Short time implied volatility of additive normal tempered stable processes

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Abstract

Empirical studies have emphasized that the equity implied volatility is characterized by a negative skew inversely proportional to the square root of the time-to-maturity.
We examine the short time-to-maturity behavior of the implied volatility smile for pure jump exponential additive processes. An excellent calibration of the equity volatility surfaces has been achieved by a class of these additive processes with power-law scaling. The two power-law scaling parameters are β, related to the variance of jumps, and δ, related to the smile asymmetry. It has been observed, in option market data, that $\beta = 1$ and $\delta = -1/2$.
In this paper, we prove that the implied volatility of these additive processes is consistent, in the short time, with the equity market empirical characteristics if and only if $\beta = 1$ and $\delta = -1/2$.

Keywords: Additive process, volatility surface, skew, small time, calibration.

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1 Introduction

Which characteristics of the implied volatility surface should be reproduced by an option pricing model? A *stylized fact* that characterizes the equity market is a downward slope in terms of strike, i.e. a negative skew, where the skew is the at-the-money (ATM) derivative of the implied volatility w.r.t. the moneyness.[1] Specifically, the short time negative skew is proportionally inverse to the square root of the time-to-maturity. The first empirical study of the equity skew dates back to Carr and Wu (2003): they find that the S&P 500 short time skew is, on average, asymptotic to $-0.25/\sqrt{t}$. Moreover, Fouque *et al.* (2004) arrive at a similar conclusion considering only options with short time-to-maturity (i.e. up to three months). In this paper, we show that a pure jump additive process, which also calibrates accurately the whole equity volatility surface, reproduces the power scaling market skew.

A vast literature on short time implied volatility and skew is available for jump-diffusion processes. Both the ATM (see, e.g. Alòs *et al.* 2007, Roper 2009, Muhle-Karbe and Nutz 2011, Andersen and Lipton 2013, Figueroa-López *et al.* 2016) or the OTM implied volatility (see, e.g. Tankov 2011, Figueroa-López and Forde 2012, Mijatović and Tankov 2016, Figueroa-López *et al.* 2018) are analyzed. For a jump-diffusion Lévy process, the ATM implied volatility is determined uniquely by the diffusion term; it goes to zero as the time-to-maturity goes to zero if there is no diffusion term, i.e. for a pure-jump process. For this reason, pure jumps Lévy processes are not suitable to reproduce the market short-time smile, because the short time implied volatility is strictly positive in all financial markets.
Muhle-Karbe and Nutz (2011) have shown that, for a relatively broad class of additive models, the ATM behavior of leading term for small time is the same of the corresponding Levy. In this paper, we analyze the ATM implied volatility and skew for a class of pure jump additive processes that is consistent with the equity market smile, differently from the Lévy case: this is the main theoretical contribution of this study.

An additive process is a stochastic process with independent but non-stationary increments; a detailed description of the main features of additive processes is provided by Sato (1999). In this paper, we focus on a pure jump additive extension of the well-known Lévy normal tempered stable process (for a comprehensive description of this set of Lévy processes, see, e.g., Cont and Tankov 2003, Ch.4).

Pure jump processes present a main advantage w.r.t. jump-diffusion models: they generally describe underlying dynamics more parsimoniously. In a jump-diffusion, both small jumps and the diffusion term describe little changes in the process (see, e.g. Asmussen and Rosiński 2001). Because both components of the jump-diffusion process are qualitatively similar, when calibrating the model to the plain vanilla option market, it is rather difficult to disentangle the two components and several set of parameters achieve similar results.

[1] The moneyness is the logarithm of the strike price over the forward price. For a description of the equity volatility surface and a definition of skew, see, e.g., Gatheral (2011).

Recently, it has been introduced a class of pure jump additive processes, the power-law scaling additive normal tempered stable process (hereinafter ATS), where the two key time-dependent parameters –the variance of jumps per unit of time, k_t, and the asymmetry parameter, η_t– present a power scaling w.r.t. the time-to-maturity t. It has been shown the excellent calibrating performances of this class of processes (see, e.g. Azzone and Baviera 2021). On the one hand, this class of pure jump additive processes allows calibrating the S&P 500 and EURO STOXX 50 implied volatility surfaces with great accuracy, reproducing "exactly" the term structure of the equity market implied volatility surfaces. On the other hand, the observed reproduction of the skew term structure appears remarkable.
Moreover, an interesting self-similar characteristic w.r.t. the time-to-maturity arises. Specifically, among all allowed power laws, the power scaling of k_t, β, is close to one, while the power scaling of η_t, δ, is statistically consistent with minus one half (see, e.g. Azzone and Baviera 2021).

Consider an option price with strike K and time-to-maturity t. We define $I_t(x)$ the model implied volatility, where $x := \log \frac{K}{F_t}$ is the log-moneyness and F_t is the underlying forward price with time-to-maturity t. In particular, we consider the *moneyness degree* y, s.t. $x =: y\sqrt{t}$, introduced by Medvedev and Scaillet (2006). It has been observed that the *moneyness degree* y can be interpreted as the distance of the option moneyness from the forward price in terms of the B&S Brownian motion standard deviation (see, e.g., Carr and Wu 2003, Medvedev and Scaillet 2006). The implied volatility w.r.t y is

$$\mathcal{I}_t(y) := I_t(y\sqrt{t}) \quad ,$$

and its first order Taylor expansion w.r.t. y in $y = 0$ is

$$\mathcal{I}_t(y) = \mathcal{I}_t(0) + y \left. \frac{d\mathcal{I}_t(y)}{dy} \right|_{y=0} + o(y) =: \hat{\sigma}_t + y\,\hat{\xi}_t + o(y) \quad .$$

We call $\hat{\xi}_t$ the skew term. We define $\hat{\sigma}_0$ and $\hat{\xi}_0$ as the the limits for t that goes to zero of $\hat{\sigma}_t$ and $\hat{\xi}_t$. Their financial interpretation is straightforward: $\hat{\sigma}_0$ corresponds to the short time ATM implied volatility, while $\hat{\xi}_0$ is related to the short time skew, because it is possible to write the skew as

$$\left. \frac{dI_t(x)}{dx} \right|_{x=0} = \frac{\hat{\xi}_t}{\sqrt{t}} \quad .$$

In Figure 1, we present an example of the S&P 500 for the short time implied volatility and the skew at a given date, the 22^{nd} of June 2020 (the business day after a triple witching Friday[2]). On the left, we plot the one month (blue squares), two months (red asterisks), three months (orange stars), and four months (purple triangles) market implied volatility w.r.t. the *moneyness degree* y: we observe a positive and finite short time $\hat{\sigma}_t$. On the right, we plot the market skew w.r.t. the time t: it appears to be well described by a fit $O\left(\sqrt{\frac{1}{t}}\right)$.

[2]A triple witching Friday is the third Friday of the months of March, June, September and December: in this quarterly date, stock options, stock index futures, and stock index options all expire on the same day.

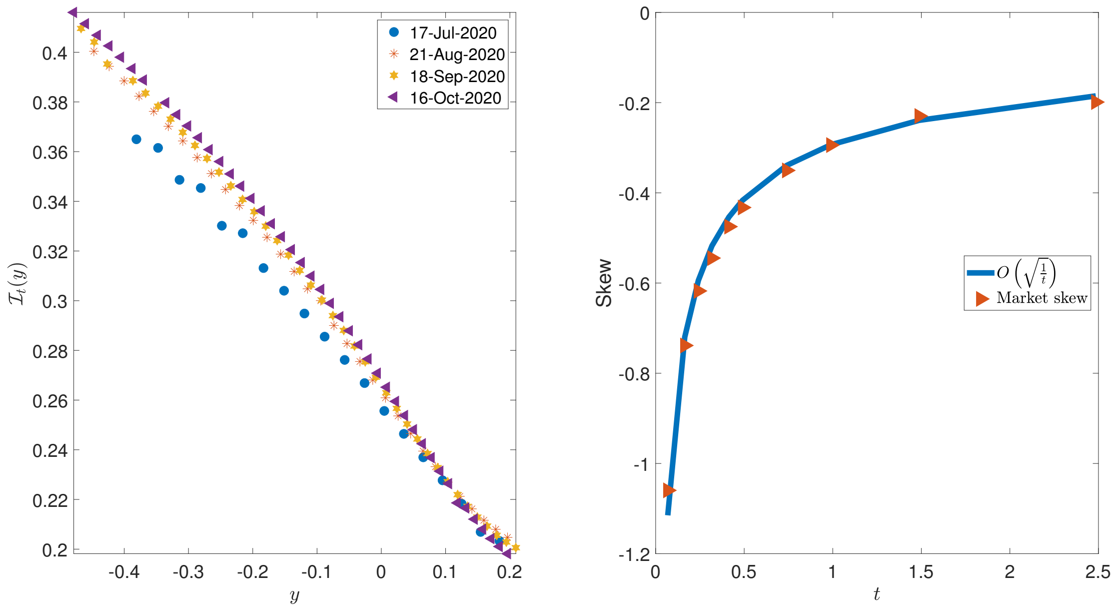

Figure 1: Example of the S&P 500 short time implied volatility and skew on the 22^{nd} of June 2020. On the left we plot the one month (blue circles), two months (red asterisks), three months (orange stars), and four months (purple triangles) market implied volatility w.r.t. the *moneyness degree* y. We observe a positive short time $\hat{\sigma}_t$. On the right we plot the market skew w.r.t. the time t and the fitted $O\left(\sqrt{\frac{1}{t}}\right)$.

As already observed in some empirical studies (see, e.g. Carr and Wu 2003, Fouque *et al.* 2004), equity market data are compatible with a positive and finite $\hat{\sigma}_0$ and a negative and finite $\hat{\xi}_0$, that leads to a skew proportionally inverse to the square root of the time-to-maturity. We aim to present a pure-jump model with these features.

We study the behavior of $\hat{\sigma}_t$ and $\hat{\xi}_t$ for the ATS process, deriving an extension of the Hull and White formula (Hull and White 1987, Eq.7) in (4, 5) (see, e.g., Alòs *et al.* 2007, for another application of this formula to the short time problem). This formula leads to two results: on the one hand, we build some relevant bounds for $\hat{\sigma}_t$; on the other hand, we obtain an expression for $\hat{\xi}_t$ in (25) via the implicit function theorem (see, e.g. Loomis and Sternberg 1990, Th.11, p.164).

Three are the main contributions of this paper. First, we deduce for a family of pure-jump additive processes, the ATS, the behavior of the short time ATM implied volatility $\hat{\sigma}_t$ and skew term $\hat{\xi}_t$ over the region of admissible parameters (see **Theorem 2.2**). Second, we prove that only the scaling parameters observed in market data ($\delta = -1/2$ and $\beta = 1$) are compatible with a finite short time implied volatility and a short time skew proportionally inverse to the square root of the time-to-maturity. Third, we demonstrate it exists a pure-jump additive process (an ATS) that presents the two features observed in market data: not only a positive short time implied volatility but also a power scaling skew.

The rest of the paper is organized as follows. Section 2 presents the ATS power scaling process and the extension of the Hull and White formula. Section 3 defines the implied volatility problem and analyzes the short time ATM implied volatility $\hat{\sigma}_t$. Section 4 computes the short time limit of the skew term $\hat{\xi}_t$. Section 5 presents the major result: the ATS process is consistent with the equity market skew if and only if $\beta = 1$ and $\delta = -1/2$. Finally, Section 6 concludes.

2 The ATS implied volatility

In this Section, we recall the ATS power-scaling sub-case characteristic function (1), we introduce a sequence of random variables (3) with the same distribution of the ATS for any fixed time t. We use this random variable to study the short time implied volatility. We discuss the volatility smile at small maturity produced by this forward model and determine the power laws of the ATS parameters that are consistent with the market data i.e. which choices of δ and β are consistent with the market short time features mentioned above.

We define the sequence of positive random variables S_t via it Laplace transform. This random variable is used in the definition of the random variable f_t.

Definition 2.1. Definition of $\{S_t\}_{t\geq 0}$
Let $\{S_t\}_{t\geq 0}$ be a sequence of positive random variable with a Laplace transform s.t.

$$\ln \mathcal{L}_t\left(u;\ k_t,\ \alpha\right) := \ln \mathbb{E}\left[e^{-uS_t}\right] = \begin{cases} \dfrac{t}{k_t}\dfrac{1-\alpha}{\alpha}\left\{1-\left(1+\dfrac{u\,k_t}{(1-\alpha)t}\right)^{\alpha}\right\} & \text{if } \ 0<\alpha<1 \\ -\dfrac{t}{k_t}\ln\left(1+\dfrac{u\,k_t}{t}\right) & \text{if } \ \alpha = 0 \end{cases} ,$$

where $k_t := \bar{k}t^{\beta}$ and $\bar{k},\ \beta \in \mathbb{R}^+$.

Notice that, by the Laplace transform we can compute any moment of S_t. The first two are

1. $\mathbb{E}\left[S_t\right] = 1$;
2. $Var\left[S_t\right] = k_t/t$;

We define the random variable f_t as a random variable with characteristic function

$$\mathbb{E}\left[e^{iuf_t}\right] := \mathcal{L}_t\left(iut\left(\frac{1}{2}+\eta_t\right)\bar{\sigma}^2 + t\frac{u^2\bar{\sigma}^2}{2};\ k_t,\ \alpha\right)e^{i\,u\,\varphi_t\,t} \ , \tag{1}$$

where

$$\eta_t := \bar{\eta}t^{\delta} \text{ and } \varphi_t\, t := -\ln \mathcal{L}_t\left(t\bar{\sigma}^2\eta_t;\ k_t,\ \alpha\right) \ . \tag{2}$$

$t,\ \bar{\sigma},\ \bar{\eta}\ \in \mathbb{R}^+$ and $\delta\ \in \mathbb{R}$.
The characteristic function is the same of the power scaling sub-case of the ATS process in Azzone and Baviera (2021, eq.2). Thus, we can use directly f_t to evaluate European payoffs. Notice that there is a slight difference in the notation. To simplify the following theorems' and propositions' proofs we have defined the Laplace transform in a way that, when is evaluated in the same input of the one in Azzone and Baviera (2021) multiplied by t, it has the same output.

We report the results of Azzone and Baviera (2021) on the existence of the power scaling ATS.

Theorem 2.2. Power-law scaling ATS

It exits an ATS

$$k_t = \bar{k}\ t^{\beta}, \qquad \eta_t = \bar{\eta}\ t^{\delta}$$

where $\alpha \in [0,1)$, $\bar{\sigma}, \bar{k}, \bar{\eta} \in \mathbb{R}^+$ and $\beta, \delta \in \mathbb{R}$ with either $\beta = \delta = 0$ or

1. $0 \leq \beta \leq \dfrac{1}{1-\alpha/2}$;

2. $-\min\left(\beta, \dfrac{1-\beta\,(1-\alpha)}{\alpha}\right) < \delta \leq 0$;

where the second condition reduces to $-\beta < \delta \leq 0$ *for* $\alpha = 0$ □

The region of admissible values for the scaling parameters β and δ is shown in Figure 2. In particular, we mention that, for all α in $[0, 1)$, the scaling parameters observed in the market, $\{\delta = -1/2,\ \beta = 1\}$, are always inside the ATS admissible region. In Figure 2, we plot the admissible region for the scaling parameters β and δ. In this paper, we prove that the ATS implied volatility at short time is qualitatively different for different sets of scaling parameters. We separate the admissible region in five Cases:
Case 1 (grey area) with $\hat{\sigma}_0 = 0$:

$$\left\{-\min\left(\frac{1}{2}, \beta\right) < \delta \leq 0,\ \ \beta < 1\right\} \cup \{\delta = \beta = 0\} \quad .$$

Case 2 (orange area) with $\hat{\sigma}_0 = \infty$:

$$\left\{-\min\left(\beta, \frac{1-\beta(1-\alpha)}{\alpha}\right) < \delta < -\max\left(\frac{\beta}{2}, \frac{1}{2}\right)\right\} \quad .$$

Case 3 (light green area) with finite $\hat{\sigma}_0$ and $\hat{\xi}_0 = 0$:

$$\left\{-\frac{\beta}{2} \leq \delta \leq 0,\ \ \beta \geq 1\right\} \setminus \left\{\beta = 1,\ \ \delta = -\frac{1}{2}\right\} \quad .$$

Case 4 (continuous dark green line) with finite $\hat{\sigma}_0$ and $\hat{\xi}_0 = -\sqrt{\frac{\pi}{2}}$:

$$\left\{\delta = -\frac{1}{2},\ \ \beta < 1\right\} \quad .$$

Case 5 (red dot) with finite $\hat{\sigma}_0$ and negative and finite $\hat{\xi}_0$:

$$\left\{\delta = -\frac{1}{2},\ \ \beta = 1\right\} \quad .$$

Notice that Case 3 includes all its boundaries, identified by the green circles, with the exception of the point $\{\delta = -1/2,\ \beta = 1\}$ (red); Case 1 includes just its upper bound, identified by the grey squares.

