field of research. In fact, substantialist ideas are also overflowing in educational discourses related to the **cognition** of the student: I call that the *cognitive substantialism*. Let me take an example from Japan, the notion of *mathematical thinking* (*sūgakurekina kangaekata*), which more or less has been set as a fundamental notion of the "official" didactic theory since the era of the so-called New Math movement. Kurosawa (2019) includes useful archives of different noospheric definitions of mathematical thinking, as well as by itself exemplifies the enthusiasm for substantiating the idea of mathematical thinking. One of the symptoms of the substantialist disposition is that it first indicates a type of objects, and then next is obsessed with categorizing its subtypes. I personally call that the *table complex*—after the French word *tableau* in Foucault's terminology—, which aspires to complete the (folk) taxonomy of didactic world. Let me give here an extreme example from a didactic theory of mathematical thinking by the Japanese mathematics educator Shigeo Katagiri, which is one of the most diffusing theories in the Japanese noosphere of school mathematics education. His theory has 24 subtypes (!) of mathematical thinking (cf. Ishoda & Katagiri, 2012, pp. 51–53)

I examine another spontaneous prescientific disposition identified by Bachelard, which is to use uninhibitedly familiar images. An applicative and popular "material" in the noosphere is the *cyclic image of didactic time*. Many kinds of cyclic models of teaching and learning process can be found in paradidactic documents. There is probably no need to give examples. Such a cyclic image is also quite valued in Japan. In Japan, several cyclic models of mathematical modeling and statistical inquiry are popular now. The usage of the cyclic image deeply connects to the substantialist disposition through *cyclic diagrams* some of which is probably easy to imagine for readers. The diagramming of the time flow—whether cyclic, liner ,or not—is a spatial metaphor of it, which furtively replaces a sequence of **moments** with a *path*. As a result, different images about a changing process of **didactic moments** are "reified" as *didactic paths* for curricular developers and schoolteachers, and thereby become easier to be used for didactic design as a kind of *prescription*. Such reification leads them to what I call the *path apriorism*, which is the other side of the valuing of small works or *monuments* as didactic stakes, that is, the epistemological **monumentalism** (Chevallard, 2015). The belief of the path apriorism regards different formats of didactic process as crucial prescriptions for teaching. It appears in a form of the *Dienes effect*

reported in the TDS, which means the teachers' superficial imitation of preestablished *didactic organizations* and its malfunctional tendency[5]. Based on data from my personal observation, many preservice teachers passionately try to realize some variation of paths of "good" lessons in the Japanese format of problem-solving without taking given didactic stakes into account. I provisionally introduce here an antonym of the path apriorism, the <u>*path aposteriorism*</u> with a neologism. This point of view regards the flow of didactic time as the *process for answering questions at stake*, where any work which functions as the **answers** to the questions is given its **raison d'être**[6]. In such a case, the sequence of different moments of a certain process cannot be predetermined. As being represented in a famous poem of Antonio Machado, paths are made of footprints when walking (this poem is filled with sorrow, but I want to apply it cheerfully: *footprints of the inquirer are questions!*).

The aforesaid example of the Dienes effect is a particular type of it, which excessively focuses on "domain-free" pedagogy independent of didactic stakes. Such a paradidactic disposition has been traditionally the *pedagogical generalism* within the framework of the ATD. The paradidactic system with this tendency can be modeled in the nested model of paradidactic system as $S(\dot{X}, \dot{Y}, S(X, Y, \emptyset))$. Let me emphasize that this analysis implies that the **scale of didactic codeterminacy levels** is useful for identifying the kinds of paradidactic stakes. A version of it includes the different five sizes of didactic systems in detail: **disciplines**, **domains**, **sectors**, **themes**, and **subjects**. This typology is related to one more implicitly paradidactic phenomenon which has been found within the framework of the ATD. That is the phenomenon of *thematic confinement*. Schoolteachers often focus on didactic systems $S(X, Y, w)$ with relatively small pieces w of knowledge as didactic stakes for preparing their teaching without designing and analyzing didactic systems $S(X, Y, \mathcal{W})$ with large bodies of knowledge $\mathcal{W}$.

$$\ldots \rightleftarrows \text{Pedagogies} \rightleftarrows \text{Disciplines} \rightleftarrows \text{Domains} \rightleftarrows \text{Sectors} \rightleftarrows \text{Themes} \rightleftarrows \ldots$$

In my view, the paradigm is also a notion for psychoanalysis of knowledge. The word "paradigm" within the framework of the ATD indicates the **didactic paradigm**, **pedagogical paradigm**, **study**

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[5] By using of the notion of judgement, this phenomenon is modeled as: $Y \vdash S(X, Y, w) \approx S(X, Y, \emptyset)$.

[6] As far as I know, in the ATD (and maybe even in didactics), the notion of *raison d'être* ("reason for being" in a direct translation) has a semi-theorized status. So, I have done previously a small and rough historical research on this concept. Of course, I cannot precisely recognize its usual meaning and connotation in French, but maybe it has some philosophical "smell" of the so-called *existentialism*, which is originated from the Danish philosopher Søren Kierkegaard, and the French philosopher Jean-Paul Sartre has popularized. At least, in Japan, the concept of *raison d'être* is registered as a philosophical one in dictionaries, and usually explained that it was imported together with existentialism. *However*, I could not find any Japanese textbook of existentialism which includes this concept as an important technical term. In my poor research, I did not read any original work in existentialism, but I found out the Japanese expression of *raison d'être*, "sonzai-ryū", in a Japanese translation of Sartre's *La nausée*.

paradigm, or ***school paradigm*** (cf. Chevallard, 2015; Bosch, 2019; Chevallard & Bosch, 2020; Chevallard, 2021). I prefer the name of "didactic" paradigm for some reasons which will be gradually revealed. A didactic paradigm can be regarded more or less as any *normative* theory for the formulation and evaluation of a given ***curricular project***, which is any set of didactic stakes and didactic gestures prescribed for realizing a targeted ***curriculum***, that is, a ***curricular trail*** alongside a determined ***positional path***. Notwithstanding that, I want to redefine it here, being based on the framework of *didactic contract* in the TDS (cf. Brousseau, 1997), and on the generic definition of the notion of paradigm as a contract by the ATD (Chevallard, 2021; see also Strømskag & Chevallard, to appear). For shortening long story, I define a didactic contract as any *set of clauses in a given didactic system* (the analysis of didactic contract in my view is the psychoanalysis of didactic systems). A reason why I try to use the notion of didactic contract for defining didactic paradigms is because many previous research works about SRP, i.e. ***study and research paths*** (cf. Bosch, 2019), implicitly or explicitly discuss not only didactic paradigms but also didactic contracts. Indeed, the transition of paradigms inevitably brings about some changes of the contracts. In my personal experience of didactic engineering of SRP, the experimentation usually has begun with the establishment of several *study and research clauses*, e.g., about the availability of the Internet. Based on such interconnectedness, I want to redefine here a didactic paradigm as an *a priori didactic contract*, which is pertinent to the reality of knowledge to be taught. Note that a concept generally can be defined by different styles with no contradiction, nonetheless to remind you of examples from mathematics. This new definition of the notion of a didactic paradigm emphasizes that the entity of didactic paradigm is located within paradidactic reality. I will return to this point in the next section.

