Nilpotent and Solvable Approximations of an Almost-Riemannian structure at singular points

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An Almost Riemannian Structure (ARS in short) on an n-dimensional differential manifold is a rank-varying sub-Riemannian structure that can be defined, at least locally, by a set of n vector fields satisfying the Lie algebra rank condition. The set of points where the dimension of the linear span of the vector fields is not full it is called the singular locus (or the singular set).

Interesting models of ARS's can be described on Lie groups by means of invariant and linear vector fields (see [3] and [2]). They are referred to as linear (or simple) ARSs on Lie groups (see [1] and [2]).

Very often nilpotent approximations (N.A in short) are used to study locally the behaviour of such structures. However there are cases where the nilpotent approximation of an ARS turns out to be a constant rank Sub-Riemannian one. The solvable approximation (S.A in short) is a local approximation of a ARS at singular point where the nilpotent approximation no longer an ARS but a constant rank sub-Riemannian structure (see [3]).

Our main result concerning the structures is the following: **The nilpotent (when it is still an ARS) or the solvable approximation of an ARS is always a linear ARS on a nilpotent Lie group (or a homogeneous space of a nilpotent Lie group), excepted in some completely degenerated cases**.

On th other hand, the original structure, the S.A, and the N.A give rise to the distances denoted by d, $\widetilde{d}$, and $\widehat{d}$ respectively. In the 3D-generic cases, where is possible defined the S.A, the N. A is the Heisenberg sub-Riemannian structure (see [4]). We show that in some 3D-generic cases the order of $|d-\widetilde{d}|$ is strictly better than the one of $|d-\widehat{d}|$. That is, the order of $|d-\widehat{d}|$ is $d^{\frac{3}{2}}$ and the one of $|d-\widetilde{d}|$ is d^2.

References

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