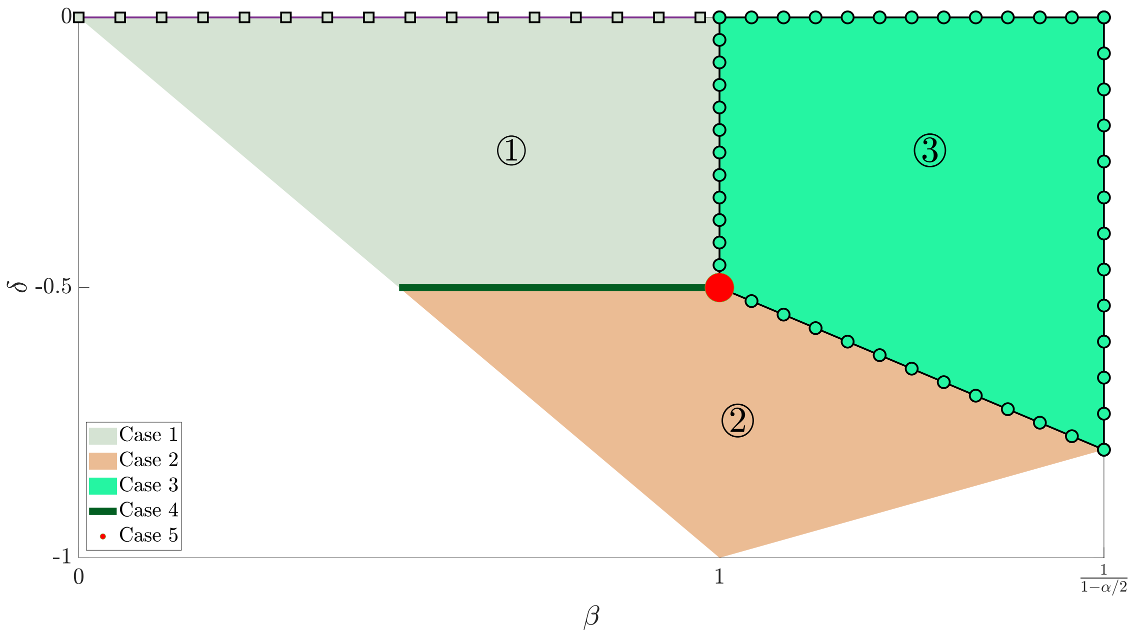

Figure 2: ATS admissible region for the scaling parameters. We separate the region in five Cases. i) Case 1 (grey area) with $\hat{\sigma}_0 = 0$. ii) Case 2 (orange area) with $\hat{\sigma}_0 = \infty$. iii) Case 3 (light green area) with finite $\hat{\sigma}_0$ and $\hat{\xi}_0 = 0$. iv) Case 4 (continuous dark green line) with finite $\hat{\sigma}_0$ and $\hat{\xi}_0 = -\sqrt{\frac{\pi}{2}}$. v) Case 5 (red dot) with finite $\hat{\sigma}_0$ and negative and finite $\hat{\xi}_0$.
Notice that Case 3 includes all its boundaries, identified by the green circles, with the exception of the point $\{\delta = -1/2, \beta = 1\}$ (red), that corresponds to Case 5. Moreover, Case 1 includes just its upper bound, identified by the grey squares. We emphasize that for all α in $[0, 1)$ the point $\{\delta = -1/2, \beta = 1\}$ is inside the admissible region.

A summary of the ATS short time behavior, w.r.t. the different Cases is available in Table 1.

Case 1	$\hat{\sigma}_0 = 0$
Case 2	$\hat{\sigma}_0 = \infty$
Case 3	$\hat{\sigma}_0 > 0$ and $\hat{\xi}_0 = 0$
Case 4	$\hat{\sigma}_0 > 0$ and $\hat{\xi}_0 = -\sqrt{\frac{\pi}{2}}$
Case 5	$\hat{\sigma}_0 > 0$ and $\hat{\xi}_0 < 0$

Table 1: Summary of ATS short time behavior, w.r.t. the scaling parameters δ and β in the additive process boundaries.

A summary of the ATS short time behavior, w.r.t. the scaling parameters δ and β in the additive process boundaries is available in Table 2.

	$0 < \beta < 1$	$\beta = 1$	$\beta > 1$
$\delta > -\frac{1}{2}$	$\hat{\sigma}_0 = 0$	$\hat{\xi}_0 = 0$	$\hat{\xi}_0 = 0$
$\delta = -\frac{1}{2}$	$\hat{\xi}_0 = -\sqrt{\frac{\pi}{2}}$	$\boldsymbol{\hat{\xi}_0 < 0}$	$\hat{\xi}_0 = 0$
$\delta < -\frac{1}{2}$	$\hat{\sigma}_0 = \infty$	$\hat{\sigma}_0 = \infty$	$\hat{\sigma}_0 = \infty$ or $\hat{\xi}_0 = 0$

Table 2: Summary of ATS short time behavior, w.r.t. the scaling parameters δ and β in the additive process boundaries.

It can be easily proven that, for every time t, the random variable

$$f_t = -\left(\eta_t + \frac{1}{2}\right)\ \bar{\sigma}^2\ S_t\ t +\ \bar{\sigma}\sqrt{S_t t}G + \varphi_t t \tag{3}$$

has the characteristic function in (1), where G is a standard normal random variable independent from S_t. We use this expression of f_t to compute the option prices.
Consider a European call (put) option discounted payoff $B_t\left(F_t e^{f_t} - F_t e^{x}\right)^+$ $\left(B_t\left(F_t e^{x} - F_t e^{f_t}\right)^+\right)$ where F_t is the underlying forward price at time 0, $F_t e^{f_t}$ is the underlying forward price at time t, t the option maturity, K the option strike price, B_t is the discount factor between 0 and t and $x = \ln\frac{K}{F_0}$ the asset log-moneyness. We can write the expected European call and put option price at time zero conditioning to S_t

$$\begin{aligned}
C_t(x) &= \mathbb{E}\left[\left(e^{f_t} - e^{x}\right)^+\right] \\
&= \mathbb{E}\left[e^{\varphi_t t - t\bar{\sigma}^2\eta_t S_t} N\left(\frac{-x}{\bar{\sigma}\sqrt{tS_t}} + l_t^{S_t} + \frac{\bar{\sigma}\sqrt{S_t t}}{2}\right) - e^{x} N\left(\frac{-x}{\bar{\sigma}\sqrt{tS_t}} + l_t^{S_t} - \frac{\bar{\sigma}\sqrt{S_t t}}{2}\right)\right] \qquad (4) \\
P_t(x) &= \mathbb{E}\left[\left(e^{x} - e^{f_t}\right)^+\right] \\
&= \mathbb{E}\left[e^{x} N\left(\frac{x}{\bar{\sigma}\sqrt{tS_t}} - l_t^{S_t} + \frac{\bar{\sigma}\sqrt{S_t t}}{2}\right) - e^{\varphi_t t - t\bar{\sigma}^2\eta_t S_t} N\left(\frac{x}{\bar{\sigma}\sqrt{tS_t}} + l_t^{S_t} - \frac{\bar{\sigma}\sqrt{S_t t}}{2}\right)\right], \qquad (5)
\end{aligned}$$

where

$$l_t^z := -\bar{\sigma}\eta_t\sqrt{z\,t} + \frac{\varphi_t\sqrt{t}}{\bar{\sigma}\sqrt{z}} \tag{6}$$

and N is the standard normal cumulative distribution. Notice that, for simplicity, we define directly the option prices with $F_t = 1$ and $B_t = 1$. To do so is without any loss of generality because we can always cancel these therms from both sides of the implied volatility equation. Notice that the quantity inside the expected values are, in both equations (4) and (5), positive (because they are the price of an option under the B&S model multiplied by a positive constant). A similar result is obtained by Hull and White (1987) for option on an asset with stochastic volatility.
We can re-write equations (4) and (5) w.r.t to the *moneyness degree* y

$$\begin{aligned}
C_t(y\sqrt{t}) &= \mathbb{E}\left[e^{\varphi_t t - t\bar{\sigma}^2\eta_t S_t} N\left(-\frac{y}{\bar{\sigma}\sqrt{S_t}} + l_t^{S_t} + \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right) - e^{y\sqrt{t}} N\left(-\frac{y}{\bar{\sigma}\sqrt{S_t}} + l_t^{S_t} - \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right)\right] \\
P_t(y\sqrt{t}) &= \mathbb{E}\left[e^{y\sqrt{t}} N\left(\frac{y}{\bar{\sigma}\sqrt{S_t}} - l_t^{S_t} + \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right) - e^{\varphi_t t - t\bar{\sigma}^2\eta_t S_t} N\left(\frac{y}{\bar{\sigma}\sqrt{S_t}} - l_t^{S_t} - \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right)\right] \quad .
\end{aligned}$$

We define directly the B&S option prices w.r.t. y,

$$\begin{aligned}
c_t^{BS}(\mathcal{I}_t(y), y) &= N\left(-\frac{y}{\mathcal{I}_t(y)} + \frac{\mathcal{I}_t(y)\sqrt{t}}{2}\right) - e^{y\sqrt{t}} N\left(-\frac{y}{\mathcal{I}_t(y)} - \frac{\mathcal{I}_t(y)\sqrt{t}}{2}\right) \\
p_t^{BS}(\mathcal{I}_t(y), y) &= e^{y\sqrt{t}} N\left(\frac{y}{\mathcal{I}_t(y)} + \frac{\mathcal{I}_t(y)\sqrt{t}}{2}\right) - N\left(\frac{y}{\mathcal{I}_t(y)} - \frac{\mathcal{I}_t(y)\sqrt{t}}{2}\right) \quad ,
\end{aligned}$$

where $\mathcal{I}_t(y)$ is the implied volatility w.r.t. the *moneyness degree*. Moreover, we define $c_t(S_t, y)$ and $p_t(S_t, y)$ such that

$$\begin{aligned}
\mathbb{E}[c_t\left(S_t, y\right)] &:= C_t(y\sqrt{t}) \\
\mathbb{E}[p_t\left(S_t, y\right)] &:= P_t(y\sqrt{t}) \quad .
\end{aligned}$$

The implied volatility equation for the call options is

$$\mathbb{E}[c_t\,(S_t,y)] = c_t^{BS}(\mathcal{I}_t(y),y) \tag{7}$$

and the implied volatility equation for the put option is

$$\mathbb{E}[p_t\,(S_t,y)] = p_t^{BS}(\mathcal{I}_t(y),y) \ \ . \tag{8}$$

Alòs *et al.* (2007, Lemma 6.1, p.580) prove that, for a generalization of the Bates model, $\hat{\sigma}_t\sqrt{t} = o(1)$. The same proof holds in the ATS case because all additive processes goes to zero in distribution at short time. We use this property to study the short time asymptotic of the ATM B&S price. Notice that, if at short time $\hat{\sigma}_t\sqrt{t} = o(1)$, we can rewrite the right hand side of (7) and (8) for $y = 0$ as

$$c_t^{BS}(\hat{\sigma}_t,0) = p_t^{BS}(\hat{\sigma}_t,0) = \hat{\sigma}_t\sqrt{\frac{t}{2\pi}} + o\left(\hat{\sigma}_t\sqrt{t}\right) \ \ , \tag{9}$$

the asymptotic expansion is because $N'(0) = \sqrt{\frac{1}{2\pi}}$, where N' is the standard normal probability density function.

3 Short time ATM implied volatility

In this Section, we study the behavior of $\hat{\sigma}_t$ at short time. The idea of the proofs is simple. Equation (9) is the short time asymptotic expansion of the B&S ATM call and put prices. We can study the short time behavior of the ATS model price in (7) and (8).

1. If the model price goes to zero faster than $\sqrt{t}$, then $\hat{\sigma}_0 = 0$ (Case 1).
2. If the model price goes to zero slower than $\sqrt{t}$, then $\hat{\sigma}_0 = \infty$ (Case 2).
3. If the model price goes to zero as $\sqrt{t}$, then $\hat{\sigma}_0$ is finite (Cases 3, 4, 5).

The idea of the proofs is the following. In Case 1 we bound from above the model price and we prove that it is $o\left(\sqrt{t}\right)$. In Case 2 we bound from below the model price and we demonstrate that it goes to zero slower than $\sqrt{t}$. Finally in the remaining Cases we build upper and lower bounds on the model price and prove that both bounds are $O\left(\sqrt{t}\right)$. Furthermore, the proofs are divided in some sub-cases that correspond to particular ranges of the parameters δ and β: we indicate with bold characters the range at the beginning of each sub-case.

Proposition 3.1.
For Case 1: $\begin{cases} -\min\left(\frac{1}{2},\beta\right) < \delta \leq 0 \ \ \& \ \ \beta < 1 \ \ \textit{or} \\ \delta = \beta = 0 \end{cases}$,
then, the implied volatility is s.t.

$$\hat{\sigma}_0 = 0 \ \ .$$

Proof.

$$\boldsymbol{-\min\left(\tfrac{1}{2},\beta\right) < \delta \leq 0 \ \&\ \beta < 1 \ \textbf{or}\ \delta = \beta = 0}$$

We bound $c_t(S_t, 0)$ from above as follows.

$$
\begin{aligned}
c_t(S_t, 0) = & N\left(l_t^{S_t} + \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right) - N\left(l_t^{S_t} - \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right) - \left(e^{\varphi_t t - t\bar{\sigma}^2\eta_t S_t} - 1\right) N\left(l_t^{S_t} + \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right) \\
\leq & \sqrt{\frac{t}{2\pi}}\ \bar{\sigma}\ \sqrt{S_t} + e^{\varphi_t t} - 1 \quad .
\end{aligned} \tag{10}
$$

In the equality we have just added and subtracted the quantity $N\left(l_t^{S_t} + \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right)$. The inequality holds because, by definition of standard normal cumulative distribution function,

$$
N\left(l_t^{S_t} + \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right) - N\left(l_t^{S_t} - \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right) = \frac{1}{\sqrt{2\pi}} \int_{l_t^{S_t} - \bar{\sigma}\frac{\sqrt{S_t t}}{2}}^{l_t^{S_t} + \bar{\sigma}\frac{\sqrt{S_t t}}{2}} dz\ e^{-z^2/2} \leq \sqrt{\frac{t}{2\pi}}\ \bar{\sigma}\sqrt{S_t}\ , \tag{11}
$$

and because we bound from above the product $\left(e^{\varphi_t t - t\bar{\sigma}^2\eta_t S_t} - 1\right) N\left(l_t^{S_t} + \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right)$ with the (positive) maximums of both factors.
We bound the expected value of $c_t(S_t, 0)$ as

$$
\mathbb{E}[c_t(S_t, 0)] \leq \mathbb{E}\left[\sqrt{\frac{t}{2\pi}}\bar{\sigma}\sqrt{S_t}\right] + e^{\varphi_t t} - 1 = \sqrt{\frac{t}{2\pi}}\ \bar{\sigma}\ \mathbb{E}[\sqrt{S_t}] + o\left(\sqrt{t}\right) = o\left(\sqrt{t}\right) \quad .
$$

The first equality holds because $e^{\varphi_t t} - 1 = O(\varphi_t t) = o\left(\sqrt{t}\right)$ and the last equality because $\mathbb{E}[\sqrt{S_t}]$ goes to zero at short time (see **Lemma A.5**).