4. Toward the completion of the scale of didactic codeterminacy

I want to try to describe here the *raison d'être* of paradidactic analysis in didactics. As I have shortly mentioned before, paradidactic analysis could be set in the French tradition of epistemology (especially, epistemology of social science), even though it occupies a quite peripheral place. However, we are *not pure epistemologists but didacticians*. We are interested in epistemological problematic *only when* it is helpful for studying the possibly didactic. In my opinion, paradidactic analysis is a part of the so-called ***ecological analysis*** (cf. Chevallard, 2019a). For justifying this statement, I want to illustrate effects of paradidactic reality to didactic reality. As the first step of that, let us extend the scale of didactic codeterminacy levels, which originally includes the implicit interlevel of noospheres between the levels of societies and schools. In the

same way, I add the levels of professions and paradidactic systems as the inter-levels. This extended scale implies that there are three realities: _protodidactic reality_ (or _anthropological reality_) on humankind, civilizations and societies; _paradidactic reality_ on noospheres, professions, and paradidactic; and (narrowly) _didactic reality_ on schools, pedagogies, and didactic systems. This trichotomy is just an ecological paraphrase of the anthropological principle of the ATD. In my understanding, when the didactic transposition theory has evolved to the ATD, the "didactic" has _explicitly_ changed from twofold reality (school and noosphere) to threefold reality (school, noosphere, and society).

<div align="center">

Humankind ⇄ Civilizations ⇄ Societies

↓↑

Noospheres ⇄ Professions ⇄ Paradidactic systems

↓↑

Schools ⇄ Pedagogies ⇄ Didactic systems

</div>
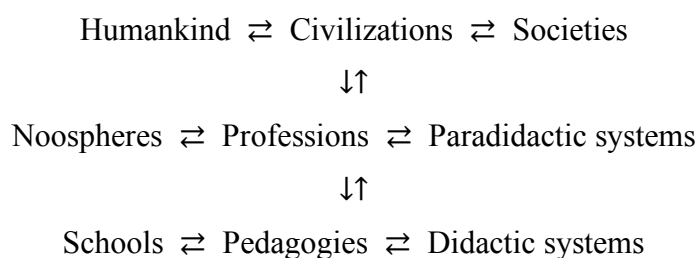
For clarifying the habitat of the didactic paradigms in this diagram, let me focus on here the word use of the expressions such as _pedagogical_ paradigm and _disciplinary_ paradigm (e.g., Gascón & Nicolás, 2021). In my personal observation, the adjectives "pedagogical" and "disciplinary" tend to be used as critical identifiers for the location of didactic paradigms in the codeterminacy levels. However, that is too simplistic and often lead us to confusion. In such cases, the adjectives do not express the ecological levels of paradigms, but the _stakes_ of the paradigms. Analogically, talking about cats does not means that we are cats. The same tendency maybe also observed in the ecological analyses related to the phenomena of _pedagogical_ generalism and _thematic_ confinement, and of knowledge to be taught described by curricular documents and textbooks. Didactic paradigms, pedagogical generalism, thematic confinement, and knowledge to be taught are not in didactic reality but in paradidactic reality, because they are the consequences or outcomes of paradidactic economy[7].

I want to progress further the completion of the scale. The scale of didactic codeterminacy levels can include more levels. In that case, different types of didactic stakes ♥ are used metonymically as _sublevels_ of didactic systems. Well-known types of ♥ are based on the _granularity of knowledge_, that is, aforesaid _discipline_, _domain_, _sector_, _theme_, and _subject_. They often explicitly appear in the scale of didactic codeterminacy levels. Such inclusion of the five sublevels of didactic systems

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[7] Another representative of paradidactic reality is the _assessment_ of different kinds. Artigue & Winsløw (2010), who study PISA and TIMSS, illustrate this fact. As far as I know, this paper is the very first occurrence—at least in English—of the explicit and systematic usage of the scale of didactic codeterminacy levels not for the ecological analysis of didactic reality but for the economic analysis of paradidactic reality.

seems to express an implicit stake of traditional didactic research under the paradigm of visiting works, which can be located at paradidactic reality. Indeed, the structure of works to study is often *naturalized* in the pedagogies within it. As a result, the profession of didactician has been constrained by it as well. Let me point out here one more kind of didactic stakes which is becoming important, the type of *question*. Within the paradigm of visiting works, questions are no less present. However, its status is lowest, that is to say, teachers *Y* bring questions into play for the teaching of works. In other word, the questions are determined by the logic of the works. Note that, precisely speaking, the didactic paradigm can be located at the level of noospheres, and then hypostatized as more "material" **conditions** at the levels of professions and paradidactic systems. For example, the image of inquiry of the profession of the curricular developer in Japan is expressed on the national curricular document of *course of study*. It includes a *spiral diagram* of inquiry process, which is a variation of cyclic diagram. Of course, it could in turn become a topic of paradidactic systems in the profession of the schoolteacher. (Note: from now one, I use the expression [*italic words*] not for adding new levels into the scale, but for describing any specific condition on the scale.)
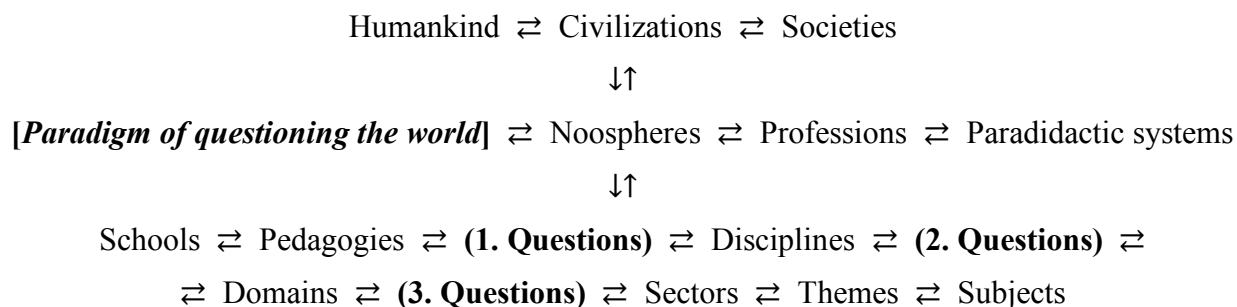
$$\text{Humankind} \rightleftarrows \text{Civilizations} \rightleftarrows \text{Societies}$$
$$\downarrow\uparrow$$
$$[\textbf{\textit{Paradigm of visiting works}}] \rightleftarrows \text{Noospheres} \rightleftarrows \text{Professions} \rightleftarrows \text{Paradidactic systems}$$
$$\downarrow\uparrow$$
$$\text{Schools} \rightleftarrows \text{Pedagogies} \rightleftarrows \text{Disciplines} \rightleftarrows \dots \rightleftarrows \text{Subjects} \rightleftarrows \textbf{Questions}$$