Summarizing, the upper bound to the ATS ATM price in (7) is $o\left(\sqrt{t}\right)$. From (9) we have that the B&S price is $O\left(\hat{\sigma}_t\sqrt{t}\right)$. Thus,

$$
\hat{\sigma}_0 = 0
$$

□

Proposition 3.2.
For Case 2: $-\min\left(\beta, \frac{1-\beta(1-\alpha)}{\alpha}\right) < \delta < -\max\left(\frac{\beta}{2}, \frac{1}{2}\right)$,
then,

$$
\hat{\sigma}_0 = \infty \quad .
$$

Proof.
We divide the proof in two sub-cases.

$$
\boldsymbol{-\beta < \delta < -\tfrac{1}{2}\ \&\ \beta \leq 1}
$$

Consider the left hand side of equation (7). We compute the derivative of $c_t(z, y)$ w.r.t. z in $y = 0$.

$$
\begin{aligned}
\frac{\partial c_t(z, 0)}{\partial z} = & - t\bar{\sigma}^2\eta_t e^{\varphi_t t - t\bar{\sigma}^2\eta_t z} N\left(l_t^z + \bar{\sigma}\frac{\sqrt{zt}}{2}\right) \\
& - \left(\frac{\varphi_t\sqrt{t}}{2\bar{\sigma}z^{3/2}} + \frac{\sqrt{t}\bar{\sigma}\eta_t}{2\sqrt{z}}\right)\left(e^{\varphi_t t - t\bar{\sigma}^2\eta_t z} N'\left(l_t^z + \bar{\sigma}\frac{\sqrt{zt}}{2}\right) - N'\left(l_t^z - \bar{\sigma}\frac{\sqrt{zt}}{2}\right)\right) \\
& + \frac{\sqrt{t}\bar{\sigma}}{4\sqrt{z}}\left(e^{\varphi_t t - t\bar{\sigma}^2\eta_t z} N'\left(l_t^z + \bar{\sigma}\frac{\sqrt{zt}}{2}\right) + N'\left(l_t^z - \bar{\sigma}\frac{\sqrt{zt}}{2}\right)\right) \quad .
\end{aligned} \tag{12}
$$

At short time, for a given $z \in \left(0, \frac{\varphi_t}{\bar\sigma^2\eta_t}\right)$, $l_t^z = \frac{\sqrt{t}\bar\sigma\eta_t}{\sqrt{z}}\left(-z + \frac{\varphi_t}{\bar\sigma^2\eta_t}\right) > 0$. We observe that $e^{\varphi_t t - t\bar\sigma^2\eta_t z} = 1 + o(1)$ and $\lim_{t\to 0} l_t^z = \infty$ due to **Lemma A.6** point 1; then, $N\left(l_t^z + \bar\sigma\frac{\sqrt{zt}}{2}\right) = 1 + o(1)$. Thus,

$$\frac{\partial c_t(z,0)}{\partial z} = -t\bar\sigma^2\eta_t + o(t\eta_t) \ ,$$

because the first term goes to zero as $t\eta_t$ while the second and the third terms go to zero as $N'(\sqrt{t}\,\eta_t)$ (i.e. as a negative exponential). Thus, for sufficiently small t, $c_t(z,0)$ is decreasing w.r.t. z in $(0, \frac{\varphi_t}{\bar\sigma^2\eta_t})$. We emphasize that the right extreme of the interval is increasing to one for sufficiently small t, see **Lemma A.6** points 2 and 3.
Fix $\tau > 0$ and $S^* \in (0, \frac{\varphi_\tau}{\bar\sigma^2\eta_\tau})$; for any $t < \tau$

$$\begin{aligned}
&\mathbb{E}\left[c_t(S_t,0)\right] \\
\geq &c_t(S^*,0)\mathbb{P}\left(S_t \leq S^*\right) \\
\geq &\left\{N\left(l_t^{S^*} + \bar\sigma\frac{\sqrt{S^*t}}{2}\right) - N\left(l_t^{S^*} - \bar\sigma\frac{\sqrt{S^*t}}{2}\right) + \left(\varphi_t t - t\bar\sigma^2\eta_t S^*\right) N\left(l_t^{S^*} + \bar\sigma\frac{\sqrt{S^*t}}{2}\right)\right\}\mathbb{P}\left(S_t \leq S^*\right) \\
\geq &\left(\varphi_t t - t\bar\sigma^2\eta_t S^*\right) N\left(l_t^{S^*} + \bar\sigma\frac{\sqrt{S^*t}}{2}\right)\mathbb{P}\left(S_t \leq S^*\right) = O(t\eta_t) \ .
\end{aligned}$$

The first inequality holds because $c_t(z,0)$ is positive for any $z \geq 0$ and because we bound from below the expected value with its minimum in the interval $(0, S^*)$ multiplied by the probability of the interval, $\mathbb{P}\left(S_t \leq S^*\right)$. The second inequality is due to the fact that $e^x \geq x+1$. Finally, the last inequality holds because, by definition of the standard normal cumulative distribution function,

$$N\left(l_t^z + \bar\sigma\frac{\sqrt{zt}}{2}\right) - N\left(l_t^z - \bar\sigma\frac{\sqrt{zt}}{2}\right) \geq 0 \ , \quad z \in \mathbb{R}^+ \ . \tag{13}$$

Recall that $N\left(l_t^{S^*} + \bar\sigma\frac{\sqrt{S^*t}}{2}\right) = 1 + o(1)$; notice that $\mathbb{P}\left(S_t \leq S^*\right)$ is constant for $\beta = 1$ and goes to one, by **Lemma A.4** point 1, for $\beta < 1$. This proves the last equality.

$$\boldsymbol{-\frac{1-\beta(1-\alpha)}{\alpha} < \delta < -\frac{\beta}{2} \ \& \ \beta > 1}$$

It exits q such that $(\beta-1)/2 < q < -\delta - 1/2$. We bound the ATM put price (5) from below for a sufficiently small t

$$\begin{aligned}
&\mathbb{E}[p_t\left(S_t,0\right)] \\
\geq &\mathbb{E}\left[\mathbb{1}_{S_t \geq 1+t^q}\left(N\left(-l_t^{S_t} + \bar\sigma\frac{\sqrt{S_t t}}{2}\right) - N\left(-l_t^{S_t} - \bar\sigma\frac{\sqrt{S_t t}}{2}\right) + N\left(-l_t^{S_t} - \bar\sigma\frac{\sqrt{S_t t}}{2}\right)\left(1 - e^{\varphi_t t - t\bar\sigma^2\eta_t S_t}\right)\right)\right] \\
\geq &\mathbb{E}\left[\mathbb{1}_{S_t \geq 1+t^q} N\left(-l_t^{S_t} - \bar\sigma\frac{\sqrt{S_t t}}{2}\right)\left(1 - e^{\varphi_t t - t\bar\sigma^2\eta_t S_t}\right)\right] \\
\geq &\mathbb{P}(S_t \geq 1+t^q)\frac{1}{3}\left(1 - e^{\varphi_t t - t\bar\sigma^2\eta_t(1+t^q)}\right) =: M_t\left(t^{1+q}\bar\sigma^2\eta_t + t\bar\sigma^4\eta_t^2 k_t/2\right) \\
\geq &M_t\, t^{1+q}\bar\sigma^2\eta_t \ .
\end{aligned} \tag{14}$$

The first inequality holds because $p_t\left(S_t,0\right)$ is non negative and because we have added and subtracted the term $N\left(-l_t^{S_t} - \bar\sigma\frac{\sqrt{S_t t}}{2}\right)$. The second because the difference between the standard normal cumulative distribution functions is non negative, analogously to (13). The third because, for $S_t \in [1,\infty)$, $1-e^{\varphi_t t - t\bar{\sigma}^2\eta_t S_t}$ is positive and non decreasing in S_t; moreover, for a sufficiently small t, $N\left(-l_t^{S_t} - \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right) > 1/3$ because

$$\lim_{t\to 0} N\left(-l_t^z - \bar{\sigma}\frac{\sqrt{zt}}{2}\right) \geq 1/2 \quad , \quad z \in [1,\infty) \ .$$

The quantity M_t is defined in (14). At short time $M_t = 1/6 + o(1)$ because i) by **Lemma B.3**, $\mathbb{P}(S_t \geq 1 + t^q)$ goes to $1/2$ as t goes to zero, and ii) by **Lemma A.6** point 1,

$$1 - e^{\varphi_t t - t\bar{\sigma}^2\eta_t(1+t^q)} = \left(t^{1+q}\bar{\sigma}^2\eta_t + t\bar{\sigma}^4\eta_t^2 k_t/2\right)(1+o(1)) \quad .$$

Notice that $t^{1+q}\eta_t$ goes to zero slower than $\sqrt{t}$.

$$\textbf{Case 2: } -\min\left(\beta, \frac{1-\beta(1-\alpha)}{\alpha}\right) < \delta < -\max\left(\frac{\beta}{2}, \frac{1}{2}\right)$$

Summing up, for both sub-cases, $-\beta < \delta < -1/2$ & $\beta \leq 1$ and $-\frac{1-\beta(1-\alpha)}{\alpha} < \delta < -\beta/2$ & $\beta > 1$, the lower bounds on the ATM option prices in (7) and (8) go to zero slower than $\sqrt{t}$.
Moreover, from (9) we have that the B&S price is $O\left(\hat{\sigma}_t\sqrt{t}\right)$. Then,

$$\hat{\sigma}_0 = \infty$$

□

Proposition 3.3.
For Case 3: $\delta \geq -\beta/2$ & $\beta \geq 1$, with the exception of the point $\{\delta = -1/2, \beta = 1\}$, then,

$$\hat{\sigma}_0 \ \text{ is finite} \ .$$

Proof.
We split the proof in three sub-cases. For each sub-case we build an upper and a lower bound, on the model price, and we demonstrate that both bounds are $O\left(\sqrt{t}\right)$ and then, that $\hat{\sigma}_0$ is finite.

$$-\frac{\beta}{2} <= \delta < -\frac{1}{2} \ \& \ \beta > 1$$

Upper bound.

Let us split the expected value of the ATS call in two parts

$$\begin{aligned} &\mathbb{E}[c_t(S_t, 0)] \\ = \ &\mathbb{E}\left[N\left(l_t^{S_t} + \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right) - N\left(l_t^{S_t} - \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right)\right] + \mathbb{E}\left[N\left(l_t^{S_t} + \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right)\left(e^{\varphi_t t - t\bar{\sigma}^2\eta_t S_t} - 1\right)\right] \\ =: \ &A_1(t) + A_2(t) \ . \end{aligned}$$

We prove that both parts are bounded from above by quantities $O\left(\sqrt{t}\right)$. The first expected value is s.t.

$$A_1(t) \leq \sqrt{\frac{t}{2\pi}}\,\bar{\sigma}\,\mathbb{E}[\sqrt{S_t}] = O\left(\sqrt{t}\right) \quad , \tag{15}$$

where the inequality holds true because of (11) and $\sqrt{t}\ \mathbb{E}[\sqrt{S_t}] = O\left(\sqrt{t}\right)$ because, by **Lemma A.5** point 1, $\mathbb{E}[\sqrt{S_t}]$ goes to one as t goes to zero.
Let us study the term $A_2(t)$.

$$\begin{aligned}
A_2(t) &< \mathbb{E}\left[\left(e^{\varphi_t t - t\bar{\sigma}^2\eta_t S_t} - 1\right)\mathbb{1}_{S_t<\varphi_t/(\bar{\sigma}^2\eta_t)}\right] \\
&= \sqrt{\frac{t}{2\pi k_t}}\int_0^{\varphi_t/(\bar{\sigma}^2\eta_t)} dz e^{-\frac{t(z-1)^2}{2k_t}}\left(e^{\varphi_t t - t\bar{\sigma}^2\eta_t z} - 1\right) && (16)\\
&\quad + \int_0^{\varphi_t/(\bar{\sigma}^2\eta_t)} dz \left(\mathcal{P}_{S_t}(z) - \sqrt{\frac{t}{2\pi k_t}}\, e^{-\frac{t(z-1)^2}{2k_t}}\right)\left(e^{\varphi_t t - t\bar{\sigma}^2\eta_t z} - 1\right) && (17)\\
&\le O\left(t^{\delta+(\beta+1)/2}\right) \quad ,
\end{aligned}$$

where $\mathcal{P}_{S_t}$ is the law of S_t. The first inequality is true because the quantity inside the expected value is positive on $\left(0, \frac{\varphi_t}{\bar{\sigma}^2\eta_t}\right)$ and negative elsewhere. The equality is obtained by adding and subtrancting the same expected value for a Gaussian random variable. We prove the second inequality in two steps, showing that both (16) and (17) are bounded by $O\left(t^{\delta+(\beta+1)/2}\right)$.
First, we consider (16)

$$\begin{aligned}
&\sqrt{\frac{t}{2\pi k_t}}\int_0^{\varphi_t/(\bar{\sigma}^2\eta_t)} dz\ e^{-\frac{t(z-1)^2}{2k_t}}\left(e^{\varphi_t t - t\bar{\sigma}^2\eta_t z} - 1\right) \\
=&\frac{1}{\sqrt{2\pi}}\int_{\mathcal{A}_t} dw\ e^{-\frac{w^2}{2}}\left(e^{\varphi_t t - t\bar{\sigma}^2\eta_t(1+w\sqrt{k_t/t})} - 1\right) && (18)\\
\le&\frac{1}{\sqrt{2\pi}}\int_{\mathcal{A}_t} dw\ e^{-\frac{w^2}{2}}e^{-\sqrt{t}\bar{\sigma}^2\eta_t\sqrt{k_t}w} - \frac{1}{\sqrt{2\pi}}\int_{\mathcal{A}_t} dw\ e^{-\frac{w^2}{2}} && (19)\\
=&e^{t\bar{\sigma}^4\eta_t^2 k_t/2}N\left(\sqrt{t}\bar{\sigma}^2\eta_t\sqrt{k_t} + \left(\frac{\varphi_t}{\bar{\sigma}^2\eta_t} - 1\right)\sqrt{\frac{t}{k_t}}\right) - N\left(\left(\frac{\varphi_t}{\bar{\sigma}^2\eta_t} - 1\right)\sqrt{\frac{t}{k_t}}\right) + o\left(t^{\delta+(\beta+1)/2}\right) && (20)\\
=&\sqrt{\frac{t}{2\pi}}\bar{\sigma}^2\eta_t\sqrt{k_t} + \frac{t\bar{\sigma}^4\eta_t^2 k_t}{4} + o\left(t^{\delta+(\beta+1)/2}\right) = O\left(t^{\delta+(\beta+1)/2}\right) \quad , && (21)
\end{aligned}$$

where $\mathcal{A}_t \equiv \left\{w \in \mathbb{R} : -\sqrt{\frac{t}{k_t}} < w < (\varphi_t/(\bar{\sigma}^2\eta_t) - 1)\sqrt{\frac{t}{k_t}}\right\}$. Equality (18) is due to a change of the integration variable $w := (z-1)/\sqrt{k_t/t}$, equality (19) to the fact that, by **Lemma A.6**, $e^{\varphi_t t - t\bar{\sigma}^2\eta_t} < 1$. Equality (20) to a change of variable $m := w + \sqrt{t}\bar{\sigma}^2\eta_t\sqrt{k_t}$ and to the fact that both $N\left(-\sqrt{\frac{t}{k_t}}\right)$ and $N\left(\sqrt{t}\bar{\sigma}^2\eta_t\sqrt{k_t} - \sqrt{\frac{t}{k_t}}\right)$ go to zero faster that any power of t. Finally, (21) holds true because of the Taylor expansion of N in zero.

Second, we consider (17)

$$\begin{aligned}
&\left|\int_0^{\varphi_t/(\bar\sigma^2\eta_t)} dz \left(\mathcal{P}_{S_t}(z) - \sqrt{\frac{t}{2\pi k_t}}\, e^{-\frac{t(z-1)^2}{2k_t}}\right)\left(e^{\varphi_t t - t\bar\sigma^2\eta_t z} - 1\right)\right| \\
\leq & \left| -\left(\mathbb{P}(S_t < 0) - N\left(-\sqrt{\frac{t}{k_t}}\right)\right)\left(e^{\varphi_t t} - 1\right)\right| \\
& + \left|\int_0^{\varphi_t/(\bar\sigma^2\eta_t)} dz \left(\mathbb{P}(S_t < z) - N\left((z-1)\sqrt{\frac{t}{k_t}}\right)\right)\bar\sigma^2\,\eta_t\, t\, e^{\varphi_t t - t\bar\sigma^2\eta_t z}\right| \\
\leq & 2\frac{2-\alpha}{1-\alpha}\sqrt{\frac{k_t}{t}}\left(e^{\varphi_t t} - 1\right) = O\left(t^{\delta+(\beta+1)/2}\right) \quad .
\end{aligned}$$

The first inequality is due to integration by part and to the triangular inequality. The second inequality is a consequence of Jensen inequality and of **Lemma B.3**.

Lower bound.

As discussed in the proof of **Proposition 3.2**, for a sufficiently small t, $c_t\left(S_t, 0\right)$ is decreasing for $S_t \in \left(0, \frac{\varphi_t}{\bar\sigma^2\eta_t}\right)$ hence,

$$\begin{aligned}
\mathbb{E}\left[c_t(S_t,0)\right] &\geq \mathbb{E}\left[c_t(S_t,0)\mathbb{1}_{S_t\leq\varphi_t/(\bar\sigma^2\eta_t)}\right] \\
&\geq c_t\left(\frac{\varphi_t}{\bar\sigma^2\eta_t}, 0\right)\mathbb{P}\left(S_t \leq \frac{\varphi_t}{\bar\sigma^2\eta_t}\right) = \sqrt{\frac{\varphi_t t}{8\pi\eta_t}} + o\left(\sqrt{t}\right) = O\left(\sqrt{t}\right) \quad .
\end{aligned}$$

The first inequality is because $c_t\left(S_t, 0\right)$ is non negative and the second is because we bound the expected value from below with the minimum of $c_t\left(S_t, 0\right)$ multiplied by the probability of the interval $\left(0, \frac{\varphi_t}{\bar\sigma^2\eta_t}\right)$. The equality holds because, by **Lemma B.3**,

$$\lim_{t\to 0}\mathbb{P}\left(S_t \leq \frac{\varphi_t}{\bar\sigma^2\eta_t}\right) = \frac{1}{2} \quad ,$$

and

$$c_t\left(\frac{\varphi_t}{\bar\sigma^2\eta_t}, 0\right) = N\left(\sqrt{\frac{\varphi_t\, t}{4\bar\sigma\eta_t}}\right) - N\left(-\sqrt{\frac{\varphi_t\, t}{4\bar\sigma\eta_t}}\right) = \sqrt{\frac{\varphi_t\, t}{2\pi\bar\sigma\eta_t}} + o\left(\sqrt{t}\right) \quad ,$$

with $\sqrt{\frac{\varphi_t\, t}{8\pi\bar\sigma\eta_t}} = O\left(\sqrt{t}\right)$.

$$\boldsymbol{\delta = -\tfrac{1}{2}\ \&\ \beta > 1}$$

Upper bound.

The upper bound on the ATS call price is the same to the one of the previous sub-case $-\beta/2 \leq \delta < -1/2,\ \beta > 1$.

Lower bound.

We bound the put price from below. It exist $H > 1$ such that for a sufficiently small t

$$\mathbb{E}\left[p_t\left(S_t, 0\right)\right] \geq \mathbb{E}\left[p_t\left(S_t, 0\right) \mathbb{1}_{S_t \in [1,H]}\right] \geq \mathbb{E}\left[\left(N\left(-l_t^{S_t} + \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right) - N\left(-l_t^{S_t} - \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right)\right) \mathbb{1}_{S_t \in [1,H]}\right]$$

$$\geq \mathbb{E}\left[N'\left(-l_t^{S_t} + \frac{\bar{\sigma}\sqrt{S_t t}}{2}\right) \bar{\sigma}\sqrt{S_t t}\, \mathbb{1}_{S_t \in [1,H]}\right]$$

$$\geq N'\left(\bar{\sigma}\bar{\eta} - \frac{\varphi_t \sqrt{t}}{\bar{\sigma}} + \frac{\bar{\sigma}\sqrt{t}}{2}\right) \bar{\sigma}\sqrt{t}\, \mathbb{P}(S_t \in [1, H]) = \sqrt{\frac{t}{8\pi}}\bar{\sigma} + o\left(\sqrt{t}\right) \quad .$$

The first inequality holds because $p_t(S_t, 0)$ is non negative. The second because $e^{\varphi_t t - t\bar{\sigma}^2 \eta_t S_t} < 1$ in $[1, H]$. The third inequality is due to the fact that we bound from above the difference of the two standard normal cumulative distribution functions with the standard normal law evaluated in the maximum between the two (positive) arguments multiplied by the (positive) difference of the two arguments. The last inequality holds because, by **Lemma B.4**, it exists $H > 1$ s.t. the quantity inside the expected value is increasing in $[1, H]$ for a sufficiently small t. The equality is because, by **Lemma B.3**, $\mathbb{P}(S_t \in [1, H])$ goes to $1/2$ as t goes to zero and

$$\lim_{t\to 0} N'\left(\bar{\sigma}\bar{\eta} - \frac{\varphi_t \sqrt{t}}{\bar{\sigma}} + \frac{\bar{\sigma}\sqrt{t}}{2}\right) = \frac{1}{\sqrt{2\pi}} \quad .$$

$$\boldsymbol{-\tfrac{1}{2} < \delta \leq 0 \;\&\; \beta \geq 1}$$

Upper bound.

We can bound $c_t(S_t, 0)$ from above as in (10).
We bound the ATS option price as

$$\mathbb{E}[c_t(S_t, 0)] \leq \mathbb{E}\left[\frac{1}{\sqrt{2\pi}}\bar{\sigma}\sqrt{S_t t}\right] + e^{\varphi_t t} - 1 \leq O\left(\sqrt{t}\right) \quad . \tag{22}$$

The last inequality holds because, by Jensen inequality with concave function $\sqrt{*}$, $\mathbb{E}[\sqrt{S_t}] \leq \sqrt{\mathbb{E}[S_t]} = 1$ and because, by **Lemma A.6** point 1, $e^{\varphi_t t} - 1 = o\left(\sqrt{t}\right)$.

Lower bound.

To bound $c_t(z, 0)$ from below we have to study its derivative in (12). Notice that, at short time, $l_t^z = O\left(\sqrt{t}\eta_t\right) = o(1)$, due to **Lemma A.6** point 1, and to the fact that $\delta > -1/2$. Moreover, again due to **Lemma A.6** point 1, $e^{\varphi_t t - t\bar{\sigma}^2 \eta_t z} = 1 + O\left(t\eta_t\right)$. Then, we have
i) The negative first term at short time is $o\left(\sqrt{t}\right)$

$$-t\bar{\sigma}^2\eta_t e^{\varphi_t t - t\bar{\sigma}^2\eta_t z} N\left(l_t^z + \bar{\sigma}\frac{\sqrt{zt}}{2}\right) = O\left(t\eta_t\right) = o\left(\sqrt{t}\right) \quad .$$

ii) The second term at short time is $o\left(\sqrt{t}\right)$

$$\left(\frac{\varphi_t\sqrt{t}}{2\bar{\sigma}z^{3/2}} + \frac{\sqrt{t}\bar{\sigma}\eta_t}{2\sqrt{z}}\right)\left(e^{\varphi_t t - t\bar{\sigma}^2\eta_t z} N'\left(l_t^z + \bar{\sigma}\frac{\sqrt{zt}}{2}\right) - N'\left(l_t^z - \bar{\sigma}\frac{\sqrt{zt}}{2}\right)\right)$$
$$= O\left(\sqrt{t}\eta_t\right)\frac{e^{-(l_t^z)^2/2 - \bar{\sigma}^2 zt/8}}{\sqrt{2\pi}}\left(\left(1 + O(t\eta_t)\right)\left(1 - \frac{l_t^z\bar{\sigma}\sqrt{zt}}{2} + o(t\eta_t)\right) - \left(1 + \frac{l_t^z\bar{\sigma}\sqrt{zt}}{2} + o(t\eta_t)\right)\right)$$
$$= O(\eta_t^2 t^{3/2}) = o\left(\sqrt{t}\right) \quad ,$$

because

$$N'\left(l_t^z \pm \bar{\sigma}\frac{\sqrt{zt}}{2}\right) = e^{-(l_t^z)^2/2-\bar{\sigma}^2 zt/8}\left(1 + \pm\frac{l_t^z\bar{\sigma}\sqrt{zt}}{2} + o(t\eta_t)\right) \quad .$$

iii) The positive third term at short time is $O\left(\sqrt{t}\right)$

$$\frac{\sqrt{t}\bar{\sigma}}{4\sqrt{z}}\left(e^{\varphi_t t - t\bar{\sigma}^2\eta_t z}N'\left(l_t^z + \bar{\sigma}\frac{\sqrt{zt}}{2}\right) + N'\left(l_t^z - \bar{\sigma}\frac{\sqrt{zt}}{2}\right)\right) = \sqrt{\frac{t}{8\pi\, z}}\,\bar{\sigma} + o\left(\sqrt{t}\right) \quad .$$

Summarizing, the leading term in (12), at short time, is the third one, which is positive. Hence, for a fixed $z > 0$ and for sufficiently small t, $c_t(z, 0)$ is increasing; thus, we can bound the expected value from below

$$\begin{aligned}
\mathbb{E}\left[c_t\left(S_t, 0\right)\right] &\geq \mathbb{E}\left[c_t\left(S_t, 0\right)\mathbb{1}_{S_t\in[1/2,3/2]}\right] > c_t\left(\frac{1}{2}, 0\right)\mathbb{P}\left(S_t \in \left[\frac{1}{2}, \frac{3}{2}\right]\right) \\
&> \left\{N\left(l_t^{1/2} + \bar{\sigma}\sqrt{\frac{t}{8}}\right) - N\left(l_t^{1/2} - \bar{\sigma}\sqrt{\frac{t}{8}}\right)\right\}\mathbb{P}\left(S_t \in \left[\frac{1}{2}, \frac{3}{2}\right]\right) \\
&> N'\left(l_t^{1/2} + \bar{\sigma}\sqrt{\frac{t}{8}}\right)\bar{\sigma}\sqrt{\frac{t}{2}}\mathbb{P}\left(S_t \in \left[\frac{1}{2}, \frac{3}{2}\right]\right) \\
&= \left(\bar{\sigma}\sqrt{\frac{t}{4\pi}} + o\left(\sqrt{t}\right)\right)\mathbb{P}\left(S_t \in \left[\frac{1}{2}, \frac{3}{2}\right]\right) = O\left(\sqrt{t}\right) \quad .
\end{aligned}$$

The first inequality holds because $c_t(S_t, 0)$ is non negative. The second because, for a sufficiently small t, $c_t(S_t, 0)$ is increasing. The third is true because, for sufficiently small t, $e^{\varphi_t t - t\bar{\sigma}^2\eta_t/2} > 1$, by **Lemma A.6** point 3. The forth is due to the fact that the difference of the standard normal cumulative distribution functions can be bounded from below by the (positive) maximum of the two arguments multiplied by the (positive) difference of the two arguments. The equality is due to the fact that $\mathbb{P}\left(S_t \in \left[\frac{1}{2}, \frac{3}{2}\right]\right)$ is constant if $\beta = 1$ and goes to 1 at short time if $\beta > 1$ because, by **Lemma A.4** point 2, S_t goes to one in distribution at short time.

Case 3: $\boldsymbol{-\frac{\beta}{2} \leq \delta \leq 0}$ **&** $\boldsymbol{\beta \geq 1 \setminus \left\{\delta = -\frac{1}{2},\ \beta = 1\right\}}$

Summing up, in all sub-cases the upper bound and the lower bounds of the ATS option prices in (7) and (8) are $O(\sqrt{t})$. Moreover, from (9) we have that the B&S price is $O\left(\hat{\sigma}_t\sqrt{t}\right)$. Thus,

$$\hat{\sigma}_0 \;\text{ is finite} \qquad \square$$

Proposition 3.4.
For Cases 4 and 5: $\delta = -\frac{1}{2}$ *&* $\beta \leq 1$*,*
then,

$$\hat{\sigma}_0 \;\textit{ is finite} \quad .$$

Proof.

$$\boldsymbol{\delta = -\tfrac{1}{2}\ \&\ \beta \leq 1}$$

Upper bound.

We can bound $c_t(S_t, 0)$ from above as in (10).
We bound the ATS option price as

$$\mathbb{E}[c_t(S_t, 0)] \leq \mathbb{E}\left[\frac{1}{\sqrt{2\pi}}\bar{\sigma}\sqrt{S_t t}\right] + e^{\varphi_t t} - 1 = O\left(\sqrt{t}\right) \quad . \tag{23}$$

The equality holds because, by Jensen inequality with concave function $\sqrt{*}$, $\mathbb{E}[\sqrt{S_t}] \leq \sqrt{\mathbb{E}[S_t]} = 1$ and because, by **Lemma A.6** point 1, $e^{\varphi_t t} - 1 = O\left(\sqrt{t}\right)$.

Lower bound.