By contrast, the status of questions within the paradigm of questioning the world is very high. This is expressed by the **Herbartian schema** which defines the notion of an inquiry within the framework of the ATD: $S(X, Y, q) \rightarrowtail a^{\heartsuit}$. (As an application of this, the traditional didactic process can be described as: $S(X, Y, w) \rightarrowtail R(X, w)$.) In the schema, the didactic stake changes from a work to a question. This means that in principle questions are prioritized rather than works of different sizes. In other words, works to study in a given didactic system are selected whether they could be helpful for the study of an initial question or not. In this case, the level of questions is located at the inter-level between the levels of pedagogies and disciplines. However, such radical inquiry seems to be relatively rare. In reality, the study of questions is often finalized more or less. There are different *degrees of finality of the inquiry into questions*. I want to distinguish three degrees for a certain convenience. The aforementioned ecological location of questions expresses the finalizing of inquiry into questions at the first degree. Such inquiry is *unfinalized SRP*. Didactic systems within the second, and third degrees of finality studies questions under a predetermined discipline or domain respectively. These kinds of SRP, ***finalized SRP*** (cf. Bosch,

2019), are often observed in Ph.D. projects. In there, relatively higher degrees of specialty are demanded.

Humankind ⇄ Civilizations ⇄ Societies

↓↑

[*Paradigm of questioning the world*] ⇄ Noospheres ⇄ Professions ⇄ Paradidactic systems

↓↑

Schools ⇄ Pedagogies ⇄ (**1. Questions**) ⇄ Disciplines ⇄ (**2. Questions**) ⇄

⇄ Domains ⇄ (**3. Questions**) ⇄ Sectors ⇄ Themes ⇄ Subjects

From this point of view, we can define three additional degrees of finality *under the paradigm of visiting works* as the following diagram. Finding out questions for realizing the didactic process of the fourth, fifth and sixth degrees of finality of study is a problematic of traditional didactics or *Brousseaunian didactics*. In the case of TDS, such a process is called the *fundamental situation* (cf. Brousseau, 1997). And then, in the ATD, these degrees of finalization of inquiry is handled by the notion of **SRA**, i.e. the **study and research activity** (cf. Chevallard, 2022a).

Humankind ⇄ Civilizations ⇄ Societies

↓↑

[*Paradigm of visiting works*] ⇄ Noospheres ⇄ Professions ⇄ Paradidactic systems

↓↑

Schools ⇄ Pedagogies ⇄ Disciplines ⇄ Domains ⇄ Sectors ⇄

⇄ (**4. Questions**) ⇄ Themes ⇄ (**5. Questions**) ⇄ Subjects ⇄ (**6. Questions**)
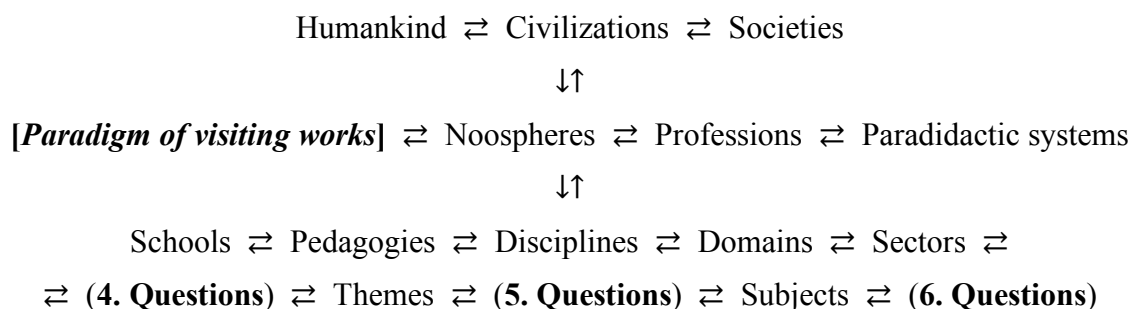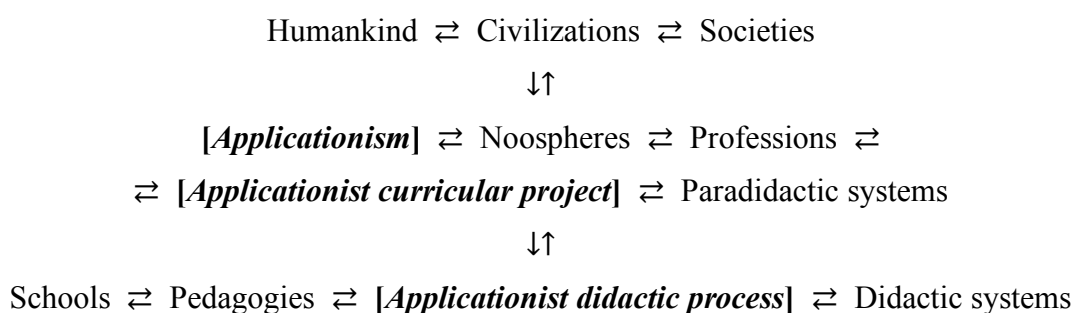
In my view, the aforesaid <u>*sensitivity of questions*</u> in didactic ecosystems is a remarkable phenomenon. The changing status of questions is conditioned by various factors. Major one of them is the variation of a given paradigm—e.g., from a trivial to a radical—, that is, the ***variant of the paradigm*** (Chevallard, 2021). However, that is not everything. Let me emphasize here that different degrees of finality of the study of questions are often imposed by **constraints** rather than intentional products. Especially in the current transitional period, the **infrastructural** *legacy* of the paradigm of visiting works could hinder the **superstructural** *dynamics* of SRP. Such hinderance can occur at the dimension of **didactic infrastructure** like the structure of didactic time and the arrangement of didactic takes to be taught, as well as the one of ***paradidactic infrastructure*** like different professional resources for the schoolteacher (cf. Winsløw, 2012; Miyakawa & Winsløw, 2013; Miyakawa & Winsløw, 2019).

About the paradidactic infrastructure, let me indicate a possible constraint on the diffusion of the
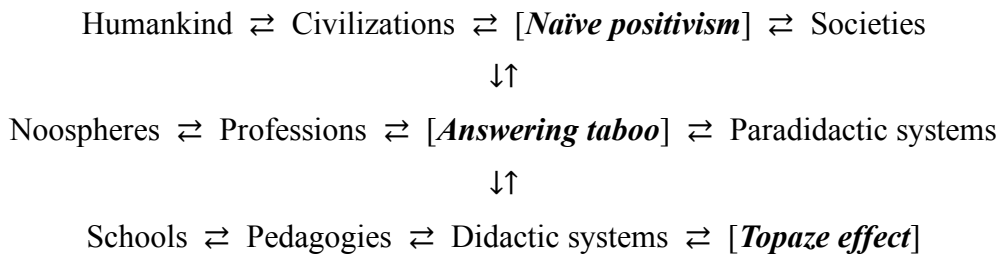
unfinalized inquiry. That is lesson study which is usually regarded as a "favorable" condition for the professional development of schoolteachers. An important part of paradidactic infrastructure for lesson study is some shared "didactic" theory in the profession of schoolteacher. In the case of current Japan, it is the *structured problem-solving approach* (cf. Asami-Johansson, 2022). This theory focuses on designing one lesson, that is to say, its unit of design is each of didactic systems $S(X, Y, \varpi)$ with small pieces ϖ of knowledge to be taught, which are predetermined by the profession of curriculum developer. This is clearly an adaptation to the paradigm of visiting works. From this point of view, the unfinalized SRP is nothing but ***isodidactic*** endeavor (probably even ***antididactic***). The adjective "structured" seems to represent so far a praise for a Japanese format of lessons within the paradigm of visiting works, but after the transition, it could become a sign of old hat.

Let me pick up another important finding in ATD's implicit paradidactic analysis which is closely related to the paradigm of visiting works, that is, the ***applicationism*** (Barquero, Bosch, & Gascón, 2013). This paradidactic disposition regards didactic process as the order of logic. For example, applicationists think that the notion of probability has to be introduced after the teaching of proportion. However, as a general rule, logical structures of well-organized theories are the complete opposite to the historical orders of them. The applicationism is hypostatized as *mathematical knowledge to be taught* as well as *taught mathematical knowledge*. In my view, the *theory of fundamental situations* in the TDS is a representative of *antiapplicationist* arguments. In fact, the abovementioned example of the notion of probability has come from a famous didactic engineering (cf. Brousseau, Brousseau, & Warfield, 2002).

Humankind $\rightleftarrows$ Civilizations $\rightleftarrows$ Societies

$\downarrow\uparrow$

[*Applicationism*] $\rightleftarrows$ Noospheres $\rightleftarrows$ Professions $\rightleftarrows$

$\rightleftarrows$ [*Applicationist curricular project*] $\rightleftarrows$ Paradidactic systems

$\downarrow\uparrow$

Schools $\rightleftarrows$ Pedagogies $\rightleftarrows$ [*Applicationist didactic process*] $\rightleftarrows$ Didactic systems

Didactic paradigms and their correlatives are not only paradidactic entities which affect didactic reality. We have already looked at in slightly implicit ways some effects of paradidactic dispositions to the functioning of didactic systems, focusing on the professions of schoolteacher and curriculum developer. The cases of the Topaze effect and the Dienes effect are relatively explicit. Let me illustrate the aforesaid ecological description of the former effect within the scale of didactic codeterminacy levels. Note that in general a condition for a phenomenon is not the

cause of it, but a part of the **ecology** of the phenomenon. The ecological analysis in this paper is also not an exception.

$$\text{Humankind} \rightleftarrows \text{Civilizations} \rightleftarrows [\textbf{\textit{Naïve positivism}}] \rightleftarrows \text{Societies}$$
$$\downarrow\uparrow$$
$$\text{Noospheres} \rightleftarrows \text{Professions} \rightleftarrows [\textbf{\textit{Answering taboo}}] \rightleftarrows \text{Paradidactic systems}$$
$$\downarrow\uparrow$$
$$\text{Schools} \rightleftarrows \text{Pedagogies} \rightleftarrows \text{Didactic systems} \rightleftarrows [\textbf{\textit{Topaze effect}}]$$

5. The complexity of teacher education

Teacher education is becoming a popular domain of didactics more and more. In the case of the didactics of mathematics, it has even a specialized journal, the *Journal of Mathematics Teacher Education* published by Springer. By the way, from what point of view can the object of *teacher education* be included in the object of study in didactics? Roughly speaking, didactics is a science of the diffusion of knowledge. Indeed, teacher education is supposed to diffuse knowledge of different types necessary for the diffusion of knowledge. Thus, it can be clearly a domain of didactics. However, the knowledge diffused in teacher education is quite complex rather than ordinary education at school. Some questions can be asked within the framework of the ATD.

The first question is the following: what kind of **institutional transposition** of knowledge is happened for teacher education? For simplifying discussion, I want to focus on the teaching of subject matters as the *mission of the teacher*, even though in reality, teachers could have many other responsible **types of tasks** (the fuzziness of the working responsibility is related to the teaching profession as a **semiprofession**; cf. Chevallard, 2022b). And for the same reason, teacher education in *well-institutionalized paradidactic infrastructure* for initiating prospective teachers into the teaching profession like teacher training courses in universities will only be taken into account. I want to label such an infrastructure the slightly metonymic name of <u>normal school</u>. At a glance, the institutional transposition for teacher education seems to be the *didactic* transposition, because such transposition has explicit intentions for the teaching of something. By contrast, when we recognize that normal school is a subinstitution of the teaching profession for preparing the teaching professional, it can also be regarded as a more primitive type of transpositions, which is similar with the transposition of mathematics from the mathematicians' institution to other scientific institutions. Such transposition has been called the **archididactic transposition**, the name of which implies that any institution can function as a kind of school for the teaching of