We bound $c_t\left(S_t, 0\right)$ from below as:

$$\begin{aligned} c_t\left(S_t, 0\right) &\geq \mathbb{1}_{S_t < \varphi_t/(\bar{\sigma}^2\eta_t)} \left(N\left(l_t^{S_t} + \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right) - N\left(l_t^{S_t} - \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right) + \left(\varphi_t t - t\bar{\sigma}^2\eta_t S_t\right)/2\right) \\ &\geq \mathbb{1}_{S_t < \varphi_t/(\bar{\sigma}^2\eta_t)} \left(\varphi_t t - t\bar{\sigma}^2\eta_t S_t\right)/2 \quad . \end{aligned}$$

The first inequality is because $c_t\left(S_t, 0\right)$ is non negative, because $e^x \geq x + 1$, and because the normal cumulative distribution function evaluated in a positive quantity is above $1/2$. The second holds because the difference between the two normal cumulative function is non negative.

$$\begin{aligned} \mathbb{E}[c_t\left(S_t, 0\right)] &\geq \mathbb{E}\left[\mathbb{1}_{S_t < \varphi_t/(\bar{\sigma}^2\eta_t)}(\varphi_t t - t\bar{\sigma}^2\eta_t S_t)/2\right] \\ &= \sqrt{t}\,\bar{\sigma}^2\,\bar{\eta}/2\;\mathbb{E}\left[\mathbb{1}_{S_t < \varphi_t/(\bar{\sigma}^2\eta_t)}(-S_t + \varphi_t/(\bar{\sigma}^2\eta_t))\right] = O\left(\sqrt{t}\right). \end{aligned}$$

The last equality is due to the fact that

$$\mathbb{E}\left[\mathbb{1}_{S_t < \varphi_t/(\bar{\sigma}^2\eta_t)}(-S_t + \varphi_t/(\bar{\sigma}^2\eta_t))\right] = \varphi_t/(\bar{\sigma}^2\eta_t)\mathbb{P}(S_t < \varphi_t/(\bar{\sigma}^2\eta_t)) - \mathbb{E}[S_t\mathbb{1}_{S_t < \varphi_t/(\bar{\sigma}^2\eta_t)}] \tag{24}$$

can be bounded from below with a positive constant for sufficiently small t. This fact can be deduced for $\beta \leq 1$. We prove it separately for the two cases $\beta < 1$ and $\beta = 1$.
For $\beta < 1$, let us observe that, at short time,

$$0 \leq \mathbb{E}[S_t\mathbb{1}_{S_t < \varphi_t/(\bar{\sigma}^2\eta_t)}] \leq \mathbb{E}[S_t\mathbb{1}_{S_t<1}] = o(1) \quad ,$$

because, by point 2 of **Lemma A.6**, $\varphi_t/(\bar{\sigma}^2\eta_t) < 1$ and, by definition of convergence in distribution, at short time $\mathbb{E}[S_t\mathbb{1}_{S_t<1}] = o(1)$, because, by **Lemma A.4** point 1, S_t converges in distribution to 0. Moreover, at short time, $\varphi_t/(\bar{\sigma}^2\eta_t)\mathbb{P}(S_t < \varphi_t/(\bar{\sigma}^2\eta_t)) = 1 + o(1)$, by point 1 of **Lemma A.6** and by point 1 of **Lemma A.4**.
For $\beta = 1$, we remind that the law of S_t does not depend from t and we observe that the limit of (24) for t that goes to zero is positive

$$\lim_{t\to 0}\left\{\varphi_t/(\bar{\sigma}^2\eta_t)\mathbb{P}(S_t < \varphi_t/(\bar{\sigma}^2\eta_t)) - \mathbb{E}[S_t\mathbb{1}_{S_t < \varphi_t/(\bar{\sigma}^2\eta_t)}]\right\} = \mathbb{P}(S_t < 1) - \mathbb{E}[S_t\mathbb{1}_{S_t<1}] > 0 \quad ,$$

where the last inequality is due to the fact that S_t has unitary mean and finite variance $\bar{k}$.

Summarizing, as in **Proposition 3.3** the upper and lower bounds of the ATM prices in (7) are $O\left(\sqrt{t}\right)$. From (9) we have that the B&S price is $O\left(\hat{\sigma}_t\sqrt{t}\right)$. Thus,

$$\hat{\sigma}_0 \;\text{ is finite}$$

□

In the propositions above we have proven that $\hat{\sigma}_0$ is finite only in Cases 3, 4 and 5. Only for these Cases we study the short time skew in the next Section.

4 Short time skew

In this Section, we focus on the skew term $\hat{\xi}_t$ when $\hat{\sigma}_0$ is finite. We obtain an expression of $\hat{\xi}_t$ in **Lemma 4.1** and study its short time limit.
In the introduction, we have mentioned that the implied volatility skew observed in the equity market is negative and it goes to zero as one over the square root of t. This behavior is equivalent to a negative and finite $\hat{\xi}_0$. In this Section, we prove that $\hat{\xi}_0$ is zero in Case 3 (**Proposition 4.2**) and is negative and finite in Cases 4 and 5 (**Proposition 4.3**). Moreover, Case 5 identifies the unique parameters' set where $\hat{\xi}_0$ can be a generic value that it is possible to calibrate from market data.

Lemma 4.1. *The skew term $\hat{\xi}_t$ is*

$$\hat{\xi}_t = \frac{N\left(-\frac{\hat{\sigma}_t\sqrt{t}}{2}\right) - \mathbb{E}\left[N\left(l_t^{S_t} - \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right)\right]}{N'\left(-\frac{\hat{\sigma}_t\sqrt{t}}{2}\right)} \ . \tag{25}$$

Proof. Applying the implicit function theorem to the implied volatility equation for the call option (7) we obtain the derivative of the implied volatility w.r.t y

$$\frac{\partial \mathcal{I}_t(y)}{\partial y} = \frac{\frac{\partial \mathbb{E}[c_t(S_t,y)]}{\partial y} - \frac{\partial c_t^{BS}(\mathcal{I}_t(y),y)}{\partial y}}{\frac{\partial c_t^{BS}(\mathcal{I}_t(y),y)}{\partial \mathcal{I}_t(y)}} \ .$$

We prove the thesis by computing the three partial derivatives separately. Notice that it is possible to exchange the expected value w.r.t. S_t and the derivative w.r.t. y using the Leibniz rule because the law of S_t does not depend from y.

1.
$$\frac{\partial \mathbb{E}\left[c_t\left(S_t,y\right)\right]}{\partial y} = -\sqrt{t}e^{\sqrt{t}y}\mathbb{E}\left[N\left(-\frac{y}{\bar{\sigma}\sqrt{S_t}} + l_t^{S_t} - \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right)\right] \ .$$

2.
$$\frac{\partial c_t^{BS}\left(\mathcal{I}_t(y),y\right)}{\partial y} = -\sqrt{t}e^{\sqrt{t}y}N\left(-\frac{y}{\mathcal{I}_t(y)} - \frac{\mathcal{I}_t(y)\sqrt{t}}{2}\right) \ .$$

3.
$$\frac{\partial c_t^{BS}\left(\mathcal{I}_t(y),y\right)}{\partial \mathcal{I}_t(y)} = \sqrt{t}e^{\sqrt{t}y}N'\left(-\frac{y}{\mathcal{I}_t(y)} - \frac{\mathcal{I}_t(y)\sqrt{t}}{2}\right) \ .$$

By substituting $y = 0$ and reminding that $\mathcal{I}_t(0) = \hat{\sigma}_t$ we get (25) □

Notice that, in Cases 3, 4 and 5, where $\hat{\sigma}_0$ is finite, the denominator of $\hat{\xi}_t$ in (25), $N'\left(-\frac{\hat{\sigma}_t\sqrt{t}}{2}\right)$, goes to $\frac{1}{\sqrt{2\pi}}$ at short time. To study the short time behavior of $\hat{\xi}_t$ it is sufficient to consider only the numerator of equation (25)

$$N\left(-\frac{\hat{\sigma}_t\sqrt{t}}{2}\right) - \mathbb{E}\left[N\left(l_t^{S_t} - \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right)\right] \ .$$

Proposition 4.2.
For Case 3: $-\beta/2 \le \delta \le 0$ & $\beta \ge 1$, with the exception of the point $\{\delta = -1/2, \ \beta = 1\}$,

$$\hat{\xi}_0 = 0 \ .$$

Proof.
We divide the proof in two sub-cases.

$$\boldsymbol{-\tfrac{1}{2} < \delta \le 0 \ \& \ \beta = 1}$$

We study the numerator of $\hat{\xi}_t$ in (25).

$$\lim_{t\to 0}\left\{N\left(-\frac{\hat{\sigma}_t\sqrt{t}}{2}\right) - \mathbb{E}\left[N\left(l_t^{S_t} - \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right)\right] = 0\right\} \ .$$

We compute the limit thanks to the dominated convergence theorem because the law of S_t does not depend on t and $l_t^z = o(1)$ in this sub-case.

$$\boldsymbol{-\tfrac{\beta}{2} \le \delta \le 0 \ \& \ \beta > 1}$$

We want to prove that

$$\mathbb{E}\left[N\left(l_t^{S_t} - \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right)\right] = \frac{1}{2} + o\left(1\right) \ . \tag{26}$$

The equality holds because

$$\begin{aligned}
&\mathbb{E}\left[N\left(l_t^{S_t} - \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right)\right] \\
=&\sqrt{\frac{t}{2\pi k_t}}\int_0^\infty dz\; e^{-t\frac{(z-1)^2}{2k_t}} N\left(l_t^z - \bar{\sigma}\frac{\sqrt{zt}}{2}\right) &(27)\\
&+\int_0^\infty dz\left(\mathcal{P}_{S_t}(z) - \sqrt{\frac{t}{2\pi k_t}}\; e^{-t\frac{(z-1)^2}{2k_t}}\right) N\left(l_t^z - \bar{\sigma}\frac{\sqrt{zt}}{2}\right) \ , &(28)
\end{aligned}$$

where $\mathcal{P}_{S_t}$ is the distribution of S_t. We study the quantities in (27) and (28) separately. First, we consider (27)

$$\begin{aligned}
&\lim_{t\to 0}\sqrt{\frac{t}{2\pi k_t}}\int_0^\infty dz\; e^{-t\frac{(z-1)^2}{2k_t}} N\left(l_t^z - \bar{\sigma}\frac{\sqrt{zt}}{2}\right) \\
=&\lim_{t\to 0}\frac{1}{\sqrt{2\pi}}\int_{-\sqrt{\frac{k_t}{t}}}^\infty dw\; e^{-\frac{w^2}{2}} N\left(\frac{\varphi_t\sqrt{t}}{\bar{\sigma}\sqrt{1+w\sqrt{k_t/t}}} - \bar{\sigma}\eta_t\sqrt{t\left(1+w\sqrt{k_t/t}\right)} - \bar{\sigma}\frac{\sqrt{t\left(1+w\sqrt{k_t/t}\right)}}{2}\right) \\
=&\lim_{t\to 0}\frac{1}{\sqrt{2\pi}}\int_{-\sqrt{\frac{k_t}{t}}}^\infty dw\; e^{-\frac{w^2}{2}} N\left(\bar{\sigma}\eta_t\sqrt{t}\left(1 - w\sqrt{k_t/t}/2\right) - \bar{\sigma}\eta_t\sqrt{t}\left(1 + w\sqrt{k_t/t}/2\right) + O\left(\sqrt{t}\right)\right) \\
=&\frac{1}{\sqrt{2\pi}}\lim_{t\to 0}\int_{\mathbb{R}} dw\; e^{-\frac{w^2}{2}}\left\{N\left(-\bar{\sigma}\bar{\eta}\sqrt{\bar{k}}t^{\delta+\beta/2}w\right) - \frac{1}{2} + \frac{1}{2}\right\} = \frac{1}{2} \ .
\end{aligned}$$

The first equality is obtained via a change of the integration variable ($w := \sqrt{t}(z-1)/\sqrt{k_t}$). The second equality is due to the asymptotic of $\varphi_t t$ in **Lemma A.6** point 1. The third equality holds because of the dominated convergence theorem. The last is trivial because $\left[N\left(-\bar{\eta}\sqrt{\bar{k}}t^{\delta+\beta/2}w\right)-1/2\right]$ is odd w.r.t. w.
Second, we consider (28)

$$
\begin{aligned}
&\int_0^\infty dz \left(\mathcal{P}_{S_t}(z) - \sqrt{\frac{t}{2\pi k_t}}\, e^{-t\frac{(z-1)^2}{2k_t}}\right) N\left(l_t^z - \bar{\sigma}\frac{\sqrt{zt}}{2}\right) \\
&= \left(\mathbb{P}(S_t<0)) - N\left(-\sqrt{\frac{t}{k_t}}\right)\right) \\
&\quad + \int_0^\infty dz \left(\mathbb{P}(S_t<z) - N\left((z-1)\sqrt{\frac{t}{k_t}}\right)\right) N'\left(l_t^z - \bar{\sigma}\frac{\sqrt{zt}}{2}\right)\left(\frac{\varphi_t\sqrt{t}}{2\bar{\sigma}\, z^{3/2}} + \frac{\bar{\sigma}\sqrt{t}\eta_t + \bar{\sigma}\sqrt{t}/2}{2\sqrt{z}}\right) = o(1) \quad .
\end{aligned}
$$

The first equality is due to integration by part. The second to the fact that i) $\mathbb{P}(S_t<0)=0$, ii) $N\left(-\sqrt{\frac{t}{k_t}}\right)$ go to zero as t goes to zero, and iii)

$$
\begin{aligned}
&\left|\int_0^\infty dz \left(N\left((z-1)\sqrt{\frac{t}{k_t}}\right) - \mathbb{P}(S_t<z)\right) N'\left(l_t^z - \bar{\sigma}\frac{\sqrt{zt}}{2}\right)\left(\frac{\varphi_t\sqrt{t}}{2\bar{\sigma}\, z^{3/2}} + \frac{\bar{\sigma}\sqrt{t}\eta_t + \bar{\sigma}\sqrt{t}/2}{2\sqrt{z}}\right)\right| \\
&\leq \frac{2-\alpha}{1-\alpha}\sqrt{\frac{k_t}{t}}\int_0^\infty dz N'\left(l_t^z - \bar{\sigma}\frac{\sqrt{zt}}{2}\right)\left(\frac{\varphi_t\sqrt{t}}{2\bar{\sigma}\, z^{3/2}} + \frac{\bar{\sigma}\sqrt{t}\eta_t + \bar{\sigma}\sqrt{t}/2}{2\sqrt{z}}\right) \\
&= \frac{2-\alpha}{1-\alpha}\sqrt{\frac{k_t}{t}} = O\left(\sqrt{\frac{k_t}{t}}\right) \quad ,
\end{aligned}
$$

where the inequality is due to **Lemma B.3** and the first equality is due the fact that

$$
\int_0^\infty dz N'\left(l_t^z - \bar{\sigma}\frac{\sqrt{zt}}{2}\right)\left(\frac{\varphi_t\sqrt{t}}{2\bar{\sigma}\, z^{3/2}} + \frac{\bar{\sigma}\sqrt{t}\eta_t + \bar{\sigma}\sqrt{t}/2}{2\sqrt{z}}\right) = -\left.N\left(l_t^z - \bar{\sigma}\frac{\sqrt{zt}}{2}\right)\right|_0^\infty = 1 \quad .
$$

This proves (26).
It is now possible to compute the short time limit of the skew term

$$
\lim_{t\to 0}\left\{\left(N\left(-\frac{\hat{\sigma}_t\sqrt{t}}{2}\right) - \mathbb{E}\left[N\left(l_t^{S_t} - \bar{\sigma}\frac{\sqrt{S_t t}}{2}\right)\right]\right\} = 0 \qquad \square
$$

Proposition 4.3.
For Case 4: $\delta = -1/2$ and $\beta < 1$,

$$
\hat{\xi}_0 = -\sqrt{\frac{\pi}{2}} \quad .
$$

For Case 5: $\delta = -1/2$ and $\beta = 1$

$$
\hat{\xi}_0 = -\sqrt{\frac{\pi}{2}}\;\mathbb{E}\left[erf\left(\bar{\sigma}\bar{\eta}\; r(S_t)\right)\right] \quad , \tag{29}
$$

where $r(S_t) := \sqrt{2}(1/\sqrt{S_t} - \sqrt{S_t})$.

Proof.
We prove separately the two Cases.

$$\boldsymbol{\delta = -\tfrac{1}{2} \ \& \ \beta < 1}$$

Thanks to **Lemma B.2** the limit of the numerator of $\hat{\xi}_t$ in (25) can be computed simply,

$$\lim_{t\to 0}\left(N\left(-\frac{\hat{\sigma}_t\sqrt{t}}{2}\right)-\mathbb{E}\left[N\left(l_t^{S_t}-\bar{\sigma}\frac{\sqrt{S_t t}}{2}\right)\right]\right)=-\frac{1}{2} \ .$$

Thus,

$$\hat{\xi}_0=\lim_{t\to 0}\frac{N\left(-\frac{\hat{\sigma}_t\sqrt{t}}{2}\right)-\mathbb{E}\left[N\left(l_t^{S_t}-\bar{\sigma}\frac{\sqrt{S_t t}}{2}\right)\right]}{N'\left(-\frac{\hat{\sigma}_t\sqrt{t}}{2}\right)}=-\sqrt{\frac{\pi}{2}} \ .$$

$$\boldsymbol{\delta = -\tfrac{1}{2} \ \& \ \beta = 1}$$

We compute the limit in $t=0$ of the numerator of $\hat{\xi}_t$ in (25)

$$\lim_{t\to 0}\left(N\left(-\frac{\hat{\sigma}_t\sqrt{t}}{2}\right)-\mathbb{E}\left[N\left(l_t^{S_t}-\bar{\sigma}\frac{\sqrt{S_t t}}{2}\right)\right]\right)=\mathbb{E}\left[1/2-N\left(\bar{\sigma}\bar{\eta}\left(1/\sqrt{S_t}-\sqrt{S_t}\right)\right)\right]$$

We obtain the equality thanks to the dominated convergence theorem because the law of S_t is constant in time. Recall that $erf(z)=2N(z/\sqrt{2})-1$, substituting in (25), obtain (29) □

Eq.(29) is one of the major results of the paper. Let us stop and comment.
First, let us notice that $\hat{\xi}_0$ in Eq.(29), is a generic function of the couple of positive parameters $\bar{\sigma}\bar{\eta}$ and $\bar{k}$; in particular the erf function is odd in its argument and $r:\mathbb{R}^+\to\mathbb{R}$. Moreover $\hat{\xi}_0$ depends on the parameter $\alpha\in[0,1)$ that selects the truncated additive process of interest.
Second,

$$-\sqrt{\frac{\pi}{2}}\leq\hat{\xi}_0\leq 0$$

i.e. the minimum value for the skew term is $-\sqrt{\pi/2}$, its value in Case 4.
To show the upper bound, we can rewrite in (29)

$$\mathbb{E}\left[erf\left(\bar{\sigma}\bar{\eta}\, d(S_t)\right)\right]=\int_0^\infty dz\,\mathcal{P}_{S_t}(z)\,erf\left(\bar{\sigma}\bar{\eta}\,r(z)\right)=\int_0^1 dz\,\left(\mathcal{P}_{S_t}(z)-\frac{\mathcal{P}_{S_t}(z)}{z^2}\right)\,erf\left(\bar{\sigma}\bar{\eta}\,r(z)\right) \ .$$

where the second equality is due to the change of variable $w=1/z$, and second to $r(1/w)=-r(w)$ and $erf(z)$ is odd. We also observe that $erf\left(\bar{\sigma}\bar{\eta}\,r(z)\right)>0$ in $(0,1)$.
For the two cases where the distribution of S_t is known analytically $\alpha=0$ (VG) and $\alpha=1/2$ (NIG), we can prove that the skew term $\hat{\xi}_0$ in Eq.(29) is negative for non zero $\bar{\sigma}\bar{\eta}$ and $\bar{k}$ (for the expression of the Gamma and Inverse Gaussian laws see, e.g., Cont and Tankov 2003, Ch.4, p.128). In both cases we can prove that $\left(\mathcal{P}_{S_t}(z)-\frac{\mathcal{P}_{S_t}(1/z)}{z^2}\right)>0$ in $(0,1)$; recall that $\mathcal{P}_{S_t}(z)$ does not depend from time because $\beta=1$.
In the $\alpha=0$ case, S_t has the law of a Gamma random variable

$$\mathcal{P}_{S_t}(z)-\frac{\mathcal{P}_{S_t}(1/z)}{z^2}=\frac{1}{\bar{k}^{1/\bar{k}}\Gamma(1/\bar{k})}z^{1/\bar{k}}e^{-z/\bar{k}}\left(1-\frac{e^{-1/\bar{k}\,(1/z-z)}}{z^{2/\bar{k}}}\right)>0 \ ,$$

where the inequality is true in $(0,1)$ because $1 - \frac{e^{-1/\bar{k}\,(1/z-z)}}{z^{2/\bar{k}}} > 0$ or equivalently $1/z - z + 2\ln z > 0$. The last inequality is trivial $\forall z \in (0,1)$, because it is equal to zero for $z = 1$ and its derivative is negative.
In the $\alpha = 1/2$ case, S_t has the law of an Inverse Gaussian random variable

$$\mathcal{P}_{S_t}(z) - \frac{\mathcal{P}_{S_t}(1/z)}{z^2} = \frac{1}{\sqrt{2\pi\bar{k}}} e^{-r(z)^2/(2\bar{k})} \left(\frac{1}{z^{3/2}} - \frac{1}{\sqrt{z}} \right) > 0 \ ,$$

where the inequality is true because $\frac{1}{z^{3/2}} - \frac{1}{\sqrt{z}} > 0, \forall z \in (0,1)$.

In all other cases, we compute numerically the skew term $\hat{\xi}_0$ for different admissible values of $\bar{k}, \bar{\sigma}\bar{\eta} \in \mathbb{R}^+$ and $\alpha \in [0,1)$, by means of inversion of the characteristic function of S_t, showing that it is either negative or equal to zero. In Figure 3, we plot the numerical estimation of the skew term for $\bar{\sigma}\bar{\eta}$ and $\bar{k}$ below 3 (an interval in line with the situation generally observed in market data) and for a grid of four values for α ($\alpha = 0, 1/4, 1/2, 3/4$); in all cases the skew term $\hat{\xi}_0$ looks rather similar: equal to zero on the boundaries ($\bar{k} = 0$ and $\bar{\sigma}\bar{\eta} = 0$), a negative quantity in all other cases and a decreasing function w.r.t. both $\bar{k}$ and $\bar{\sigma}\bar{\eta}$. In Figure 4, we plot also the skew term for the same four values of α, varying $\bar{k}$ with $\bar{\sigma}\bar{\eta} = 1$ (on the left) and varying $\bar{\sigma}\bar{\eta}$ for $\bar{k} = 1$ (on the right): all plots look rather similar with a decreasing $\hat{\xi}_0$.

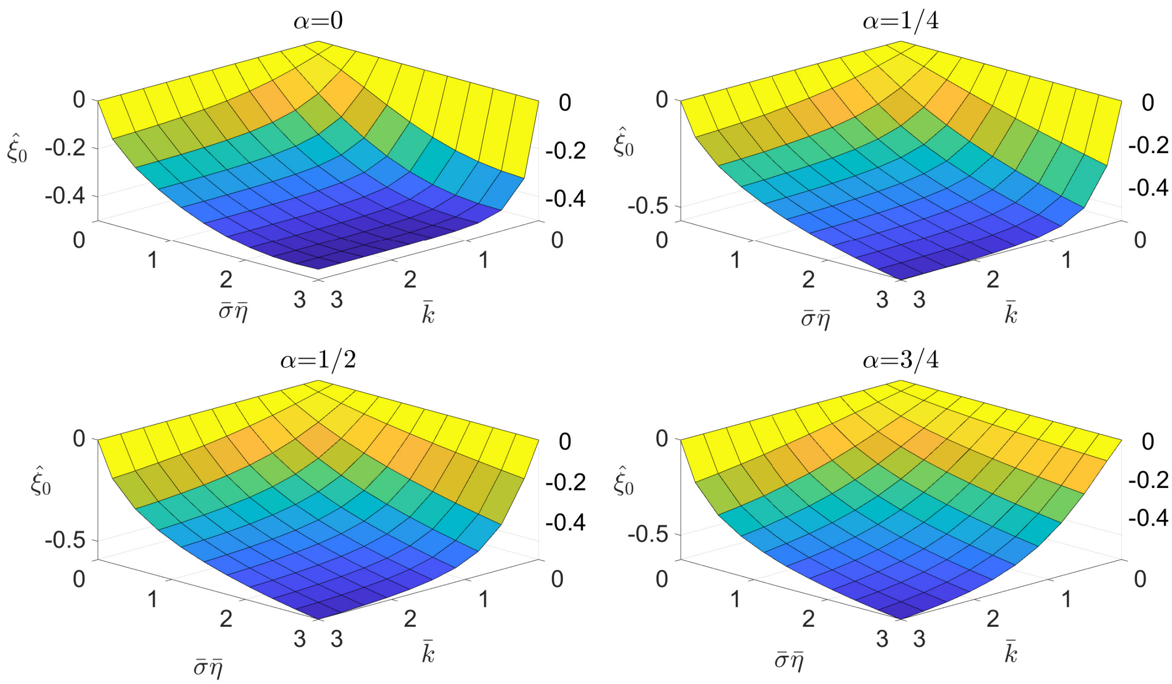

Figure 3: ATS skew term $\hat{\xi}_0$ for $\{\delta = -1/2, \beta = 1\}$. We report $\hat{\xi}_0$ for four values of α: $\alpha = 0$ in the upper left corner, $\alpha = 1/4$ in the upper right corner, $\alpha = 1/2$ in the lower left corner and $\alpha = 3/4$ in the lower right corner. We plot the skew for $\bar{k}, \bar{\sigma}\bar{\eta} \in [0,3]$. In all cases the skew is negative and decreasing w.r.t. $\bar{k}$ and $\bar{\sigma}\bar{\eta}$.

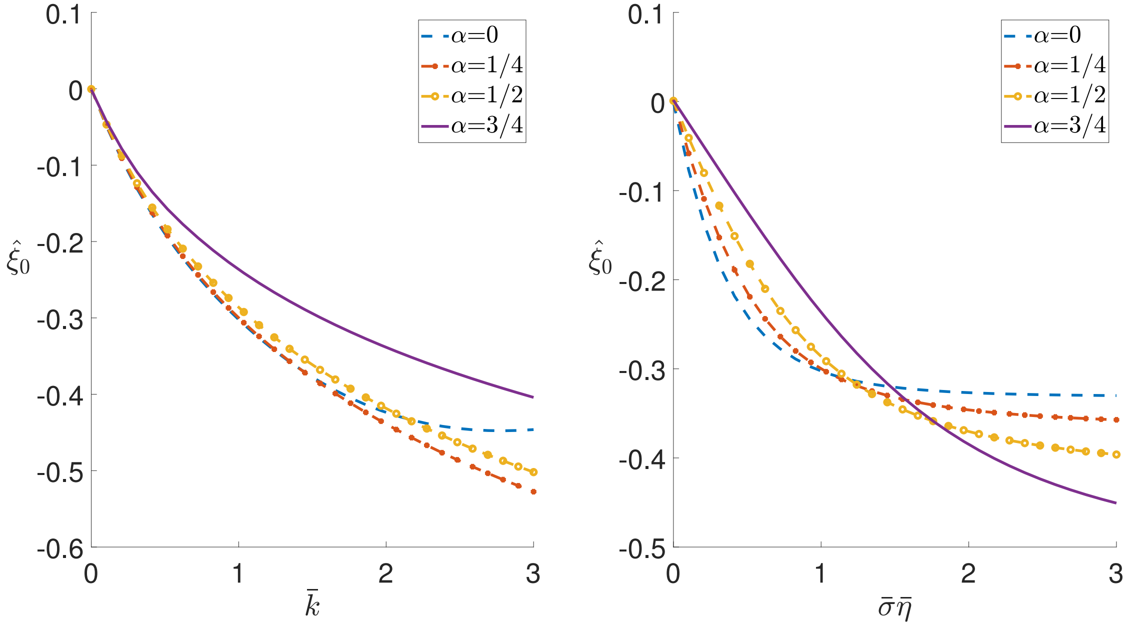

Figure 4: ATS skew term $\hat{\xi}_0$ for $\delta = -1/2$ and $\beta = 1$ for $\alpha = 0$ (blue dashed line), $\alpha = 1/4$ (red stars), $\alpha = 1/2$ (orange circles) and, $\alpha = 3/4$ (violet line). We plot the skew for $\bar{k} \in [0, 3]$ with $\bar{\sigma}\bar{\eta} = 1$ (on the left) and for $\bar{\sigma}\bar{\eta} \in [0, 3]$ for $\bar{k} = 1$ (on the right). In all cases the skew is decreasing w.r.t. $\bar{k}$ and $\bar{\sigma}\bar{\eta}$.

Finally, let us emphasize that the limits of $\hat{\xi}_0$ are zero for $\bar{\sigma}\bar{\eta}$ and $\bar{k}$ that go to zero.
On the one hand, recall that the law of S_t, $\mathcal{P}_{S_t}$, does not depend of $\bar{\sigma}\bar{\eta}$. By the dominated convergence theorem with bound $\mathcal{P}_{S_t}$, we have that

$$\lim_{\bar{\sigma}\bar{\eta}\to 0} \mathbb{E}\left[erf\left(\bar{\sigma}\bar{\eta}\, r(S_t)\right)\right] = 0 \quad .$$

On the other hand, by Kijima (1997, Th. B.9, p. 308), we have that S_t converges in distribution to 1 as $\bar{k}$ goes to zero because

$$\lim_{\bar{k}\to 0} \mathcal{L}_t(u;\ k_t,\ \alpha) = \lim_{t\to 0} e^{\frac{1}{\bar{k}}\frac{1-\alpha}{\alpha}\left\{1-\left(1+\frac{u\,\bar{k}}{(1-\alpha)}\right)^{\alpha}\right\}} = e^{-u} \quad .$$

We are computing the expected value of a bounded function of S_t that does not depend of $\bar{k}$. Thus, by definition of convergence in distribution,

$$\lim_{\bar{k}\to 0} \mathbb{E}\left[erf\left(\bar{\sigma}\bar{\eta}\sqrt{2}\left(1/\sqrt{S_t} - \sqrt{S_t}\right)\right)\right] = 0 \quad .$$

5 Main Result

In the following theorem we present the main results of this paper. We prove that only in the Case $\beta = 1$ and $\delta = -\frac{1}{2}$ the ATS has a positive and constant short time implied volatility $\hat{\sigma}_0$ and a negative and constant short time skew $\hat{\xi}_0$. We point out that a finite skew w.r.t. y correspond to a skew that goes as $\frac{1}{\sqrt{t}}$ at short time w.r.t. the log-moneyness x. The proof is based on the propositions of Sections 3 and 4.

Theorem 5.1. *The ATS short time implied volatility behave as is described in table 1.*

Proof. We prove that, for Case 1, $\hat{\sigma}_0 = 0$ in **Proposition 3.1**. We prove that, for Case 2, $\hat{\sigma}_0 = \infty$ in **Proposition 3.2**. We prove that, for Cases 3, 4 and 5, $\hat{\sigma}_0$ is finite in **Proposition 3.3** and **Proposition 3.4**.
Moreover, in **Proposition 4.2** we demonstrate that, for Case 3, $\hat{\xi}_0 = 0$ and in **Proposition 4.3** we demonstrate that, for Case 4, $\hat{\xi}_0 = -\sqrt{\frac{\pi}{2}}$ and that, for Case 5, $\hat{\xi}_0$ is negative and finite. □

6 Conclusions

In this paper, we have analyzed the short time-to-maturity smile of a family of pure jumps additive processes and we have shown that it reproduces accurately the equity market one.

An excellent calibration of the equity implied volatility surface has been achieved by the ATS, a class of power-law scaling additive processes. This class of processes builds upon the power-law scaling parameters β, related to the variance of jumps, and δ related to the smile asymmetry.
The behavior of the short time ATM implied volatility ($\hat{\sigma}_t$) and skew (related to $\hat{\xi}_t$) is investigated over the range of ATS admissible scaling parameters (cf. **Theorem 2.2**). To study the behavior of the two functions of time, we derive an extension of the Hull and White formula for this class of additive processes (Hull and White 1987, Eq. (7)) in equations (4, 5). We use this formula to constructs some relevant bounds on $\hat{\sigma}_t$. We obtain an expression of $\hat{\xi}_t$, cf. equation (25), via the implicit function theorem.
In **Theorem 5.1** we prove that only the scaling parameters observed in market data, $\delta = -1/2$ and $\beta = 1$, are compatible with a positive and bounded short time implied volatility and a short time skew that is proportional to the inverse of the square root of the time-to-maturity. Hence, it is possible to build a pure jumps process that, differently from the Lévy case, presents not only a finite short time implied volatility but also a power scaling skew.