exogenetic knowledge relative to the institution (cf. Artaud & Bourgade, 2022). In fact, normal schools organize didactic systems of exogenetic knowledge like mathematics, psychology, foreign language, law and so on. However, that is not everything. There is also the institutional transposition of **endogenetic** knowledge, that is, the teachers' didactic knowledge. This seems to be quite similar with what we call the didactic transposition, but a small difference still exists about the nature of original institutions. The so-called "didactic" transposition, implicitly focusing on school education for children, seems to be the institutional transposition of endogenetic knowledge from a given society, civilization, even humankind to its didactic subinstitution "school" in a narrow sense. That is to say, such school is not professionalized for knowledge to be transposed. By contrast, the didactic transposition for professionals like the schoolteacher, engineer, and the mathematician has a determined profession. Then, I name this particular type of didactic transposition within a given profession by itself the *initiating transposition*, where transpositive gestures are more implicit and undifferentiated with assigning professional working based on the principle that *experience is the best teacher* (maybe we can identify here an hypostatization of another paradigm which we can call the paradigm of *granting membership*). In addition, naming initiating transpositions of a special type in the teaching profession the *paradidactic transposition* is probably useful in didactics.

The second question about teacher education is related to the notion of didactic system. What kind of didactic systems is set in teacher education? There are two types of didactic systems based on the two types of didactic transposition. The first type is of *didactic systems of exogenetic disciplines*. In the case of mathematics teacher education, different courses of mathematical theories are typical examples. I call such a didactic system the *archididactic system* with the notation of $S(\dot{X}, \check{Y}, \Diamond)$, where $\dot{X}$ is any set of new comers of a given profession, $\Diamond$ is any *archididactic stake* which is an exogenetic work for the profession, and $\check{Y}$ is any set of professionals of $\Diamond$, that is, members of a given *production institution $\check{P}$* of $\Diamond$. By contrast, the initiating transposition in the teaching profession leads to the *paradidactic system $S(\dot{X}, \dot{Y}, S(X, Y, \heartsuit))$* which is the second type of didactic systems in teacher education, where $\dot{Y}$ is any set of professional mentors. A typical example is the teaching practice in school.
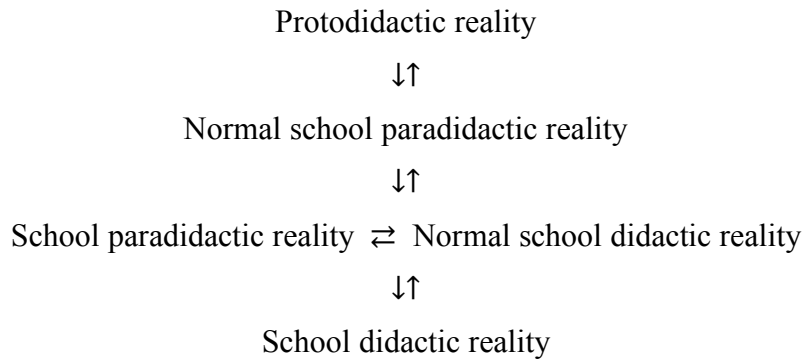
The third question: why are the two types of didactic stakes—archididactic stake $\Diamond$ and paradidactic stake $S(X, Y, \heartsuit)$—involved in teacher education? For answering that, let me here make explicit a forth component of didactic systems, that is, **didactic milieu** denoted by M. A didactic milieu M is any set of works functioning as resources for the study. This extended model of didactic systems are expressed by: $S(X, Y, \heartsuit; M)$. This sophistication of the model affects the model of paradidactic systems including the *paradidactic milieu $\dot{M}$*, which is any "toolkit" for

designing and analyzing of the didactic system at stake: $S(\dot{X}, \dot{Y}, S(X, Y, ♥); \dot{M})$. For instance, a paradidactic milieu for the teaching of a mathematical work ♥ can include various works coming from exogenetic disciplines like mathematics $\mathcal{M}$, statistics $\mathcal{S}$, psychology $\mathcal{P}$, and so on: $S(\dot{X}, \dot{Y}, S(X, Y, ♥); \mathcal{M} \cup \mathcal{S} \cup \mathcal{P} \cup \cdots)$. This is a reason why normal schools prepare archididactic systems. By contrast, the *raison d'être* of the paradidactic stake in the normal school must be clear. It is just the stake of the teaching profession.

Indeed, paradidactic systems for teacher education are located at paradidactic reality. However, that is not an absolute fact but *relative to kinds of schools* in some sense. For clarifying that, let me activate here the perspective of ***cognitive algebra***, which in my understanding emphasizes that the cognitive universe of a given reference instance $\hat{w}$ is "closed" for the operations of judgement by different instances $\hat{\imath}$ who exist for $\hat{w}$, i.e. $\hat{\imath} \in \Gamma(\hat{w})$. Σ and Σ' are respectively a school and a normal school for training its schoolteachers. Σ judges that any Σ''s paradidactic system $\dot{s}$ is in the paradidactic $\mathbb{P}$ which is the set of all the paradidactic facts: $\Sigma \vdash \dot{s} \in \mathbb{P}$. By contrast, $\dot{s}$ is an element in *the didactic* $\mathbb{D}$ in the narrow sense relative to Σ' itself: $\Sigma' \vdash \dot{s} \in \mathbb{D}$. Moreover, the judgement of Σ' is usually intended by Σ because Σ' is the school system for preparing schoolteachers of Σ at stake: $\Sigma \vdash (\Sigma' \vdash \dot{s} \in \mathbb{D})$. This means that Σ is a part of the noosphere of Σ'—I call it here the <u>*normal noosphere*</u> after the name of normal school—, who is interested in Σ'. Indeed, any normal school Σ' seems to have its own noosphere $\dot{N}_{\Sigma'}$, profession $\dot{P}_{\Sigma'}$, and "paradidactic" system $\dot{s}_{\Sigma'}$. Let me describe their particularity through looking at a type of didactic organizations which I am trying to realize in my working as a teacher educator. The (normal school) "didactic" system for it is the paradidactic system $S(\dot{X}, \dot{Y}, S(X, Y, ♥))$, which has a (school) didactic system $S(X, Y, ♥)$. In there, the didactic stake ♥ is a question q, that is to say, the paradidactic system is denoted by $S(\dot{X}, \dot{Y}, S(X, Y, q))$[8]. I want to emphasize again that this paradidactic system is the didactic system *to be targeted in the normal school*. If so, we are supposed to be able to consider its "paradidactic" system, which is denoted by $S(\ddot{X}, \ddot{Y}, S(\dot{X}, \dot{Y}, S(X, Y, ♥)))$. In my case, $\ddot{X}$ (X with diaeresis) as an engineer of $S(\dot{X}, \dot{Y}, S(X, Y, ♥))$ and $\ddot{Y}$ as a $\ddot{X}$'s teacher are me KO, $\ddot{Y}$ is empty, $\dot{X}$ is a set PT of prospective teachers in the university for which I work, $[X, Y]$ is a (fictive or hypothetical) school class at secondary school level, and ♥ is an initial question q for X: $S(KO, \emptyset, S(PT, KO, S(X, Y, q)))$. And then, I am subjected to the profession $\dot{P}_{\Sigma'}$ of the ATD within a noosphere $\dot{N}_{\Sigma'}$ who think about normal school in Japanese society. I want to summarize the description by the scale of didactic codeterminacy levels in the following, together with the awareness of its incompleteness by myself. Teacher education seems to be a