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References

Alòs, E., León, J.A., and Vives, J., 2007. On the short-time behavior of the implied volatility for jump-diffusion models with stochastic volatility, *Finance and Stochastics*, 11 (4), 571–589.

Andersen, L. and Lipton, A., 2013. Asymptotics for exponential Lévy processes and their volatility smile: survey and new results, *International Journal of Theoretical and Applied Finance*, 16 (01), 1350001.

Asmussen, S. and Rosiński, J., 2001. Approximations of small jumps of lévy processes with a view towards simulation, *Journal of Applied Probability*, 482–493.

Azzone, M. and Baviera, R., 2021. Additive normal tempered stable processes for equity derivatives and power law scaling, *arXiv preprint arXiv:1909.07139*.

Carr, P. and Wu, L., 2003. The finite moment log stable process and option pricing, *The Journal of Finance*, 58 (2), 753–777.

Cont, R. and Tankov, P., 2003. *Financial Modelling with jump processes*, Chapman and Hall/CRC Financial Mathematics Series.

Cressie, N., Davis, A.S., Folks, J.L., and Folks, J.L., 1981. The moment-generating function and negative integer moments, *The American Statistician*, 35 (3), 148–150.

Durrett, R., 2019. *Probability: theory and examples*, vol. 49, Cambridge university press.

Figueroa-López, J.E. and Forde, M., 2012. The small-maturity smile for exponential Lévy models, *SIAM Journal on Financial Mathematics*, 3 (1), 33–65.

Figueroa-López, J.E., Gong, R., and Houdré, C., 2016. High-order short-time expansions for atm option prices of exponential lévy models, *Mathematical Finance*, 26 (3), 516–557.

Figueroa-López, J.E., Gong, R., and Lorig, M., 2018. Short-time expansions for call options on leveraged etfs under exponential lévy models with local volatility, *SIAM Journal on Financial Mathematics*, 9 (1), 347–380.

Fouque, J.P., Papanicolaou, G., Sircar, R., and Solna, K., 2004. Maturity cycles in implied volatility, *Finance and Stochastics*, 8 (4), 451–477.

Gatheral, J., 2011. *The volatility surface: a practitioner's guide*, vol. 357, John Wiley & Sons.

Hull, J. and White, A., 1987. The pricing of options on assets with stochastic volatilities, *The Journal of Finance*, 42 (2), 281–300.

Kijima, M., 1997. *Markov processes for stochastic modeling*, vol. 6, CRC Press.

Küchler, U. and Tappe, S., 2013. Tempered stable distributions and processes, *Stochastic Processes and their Applications*, 123 (12), 4256–4293.

Loomis, L.H. and Sternberg, S., 1990. *Advanced calculus*, Jones and Bartlett Publishers.

Medvedev, A. and Scaillet, O., 2006. Approximation and calibration of short-term implied volatilities under jump-diffusion stochastic volatility, *The Review of Financial Studies*, 20 (2), 427–459.

Mijatović, A. and Tankov, P., 2016. A new look at short-term implied volatility in asset price models with jumps, *Mathematical Finance*, 26 (1), 149–183.

Muhle-Karbe, J. and Nutz, M., 2011. Small-time asymptotics of option prices and first absolute moments, *Journal of Applied Probability*, 48 (4), 1003–1020.

Roper, M., 2009. *Implied volatility: small time to expiry asymptotics in exponential Lévy models*, Ph.D. thesis, Thesis, University of New South Wales.

Rudin, W., 1976. *Principles of mathematical analysis*, 4.2, McGraw-Hill New York.

Sato, K.I., 1999. *Lévy processes and infinitely divisible distributions*, Cambridge University Press.

Tankov, P., 2011. Pricing and hedging in exponential lévy models: review of recent results, *Paris-Princeton Lectures on Mathematical Finance 2010*, 319–359.

Urbanik, K., 1993. Moments of sums of independent random variables, *in: Stochastic Processes*, Springer, 321–328.

Notation

Symbol	Description
B_t	discount factor between date 0 and t
$\mathbb{B}(\mathbb{R})$	Borel sigma algebra on $\mathbb{R}$
$c_t^{BS}\left(\mathcal{I}_t(y), y\right)$	B&S call option price
$c_t(S_t, t)$	quantity inside the ATS call expected value
$C_t\left(x\right)$	call option price at value date with maturity t and moneyness x
$\{f_t\}_{t\geq 0}$	Random variable that models the forward exponent
F_t	price at time 0 of a Forward contract with maturity t
$\{S_t\}_{t\geq 0}$	sequence of positive random variable
$I_t(x)$	B&S implied volatility with maturity t
$\mathcal{I}_t(y)$	B&S implied volatility w.r.t. y
$\mathbb{1}_*$	indicator function of the set $*$
k_t	variance of jumps of ATS
$\bar{k}$	constant part of variance of jumps of ATS k_t
K	option strike price
l_t^z	quantity defined in equation (6)
$N(*)$	standard normal cumulative distribution function evaluated in $*$
$N'(*)$	standard normal probability density function evaluated in $*$
$p_t^{BS}\left(\mathcal{I}_t(y), y\right)$	B&S put option price
$p_t(S_t, t)$	quantity inside the ATS put expected value
$P\left(y, t\right)$	put option price at value date with maturity t and moneyness y
t	time-to-maturity
W_t	Brownian motion
x	option moneyness, $x = \log\frac{K}{F_t}$
y	moneyness degree, $x = y\sqrt{t}$
α	tempered stable parameter of ATS
β	scaling parameter of k_t
$\Gamma(*)$	gamma function evaluated in $*$
δ	scaling parameter of η_t
η_t	*skew* parameter of ATS
$\bar{\eta}$	constant part of the *skew* parameter of ATS
$\hat{\xi}_t$	implied volatility skew term
$\hat{\xi}_0$	skew term, limit for t that goes to zero of $\hat{\xi}_t$
$\bar{\sigma}$	constant diffusion parameter of ATS
$\hat{\sigma}_t$	ATM implied volatility, equal to $\mathcal{I}_t(0)$
$\hat{\sigma}_0$	limit for t that goes to zero of $\hat{\sigma}_t$
$\mathcal{P}_{S_t}$	probability density function of S_t
φ_t	deterministic drift term of ATS

Appendices

A Basic properties

We report some useful results for the proofs in Section 3. In the following lemmas we consider S_t of **Definition 2.1** with Laplace transform $\mathcal{L}_t(u;\ k_t;\ \alpha)$, at a given time $t > 0$. The proofs that follow are for the $\alpha \in (0,1)$ case. Similar proofs hold in the $\alpha = 0$ case.

Lemma A.1. *Let $s \in (0,1)$, then*

$$\mathbb{E}\left[S_t^s\right] = \int_0^\infty \frac{\mathcal{L}_t(u;\ k_t;\ \alpha) - 1}{\Gamma(-s) u^{s+1}} du \ \ , \tag{30}$$

where Γ is the Gamma function.

Proof. By elementary calculus and Fubini's Theorem (see, e.g., Urbanik 1993, Lemma 4, p.325) □

Lemma A.2. *Let n be a positive integer, then*

$$\mathbb{E}[S_t^{-n}] = \Gamma(n)^{-1} \int_0^\infty u^{n-1} \mathcal{L}_t(u;\ k_t;\ \alpha) du \ \ .$$

Proof. By elementary calculus and Fubini's Theorem (see, e.g., Cressie *et al.* 1981, Ch.2, p.148) □

Lemma A.3.

1. *For all $t > 0$, $c \geq 1$ and $u \geq 0$*

$$1 - \mathcal{L}_t(u;\ k_t,\ \alpha) \leq 1 - e^{-cu} \ \ .$$

2. *If $\beta \geq 1$, $\mathcal{L}_t(u;\ k_t,\ \alpha)$ is non decreasing in t.*

Proof. Let us observe that

$$1 - \mathcal{L}_t(u;\ k_t,\ \alpha) \leq 1 - e^{-cu}$$
$$\frac{t}{k_t} \frac{1-\alpha}{\alpha} \left\{ \left(1 + \frac{u\, k_t}{(1-\alpha)t}\right)^{\alpha} - 1 \right\} - cu \leq 0 \ \ .$$

The last inequality is true for any $c \geq 1$ and $u \geq 0$ because the left hand side is null in $u = 0$ and its first order derivative w.r.t. u is negative:

$$\frac{1}{\left(1 + \frac{k_t u}{t(1-\alpha)}\right)^{1-\alpha}} - c < 0 \ \ .$$

This proves the first point.

We demonstrate that the logarithm of $\mathcal{L}_t(u;\ k_t,\ \alpha)$ is not decreasing. Consider a positive t, $s \in (0,t)$ and

$$h(u;\ s,\ t) := \frac{t}{k_t} \left\{ 1 - \left(1 + \frac{u\, k_t}{(1-\alpha)t}\right)^{\alpha} \right\} - \frac{s}{k_s} \left\{ 1 - \left(1 + \frac{u\, k_s}{(1-\alpha)s}\right)^{\alpha} \right\} \ \ .$$

We observe that $h(0;\ s,\ t) = 0$ and the first order derivative

$$\frac{\partial h(u;\ s,\ t)}{\partial u} = \frac{1}{\left(1 + \frac{k_s u}{s(1-\alpha)}\right)^{1-\alpha}} - \frac{1}{\left(1 + \frac{k_t u}{t(1-\alpha)}\right)^{1-\alpha}}$$

is non negative $\forall u > 0$ because k_t/t is non decreasing in t, if $\beta > 1$, and is constant in t, if $\beta = 1$. Thus, $h(u;\ s,\ t) \geq 0$, $\forall u \geq 0$, and $\mathcal{L}_t(u;\ k_t,\ \alpha)$ is non decreasing w.r.t. t. This proves point 2 □

Lemma A.4.

1. *If $\beta < 1$ S_t goes to zero in distribution as t goes to zero.*
2. *If $\beta > 1$ S_t goes to one in distribution as t goes to zero.*
3. *If $\beta = 1$ the distribution of S_t does not depend from t.*

Proof. Recall that convergence in the Laplace transform implies convergence in distribution (see, e.g., Kijima 1997, Th.B.9, p.308).
We compute the limit of S_t Laplace transform for $\beta < 1$. By using the fact that k_t/t goes to infinity as t goes to zero we obtain

$$\lim_{t\to 0} \mathcal{L}_t(u;\ k_t,\ \alpha) = \lim_{t\to 0} e^{\frac{t}{k_t}\frac{1-\alpha}{\alpha}\left\{1-\left(1+\frac{u\,k_t}{(1-\alpha)t}\right)^{\alpha}\right\}} = 1 \ .$$

Thus, S_t converges in distribution to the constant zero. This proves point 1.

We compute the limit of S_t Laplace transform for $\beta > 1$. By using the fact that k_t/t goes to zero as t goes to zero we obtain

$$\lim_{t\to 0} \mathcal{L}_t(u;\ k_t,\ \alpha) = \lim_{t\to 0} e^{\frac{t}{k_t}\frac{1-\alpha}{\alpha}\left\{1-\left(1+\frac{u\,k_t}{(1-\alpha)t}\right)^{\alpha}\right\}} = e^{-u} \ .$$

Thus, S_t converges in distribution to the constant one. This proves point 2.

Point 3 follows from the fact that, if $\beta = 1$, $\mathcal{L}_t(u;\ k_t,\ \alpha)$ is constant in t

□

Lemma A.5.

$$\lim_{t\to 0} \mathbb{E}[\sqrt{S_t}] = \begin{cases} 0 & \text{if } \beta < 1 \\ 1 & \text{if } \beta > 1 \\ D & \text{if } \beta = 1 \end{cases} , \tag{31}$$

where D is a positive constant.

Proof. Recall that S_t is a positive r.v. and $\mathbb{E}[S_t] = 1$. Then, its moment of order 1/2 is finite. By **Lemma A.1**

$$\mathbb{E}\left[\sqrt{S_t}\right] = \int_0^\infty \frac{\mathcal{L}_t(u;\ k_t,\ \alpha) - 1}{\Gamma(-1/2)u^{3/2}} du \ ,$$

where $\frac{-1}{\Gamma(-1/2)} \approx 3.45$. By **Lemma A.3** point 1 with $c = 2$, the positive quantity $(1 - \mathcal{L}_t(u;\ k_t,\ \alpha))/u^{3/2}$ is lower or equal than $(1 - e^{-2u})/u^{3/2}$. Thus,

$$0 \leq \mathbb{E}\left[\sqrt{S_t}\right] \leq \frac{-1}{\Gamma(-1/2)} \int_0^\infty \frac{1 - e^{-2u}}{u^{3/2}} du = \frac{-4}{\Gamma(-1/2)} \int_0^\infty \frac{e^{-2u}}{u^{1/2}} du = \sqrt{2} \ , \tag{32}$$

where the first equality is obtained from integration by parts and the second from the definition of Γ. Inequality (32) has two consequences. First, if $\beta = 1$,

$$\lim_{t\to 0} \mathbb{E}[\sqrt{S_t}] = \mathbb{E}[\sqrt{S_t}] := D \leq \sqrt{2} \ , \tag{33}$$

because, by **Lemma A.4** point 3, $\mathbb{E}[\sqrt{S_t}]$ is constant w.r.t. to time. Second, we can apply the dominated convergence theorem to (31) for all values of β. Recall that the limits for t that goes to zero of $\mathcal{L}_t(u;\ k_t,\ \alpha)$ for $\beta < 1$ and for $\beta > 1$ are computed in the proof of **Lemma A.4**.
If $\beta < 1$

$$\lim_{t\to 0} \mathbb{E}[\sqrt{S_t}] = \lim_{t\to 0} \frac{-1}{\Gamma(-1/2)} \int_0^\infty \frac{1 - \mathcal{L}_t(u;\ k_t,\ \alpha)}{u^{3/2}} du = 0 \ . \tag{34}$$

If $\beta > 1$

$$\begin{aligned} \lim_{t\to 0} \mathbb{E}[\sqrt{S_t}] &= \lim_{t\to 0} \frac{-1}{\Gamma(-1/2)} \int_0^\infty \frac{1 - \mathcal{L}_t(u;\ k_t,\ \alpha)}{\Gamma(-1/2) u^{3/2}} du \\ &= \frac{-1}{\Gamma(-1/2)} \int_0^\infty \frac{1 - e^{-u}}{u^{3/2}} du = \frac{-2}{\Gamma(-1/2)} \int_0^\infty \frac{e^{-u}}{u^{1/2}} du = 1 \ , \end{aligned} \tag{35}$$

where the third equality is obtained from integration by parts and the third by the definition of Γ. Equalities (33), (34) and (35) prove the thesis

□

Lemma A.6.
*Consider φ_t in (2). For every δ and β in the additive process boundaries of **Theorem 2.2***

1. $$\varphi_t t = t\bar{\sigma}^2 \eta_t - t\bar{\sigma}^4 \eta_t^2 k_t/2 + O\left(t\eta_t^3 k_t^2\right) \ , \tag{36}$$

   where the second term $t\bar{\sigma}^4\eta_t^2 k_t/2$ goes to zero faster than $t\bar{\sigma}^2\eta_t$ as t goes to zero.