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[8] We can model a special form of didactic systems for SRP, the so-called ***SRP-TE*** (cf. Barquero, Florensa, & Ruiz-Olarría, 2019), or a type of it at least, by the following: $S(\dot{X}, \dot{Y}, S(\dot{X}, \dot{Y}, q))$.

catchy theme in didactics, but its ecosystem is very complex, because of the *fourfold nature of the didactic in teacher education* of "school", "noosphere/normal school", "normal noosphere", and "society". In my view, the theoretical power for studying teacher education is definitely not enough within didactics.

<div align="center">

Protodidactic reality

↓↑

Normal school paradidactic reality

↓↑

School paradidactic reality ⇄ Normal school didactic reality

↓↑

School didactic reality

</div>

6. Reflexive didactics

This paper has be devoted to insisting on the realism of the paradidactic, while pursuing the relativization of didactic reality. In this section, as a final touch for that, I want to relativize people who are supposed to relativize didactic reality, that is, *didacticians*. Let us focus here on an ATD's metatheoretical notion, the *reference model*. For simplifying the discourse, I pick up the notion of **reference praxeological models**, or more generously, **reference epistemological model** (cf. Bosch & Gascón, 2006; Chevallard, 2020), and thereby ignore the notion of **reference didactic model**[9] (cf. Gascón & Nicolás, 2019b). In the ATD, creating reference praxeological models is emphasized. Roughly speaking, a reference praxeological model $\mathfrak{W}$ is any *didacticians'* model of a given didactic stake. I generalize this definition from the perspective of ATD's *relativity principle*: (we recognize, at least methodologically) *different instances can have different knowledge of the same object* (cf. Barbé, Bosch, Espinoza, & Gascón, 2005; Bosch et al., 2019). Any reference model $\mathfrak{W}$ originally means a reference model of a work w within the institution

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[9] Let me give here several examples of the concept of reference didactic model, which relatively does not stand out. Within the ATD framework, the format of SRP often is used as a reference didactic model. In other cases, the flow of didactic moments can function for that. By contrast, the study in the TDS usually regards the format of fundamental situation as such a model, which means that a given didactic stake could grow through the following process: protomathematical mode (need) → paramathematical mode (tool) → mathematical mode (target). Of course, every reference didactic model is deeply related to some reference epistemological model. On the one hand, the ATD creates such a model with the **Q-A map** (cf. Winsløw, Matheron, & Mercier, 2013) and/or the **praxeology** model. On the other hand, the TDS uses the model of mathematical situations: *action*, *communication*, and *validation*. About the relationship between epistemological models and didactic ones, we should probably not forget that didactics has an epistemological aspect, *as well as* epistemology has a didactic aspect conversely.

$\dot{D}$ of didacticians, denoted by $\mathfrak{W}_{\dot{D}}(w)$. Likewise, we can consider that the school system Σ has its own reference model of w, $\mathfrak{W}_{\Sigma}(w)$, as well as the production institution $\check{P}$ has $\mathfrak{W}_{\check{P}}(w)$ (in the case of $\mathfrak{W}$ about mathematics education, $\check{P}$ clearly is the *institution of mathematicians*).

The abovementioned relativization is useful for explaining a reason why the working of researchers of mathematics education often quite similar to the one of curriculum developers (at least in Japan), and why they sometimes respect historical epistemology of mathematics like the French tradition of the didactics of mathematics. Let us begin with the full unit of didactic analysis, a whole sequence of a certain didactic transposition. Given a *didactic* transposition Φ of a work w, which can be defined as $(o, R_I(o))$ where o is an object as well as I is an institution, brings into existences a didactic stake $w^{\blacktriangledown}$ as its school counterpart $\Phi(w)$, that is: Φ: $w \mapsto w^{\blacktriangledown}$. This means that there is the transpositive correspondence from the **cognitive equipment** $\Gamma(\check{P})$ $(\ni w := (o, R_{\check{P}}(o))$ of the production institution $\check{P}$ of w into the one $\Gamma(\Sigma)$ $(\ni w^{\blacktriangledown} := (o, R_{\Sigma}(o))$ of a school Σ at least: Φ: $\Gamma(\check{P}) \rightarrow \Gamma(\Sigma)$, i.e. $(o, R_{\check{P}}(o)) \mapsto (o, R_{\Sigma}(o))$[10]. In other words, there exists the *institutional route* of w denoted by: $\check{P} \rightsquigarrow \Sigma$. And then, the noosphere $\dot{N}$ often is included in the description of it in detail: $\check{P} \rightsquigarrow [\dot{N} \rightsquigarrow \Sigma]$. This sophistication is from a sensitivity of distinction between two subinstitutions in school of the teaching part and the prescribing part. In this view, the former part is called school *in a narrow sense*, whereas the latter is named noosphere. From this perspective, Σ's reference *epistemological* model of w, $\mathfrak{W}_{\Sigma}(w)$, are precisely a tuple of $(o, R_{\Sigma}(o))$ and $\dot{N}$'s relations to $(o, R_{\Sigma}(o))$: $\mathfrak{W}_{\Sigma}(w) \stackrel{\text{def}}{=} ((o, R_{\Sigma}(o)), R_{\dot{N}}((o, R_{\Sigma}(o)))$. This definition clarifies that the cognitive equipment of the noosphere of a given school is the *habitat* of reference models of works involved in the school's cognitive equipment: $((o, R_{\Sigma}(o)), R_{\dot{N}}((o, R_{\Sigma}(o))) \in \Gamma(\dot{N})$. And then, that emphasizes that creating the reference model is at a "transcendental" level with *relations to relations*. In short, $\mathfrak{W}_{\Sigma}(w)$ is precisely $\mathfrak{W}_{\dot{N}}(w)$.

Following the line of this relativization, I add here one more institution in the institutional route reminding you of the potential generality of the word "noosphere", that is, a *study* noosphere of a production institution $\check{P}$ denoted by L, which I personally name metaphorically a *library*, whereby in turn $\check{P}$ can be analyzed in detail: $[\check{P} \rightsquigarrow L] \rightsquigarrow [\dot{N} \rightsquigarrow \Sigma]$. Indeed, works of knowledge are able to be displayed and to reach to a noosphere of a school Σ, $\dot{N}$, after passing through selection and arrangement—i.e. making "noble" knowledge produced in $\check{P}$ more *diffusive* and *teachable*—by some epistemological institution, whether implicitly or explicitly. Note that all the members of $\check{P}$ more or less are epistemologists or *librarians* for $\check{P}$, that is to say, each of them has some answer,

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[10] As an application of this characterization of the transposition, I want to define the notion of **institutionalization** as any transposition Φ' from a personal cognitive equipment to an institutional cognitive equipment: Φ': $\Gamma(x) \rightarrow \Gamma(I)$.

whether naïve or sophisticated, to the question "what is the product of $\check{P}$?" in her mind, may be especially at weekends or in preparing lectures for their students who are newcomers of $\check{P}$. One of typical examples of the libraries in mathematics is the institution of Nicolas Bourbaki, who initially aimed to write treatises of mathematical analysis as you know. In the same manner to refinement in the case of Σ, $\check{P}$'s reference model of w, $\mathfrak{W}_{\check{P}}(w)$, are defined as: $\mathfrak{W}_{\check{P}}(w) = \mathfrak{W}_L(w) \overset{\text{def}}{=} ((o, R_{\check{P}}(o)), R_L((o, R_{\check{P}}(o)))).$ Let me remind you here that the reason why I make the complicated description of the institutional route and the locations of reference models of different types is to clarify the relationship between the three professions of didactician, curriculum developer, and epistemologist. The didactician is similar to the curriculum developer and the epistemologist from the viewpoint of the possessing of own reference models, which can be denoted by: $\mathfrak{W}_{\dot{D}}(w) \overset{\text{def}}{=} ((o, R_{\check{P}}(o) \cup R_{\Sigma}(o)), R_{\dot{D}}((o, R_{\check{P}}(o) \cup R_{\Sigma}(o)))).$ In this line, we can easily understand why didactics often approximates the idea of *genetic epistemology*, or it has sometimes the byname of *experimental epistemology*; as well as why it is frequently regarded as a kind of *normative* research programs.

I guess that many readers regard the notion of library as an excessive theoretical construct. There is a pragmatic reason for creating it. In my view, the didactic transposition of works of a special kind which is attracting our interest more and more cannot be explained without this notion: e.g., *metacognition*, *modeling*, and *inquiry*. They seem to be *autodidactic* praxeologies—i.e. didactic praxeologies by and for oneself—, which has no proper discipline in "usual" sense in the context of school education. So, where do such notions come from into noospheres and schools? My temporary answer is "from the library".

In principle, the institution of didacticians $\dot{D}$ is supposed to study the whole process of didactic transposition in some way so that the institutions of the institutional route, $\check{P} \rightsquigarrow L \rightsquigarrow \dot{N} \rightsquigarrow \Sigma$, involved in the process, and $\dot{D}$ are *not* essentially equal. Such a situation is denoted by : $\dot{D} \vdash \check{P} \not\approx \dot{D} \wedge L \not\approx \dot{D} \wedge \dot{N} \not\approx \dot{D} \wedge \Sigma \not\approx \dot{D}.$ This inter-institutional positioning as the outsider is obligated to us for improving and keeping objectivity of didactic research. If we default to construct $\dot{D}$'s reference models, then we are easily involved in $\check{P}$ and/or Σ roughly speaking, that is, superposed with L and/or $\dot{N}$ precisely speaking. The institutional route by the former superposition is denoted by: $\check{P} \rightsquigarrow L (\approx \dot{D}) \rightsquigarrow \dot{N} \rightsquigarrow \Sigma.$ And the latter is: $\check{P} \rightsquigarrow L \rightsquigarrow \dot{N} (\approx \dot{D}) \rightsquigarrow \Sigma.$ In such cases, $\dot{D}$ becomes not to scientifically understand a part of a given transpositive history. This is a reason why the ATD emphasizes the creation of $\dot{D}$'s reference models.

As I have already mentioned just a little, the library L, the noosphere $\dot{N}$, and the *production institution of didactics $\dot{D}$* share a kind of "transcendentality" which is the nature of taking into

account social systems for studying, that is, didactic systems $S(X, Y, ♥)$. That is, they are *fundamental institutions* of paradidactic systems $S(\dot{X}, \dot{Y}, S(X, Y, ♥))$ in the same way from such perspective. This reflection leads us to a de-centralization of didacticians and epistemologists. In the broadest meaning of the word "noosphere"—people who are related to school which is the bundle of didactic systems—, $\dot{D}$ and $Ł$ are particular types of it. The high degree of such broadness depends on the anthropological sense of the adjective "didactic", in which historical epistemology studies "study systems" of researchers. All in all, everyone who are related to didactic systems of some type is a noospherian.

After the relativization of the didacticians, I try to do a small psychoanalysis of the didacticians for reminding you of a well-known epistemological obstacle of didactics—here, I explicitly call it *obstacle*. For that, please remember ATD's relativity principle. This principle seems to be always exposed to the will of disproving it, which probably is strongest (and most aggressive) when mathematical knowledge becomes stakes. Such will is fed with the <u>*universality illusion about disciplinary knowledge*</u> (see also, Bosch & Gascón, 2006; Chevallard, 2007), which can be easily observed in discourses about *mathematics in history* and about *mathematics at school*. A remarkable example from the historical domain is found in research on antient Greek mathematics (cf. Christianidis [Ed.], 2004, Chap. 4; Saito, 1997). During the long years, the second book of Euclid's *Elements* has been interpreted as *geometric algebra* which could be originated from Babylonian mathematics. However, the Israeli historian Sabetai Unguru has contradicted this argument, arguing that historical analysis should not be based on the perspective of modern mathematics and should reconstruct knowledge at that time and space. And then, Unguru's statements have occurred emotional counterarguments of some great mathematicians, which is an appearance of the universality illusion.

The epistemological illusion of unique mathematics is also related to the diffusion and "distortion" of didactic transposition theory. In fact, this theory sometime is criticized from "math people" (Chevallard, 2019) who include the same mathematician as in the case of Unguru's historical discourse (see also, Winsløw, 2011). They probably misunderstand the notion of didactic transposition as the pushing of pure and rigorous scholarly mathematics to schoolers. Many of math people could not recognize the reality of *school* mathematics, because they are obsessed with the Bourbakist philosophy of "mathematics as a singular entity". As a consequence, math people tend to comprehend the word "transposition" as the *imposition* of advanced mathematics to school. Indeed, on the one hand, an epistemological principle of the ATD—I want to call it <u>*praxeologist principle*</u>—could be widely shared by math people: *mathematics is a particular type of human activities*. Nonetheless, on the other hand, the valuing of scholarly mathematics seems

to hinder that they understand the phenomenon of didactic transposition of knowledge. We seem be reuniting here the substantialist obstacle. We tend to carelessly substantialize the reality of *mathematics* together with its scholarly images authorized by its long history, broad usefulness, and strict logic. However, such *taken-for-grantedness* is a major obstacle of the didactics of mathematics. How can we resist this allurement? At this step, we finally led a point to abandon the superficial understanding of the notion of *praxeology*, which is accompanied by the well-known quadrupled model. The praxeological model is *completely not* the tool for drawing our spontaneous images of knowledge in detail. By contrast, it allows us to leave substantialist obstacles then to recognize a reality as a system consisted of a number of things. As historical epistemology tells us, any field of science, whether natural or social, modifies their epistemology from substantialism to *relationalism*. In my view, the praxeological model is an emblem of ATD's relationalist stance.

In principle, ATD-didacticians are supposed to overcome the universality illusion about disciplinary knowledge (especially mathematics) since the emergence of the didactic transposition theory. However, in practice, it is very difficult even if it is not impossible. We must be vigilant all the time. About that, before finishing this section, I want to confess a straggle during writing this manuscript, which inflicts "wounds" to me—maybe a kind of narcissistic injury. I notice that all the analysis in the previous discourses include myself as a part (even part and parcel!) of its stake. To tell you the truth, that is intended. I have tried to write this paper such that *what I said will come back to me* as much as possible, following the idea of *reflexive anthropology* by the French sociologist Pierre Bourdieu. Such reflexivity seems to be crucial in social scientific theories. In my opinion, the possibility that a certain theory can criticize itself is a proof that the theory can keep growing up to more scientific form. Note that this certainly is not a naïve reflection which is in most case unproductive. It is because one's own self is the most transparent reality. We need strong theories of the paradidactic as equipment for epistemological vigilance obligated to the ATD-didactician by the ***emancipatory principle*** (cf. Bosch et al., 2019).

7. Final remarks: on the professionalization of the teaching profession[11]

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[11] Careful readers probably notice that I use here the word "profession" with two meanings. The first meaning in the expression "teaching profession" is what has been explained in the second section, that is, the profession as a general and neutral term which is specific in our theoretical framework. By contrast, the second is the usual one which appears in the word "professionalization", that is, the profession as a *full-fledged* status of trades. I believe that the meaning of this word in each part is sufficiently clear by its context. But, just in case, I want to define within our framework the word *professionalization* as "becoming a full-fledged profession", and the word *semiprofession* as "unfledged profession".

This paper has been devoted to defining and generalizing the notion of the paradidactic reality of noospheres. I want to go back here to the starting point of the paradidactic problematic, that is, the teaching profession. A teaching profession in a given society is often recognized as a "semiprofession" by other noospheric professionals like politicians responsible to education and noospheric counterparts of the scholars of a determined discipline, e.g., mathematics. A property of the semiprofession is its _handy-andy status_ which is much less specialized. I think that the Japanese situation of the teaching profession gives a typical example of such semiprofessionalism. The **_problem of the profession_** (cf. Chevallard, 2013)—i.e. the crucial question for its professionals—of the lower secondary schoolteachers of all the disciplines in Japan include more or less questions about how to manage a school (sports) club-team (_bukatsudō_ in Japanese). In my personal experience, many prospective teachers even say that they want to become schoolteachers for engaging in _bukatsudō_, nevertheless, of course, the main problem of the (traditional) teaching profession is to teach disciplines, or more precisely, to solve the so-called _paradoxes of the didactic contract_, in TDS's terminology, for teaching disciplines—e.g., the _paradox of devolution_; teachers cannot devolve their responsibilities upon their students by direct and explicit ways of devolving them.

Another occurrence of the semiprofession is the _lack of respect_ to it from the society where it belongs. In the extreme case, teaching professionals are despised by the occupants of other noospheric professions with the following message more or less—schoolteachers teach a disciplinary topic in a stupid and wrong way, because they do not know the discipline well. In my view, this is deeply rooted in the cognitive nature of humankind. Let us here look at an example of it calling upon Bourdieu's notion again, the _epistemocentrism_. This word means that analysts of social systems of different kinds tend to ignore the internal practical logic, and thereby impose their "more reasonable and rational" interpretation to the stakeholders of the systems. The teaching profession is pertinent to a noospheric variation of this tendency, where the validity of such epistemocentristic views is attributed to the absolute authority and legitimacy of scholarly knowledge over the didactic stake which is a result of didactic transposition of it. We can observe here an institutional hierarchy, which could include more or less some "class conflict" between noospheric professions. In such situations, the professionalization of the teaching profession into a full-fledged profession seems like a _revolution_. How can schoolteachers accomplish it peacefully? One hopeful way probably is to be armed with relevant theoretical equipment. The ATD can be such epistemological weapons for professionalizing the teaching profession. I believe that this is a practical obligation of ATD-didacticians. In this sense, we must consider the _teaching of the ATD_ seriously. I hope that this paper would be useful for that. But, of course, my priority

when writing this paper which gives the new paradidactic theory ($\Theta^{\heartsuit}$) is in that, in my (KO's) view, $\Theta^{\heartsuit}$ would be helpful for the growing of the ATD ($\Theta^{\diamond}$) as a "food" *eaten* in the *trophic relation* of $\Theta^{\heartsuit}$ to $\Theta^{\diamond}$, denoted by: KO $\vdash$ $\Theta^{\diamond}$ $\hookleftarrow$ $\Theta^{\heartsuit}$ (cf. Chevallard, 2022a). Gràcies!

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[12] The judging of what should be included in references is always difficult. I decide basically not to list up *generally* famous scholars' works for the simplicity sake. I remember that Bourdieu said kind of the following statement somewhere: commonization is a destiny of useful notions, that is, a great honor.

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