2. $$\frac{\varphi_t}{\bar{\sigma}^2 \eta_t} \leq 1 \ .$$

3. $$\lim_{t\to 0} \frac{\varphi_t}{\bar{\sigma}^2 \eta_t} = 1 \ , \ \ \textit{for} \quad \delta > -\min(1, \beta) \ .$$

Proof. We prove the asymptotic expansion (36). In the additive process boundaries of **Theorem 2.2** at least either $\beta = \delta = 0$ or $\delta > -\min(1, \beta)$. In the former case (36) is trivial. In the latter, thanks to (2), both $t\eta_t = t^{1+\delta}\bar{\eta}$ and $\eta_t k_t = t^{\beta+\delta}\bar{\eta}\bar{k}$ go to zero as t goes to zero. Using the Taylor series expansion

$$\begin{aligned} \varphi_t t &= \frac{t(1-\alpha)}{k_t} \left\{ \frac{\bar{\sigma}^2\, \eta_t\, k_t}{1-\alpha} - \frac{\bar{\sigma}^4\, \eta_t^2\, k_t^2}{2(1-\alpha)} + O\left(\eta_t^3\, k_t^3\right) \right\} \\ &= t\, \bar{\sigma}^2\, \eta_t - t\, \bar{\sigma}^4\, \eta_t^2\, k_t/2 + O\left(t\, \eta_t^3\, \kappa_t^2\right) \ . \end{aligned}$$

This proves point 1.

We prove that $\varphi_t/(\bar{\sigma}^2\eta_t) \leq 1$. We substitute the definition of φ_t in (2), for $\alpha > 0$, in (36) and we get

$$\varphi_t/(\bar{\sigma}^2\eta_t) = \frac{(1-\alpha)}{\alpha\bar{\sigma}^2\eta_t k_t}\left(\left(1+\frac{\bar{\sigma}^2\eta_t k_t}{1-\alpha}\right)^{\alpha} - 1\right) \leq 1 \ . \tag{37}$$

We define $z := \frac{\bar{\sigma}^2\eta_t k_t}{1-\alpha}$. Then, (37) is equivalent to

$$(1+z)^{\alpha} \leq 1 + \alpha z \ \ ,$$

which is a well known inequality. This proves point 2.

Point 3 is straightforward, given point 1, because, if $\delta > -\min(1,\beta)$, $\eta_t k_t$ goes to zero as t goes to zero □

B Short time limits

Lemma B.1. *Consider a family of positive random variables X_t s.t. $\lim_{t\to 0} X_t = X$ in distribution and a sequence of functions $g_t(z) \geq 0$ and uniformly bounded s.t. $\lim_{t\to 0} g_t(z) = g(z)$.*
If $\exists\, \tau > 0$ s.t. for $t \in (0,\tau)$

i) $g_t(z)$ is Lipschitz continuous with bounded Lipschitz constant,

ii) $|g_t(z) - g(z)| < h(z)$ with $\lim_{z\to\infty} h(z) = 0$,

then

$$\lim_{t\to 0} \mathbb{E}[g_t(X_t)] = \mathbb{E}[g(X)] \ \ .$$

Proof. It is possible to apply the Ascoli-Arzelá theorem (see, e.g.,, Rudin 1976, Th.7.25, p.158) on every compact set $[0,K]$, $K > 0$, because a sequence of Lipschitz continuous functions with bounded Lipschitz constant is equicontinous on any compact set. Thus, a sub-sequence of $g_t(z)$ converges uniformly to $g(z)$ in any $[0,K]$. For every $\epsilon > 0$, $\exists\, K$ s.t.

$$\lim_{t\to 0} \mathbb{E}[|g_t(X_t) - g(X_t)|] = \lim_{t\to 0} \mathbb{E}\left[|g_t(X_t) - g(X_t)|\, \mathbb{1}_{X_t<K}\right] + \lim_{t\to 0} \mathbb{E}\left[|g_t(X_t) - g(X_t)|\, \mathbb{1}_{X_t>K}\right] < \epsilon \ \ .$$

The first expected value goes to zero because $g_t(z)$ converges uniformly to $g(z)$ on $[0,K]$, as proven above via Ascoli-Arzelá theorem. It exists K s.t. it is possible to bound the second with ϵ because $h(z)$ goes to zero as z goes to infinity.
Moreover, $g(z)$ is bounded because it is the limit of a uniformly bounded sequence and

$$\lim_{t\to 0} \mathbb{E}\left[|g(X_t) - g(X)|\right] = 0 \ \ .$$

by definition of convergence in distribution, because $g(z)$ is bounded. We have that

$$0 \leq \lim_{t\to 0} \mathbb{E}[|g_t(X_t) - g(X)|] \leq \lim_{t\to 0} \left\{\mathbb{E}[|g_t(X_t) - g(X_t)|] + \mathbb{E}\left[|g(X_t) - g(X)|\right]\right\} = 0 \ \ ,$$

this proves the thesis

□

Lemma B.2. *For $\delta = -1/2$, let X_t be a sequence of positive random variable s.t. $X_t \to X$ in distribution for t that goes to zero.*
Then,

$$\lim_{t\to 0} \mathbb{E}\left[N\left(\bar\sigma\bar\eta\left(-\sqrt{X_t}+\varphi_t/(\bar\sigma^2\sqrt{X_t}\eta_t)\right)-\bar\sigma\sqrt{tX_t}/2\right)\right] = \mathbb{E}\left[N(\bar\sigma\bar\eta(-\sqrt{X}+1/\sqrt{X}))\right] \quad .$$

Proof. Define

$$g_t(z) := N\left(\bar\sigma\bar\eta\left(-\sqrt{z}+\varphi_t/(\bar\sigma^2\sqrt{z}\eta_t)\right)-\bar\sigma\sqrt{t\,z}/2\right) \quad \text{and} \quad g(z) := N(\bar\sigma\bar\eta(-\sqrt{z}+1/\sqrt{z})) \quad .$$

We emphasize that $g_t(z)$ is uniformly bounded by one and $g_t(z)$ converges point-wise to $g(z)$ because, thanks to **Lemma A.6** point 3, $\lim_{t\to 0}\varphi_t/(\bar\sigma^2\eta_t) = 1$.
We prove that $\exists\,\tau \in (0,1)$ s.t. the derivative of $g_t(z)$ is uniformly bounded, if $t \in (0,\tau)$. Fix $\tau \in (0,1)$ s.t.

$$\begin{cases} \dfrac{\varphi_\tau}{\bar\sigma^2\,\eta_\tau} & > \frac{2}{3} \\ \dfrac{\bar\sigma\bar\eta}{\bar\sigma\bar\eta+\bar\sigma\sqrt{\tau}/2} & > \frac{3}{4} \end{cases} \quad .$$

The following hold for $t < \tau$,

$$\begin{aligned}
&\left|\frac{\partial g_t}{\partial z}\right| \\
&= N'\left(\bar\sigma\bar\eta\left(-\sqrt{z}+\varphi_t/(\bar\sigma^2\eta_t\sqrt{z})\right)-\bar\sigma\sqrt{zt}/2\right)\left|\bar\sigma\bar\eta\left(-1/(2\sqrt{z})-\varphi_t/(2\bar\sigma^2\eta_t\,z^{3/2})\right)-\bar\sigma\sqrt{t}/(4\sqrt{z})\right| \\
&= N'\left(\bar\sigma\bar\eta\left(-\sqrt{z}+\varphi_t/(\bar\sigma^2\eta_t\sqrt{z})\right)-\bar\sigma\sqrt{zt}/2\right)\left(1+\varphi_t/(\bar\sigma^2\eta_t z)+\sqrt{t}/(2\bar\eta)\right)\bar\sigma\bar\eta/\left(2\sqrt{z}\right) \\
&\le N'\left(\bar\sigma\bar\eta\left(-\sqrt{z}+\varphi_t/(\bar\sigma^2\eta_t\sqrt{z})\right)-\bar\sigma\sqrt{zt}/2\right)\left(1+1/z+1/(2\bar\eta)\right)\bar\sigma\bar\eta/\left(2\sqrt{z}\right) \qquad (38) \\
&\le \left[\frac{1}{\sqrt{2\pi}}\mathbb{1}_{D_2}+N'\left(\bar\sigma\bar\eta\left(-\sqrt{z}+2/(3\sqrt{z})\right)-\tau\bar\sigma\sqrt{z}/2\right)\mathbb{1}_{D_1}+N'\left(\bar\sigma\bar\eta\left(-\sqrt{z}+1/\sqrt{z}\right)\right)\mathbb{1}_{D_3}\right]\cdot \\
&\quad \left(1+1/z+1/(2\bar\eta)\right)\bar\sigma\bar\eta/\left(2\sqrt{z}\right) := M(z) \quad . \qquad (39)
\end{aligned}$$

Inequality (38) holds because, by **Lemma A.6** point 2, $\varphi_t/(\bar\sigma^2\eta_t) < 1$ and $\tau \in (0,1)$. Let us observe that (38) is the product of positive quantities. In (39) we bound from above only the first factor, the only one that still depends from t. Inequality (39) is deduced by dividing the domain of $z \in \mathbb{R}^+$ in the three sets $D_1 \equiv (0,1/2]$, $D_2 \equiv (1/2,3/2]$ and $D_3 \equiv (3/2,\infty)$.
For $z \in D_2$, we bound the first factor with its maximum $\frac{1}{\sqrt{2\pi}}$.
For $z \in D_1$, we observe that for $t < \tau$

$$\bar\sigma\bar\eta\left(-\sqrt{z}+\varphi_t/(\bar\sigma^2\eta_t\sqrt{z})\right)-\bar\sigma\sqrt{zt}/2 > \bar\sigma\bar\eta\left(-\sqrt{z}+2/(3\sqrt{z})\right)-\tau\bar\sigma\sqrt{z}/2 > 0 \quad .$$

Hence, because N' is a decreasing function of its argument in $\mathbb{R}^+$,

$$N'\left(\bar\sigma\bar\eta\left(-\sqrt{z}+\varphi_t/(\bar\sigma^2\eta_t\sqrt{z})\right)-\bar\sigma\sqrt{zt}/2\right) \le N'\left(\bar\sigma\bar\eta\left(-\sqrt{z}+2/(3\sqrt{z})\right)-\tau\bar\sigma\sqrt{z}/2\right) \quad , \quad z \in D_1 \quad .$$

Finally, for $z \in D_3$

$$\bar\sigma\bar\eta\left(-\sqrt{z}+\varphi_t/(\bar\sigma^2\eta_t\sqrt{z})\right)-\bar\sigma\sqrt{tz}/2 < \bar\sigma\bar\eta\left(-\sqrt{z}+1/\sqrt{z}\right) < 0 \quad . \qquad (40)$$

Thus, because N' is an increasing function of its argument in $\mathbb{R}^-$

$$N'\left(\bar\sigma\bar\eta\left(-\sqrt{z}+\varphi_t/(\bar\sigma^2\eta_t\sqrt{z})\right)-\bar\sigma\sqrt{tz}/2\right) < N'\left(\bar\sigma\bar\eta\left(-\sqrt{z}+1/\sqrt{z}\right)\right) \quad , \quad z\in D_3 \ .$$

Notice that $M(z)$ is positive and bounded on $\mathbb{R}^+$; this implies that the derivatives of $g_t(z)$ is uniformly bounded. Thus, the sequence $g_t(z)$ is Lipschitz continuous in z with bounded Lipschitz constant on $(0,\tau)$.
Moreover, for $t<\tau<1$ we have that

$$\begin{aligned}|g_t(z)-g(z)| &\le \mathbb{1}_{z\in(0,1]} + N'\left(\bar\sigma\bar\eta\left(-\sqrt{z}+1/\sqrt{z}\right)\right)\left(\bar\sigma\sqrt{zt}/2+\bar\sigma\bar\eta(1-\varphi_t/(\bar\sigma^2\eta_t))/\sqrt{z}\right)\mathbb{1}_{z\in(1,\infty)}\\ &\le \mathbb{1}_{z\in(0,1]} + N'\left(\bar\sigma\bar\eta\left(-\sqrt{z}+1/\sqrt{z}\right)\right)\left(\bar\sigma\sqrt{z}/2+\bar\sigma\bar\eta/\sqrt{z}\right)\mathbb{1}_{z\in(1,\infty)} := h(z) \ .\end{aligned}$$

In the first inequality we divide the domain of $z\in\mathbb{R}^+$ in two sets, $D_1\equiv(0,1]$ and $D_2\equiv(1,\infty)$. In the first domain the difference is bounded by one. In the second set, notice that (40) is still valid for $z>1$; then, the difference is lower than N' computed on the max of the arguments of N multiplied by the positive difference of the arguments of N. The second inequality holds because $\varphi_t/(\bar\sigma^2\eta_t)$ is positive and $t<1$. We observe that $h(z)$ goes to zero as z goes to infinity.
Notice that X_t converges to X in distribution, $g_t(z)$ is a sequence of positive function uniformly bounded, Lipschitz continuous with bounded Lipschitz constant on $(0,\tau)$, and $\lim_{z\to\infty} h(z)$=0. Thus, we prove the thesis via **Lemma B.1** □

Lemma B.3. *For $t>0$,*

$$\sup_z\left|\mathbb{P}(S_t<z)-N\left((z-1)\sqrt{\frac{t}{k_t}}\right)\right| \le \frac{2-\alpha}{1-\alpha}\sqrt{\frac{k_t}{t}} \ , \tag{41}$$

where S_t is the random variable of ***Definition 2.1*** *with Laplace transform $\mathcal{L}_t(u;\,k_t,\,\alpha)$. Moreover, if $\beta>1$,*

1.
$$\lim_{t\to0}\mathbb{P}(S_t<1)=\lim_{t\to0}\mathbb{P}(S_t\ge1)=\lim_{t\to0}\mathbb{P}\left(S_t\le\frac{\varphi_t}{\bar\sigma^2\eta_t}\right)=\frac{1}{2} \ .$$

2.
$$\lim_{t\to0}\mathbb{P}(S_t\le 1-t^q)=\begin{cases}1/2 & \text{if } q>\frac{\beta-1}{2}\\ N(-1/\sqrt{k}) & \text{if } q=\frac{\beta-1}{2}\\ 0 & \text{if } q<\frac{\beta-1}{2}\end{cases} \ .$$

3.
$$\lim_{t\to0}\mathbb{P}(S_t\le 1+t^q)=\begin{cases}1/2 & \text{if } q>\frac{\beta-1}{2}\\ N(1/\sqrt{k}) & \text{if } q=\frac{\beta-1}{2}\\ 1 & \text{if } q<\frac{\beta-1}{2}\end{cases} \ .$$

Proof. We use an approach similar to Küchler and Tappe (2013, Th.4.7, p.4271). Given $t>0$, $n\in\mathbb{N}$ we define $X_t^i := S_t^i-1$ for $i=1,2,...,n$ with S_t^i independent positive random variables with Laplace transform $\mathcal{L}_t(u;\,k_t\,n,\,\alpha)$. The standard deviation of S_t^i is $\Sigma_t^n := \sqrt{k_t n/t}$. We define $Q_t^n := \sum_{i=1}^n X_t^i/(\sqrt{n}\Sigma_t^n)$. Notice that $Q_t^n+\sqrt{n}/\Sigma_t^n$ has the same law of $\sqrt{t/k_t}S_t$ by identity in Laplace transform because

$$\mathbb{E}\left[e^{-u\left(P_t^n+\sqrt{n}/\Sigma_t^n\right)}\right]=\mathbb{E}\left[e^{-u\sum_{i=1}^n S_t^i/(\sqrt{n}\Sigma_t^n)}\right]=e^{\sum_{i=1}^n\frac{t(1-\alpha)}{k_t\alpha n}\left(1-\left(1+\frac{u\sqrt{k_t}}{\sqrt{t}(1-\alpha)}\right)^{\alpha}\right)}=\mathbb{E}\left[e^{-u\sqrt{\frac{t}{k_t}}S_t}\right] \ .$$

Thus, $\sqrt{k_t/t}\, Q_t^n + 1$ is equal in distribution to S_t. Moreover, for any $t > 0$

$$\sup_z |\mathbb{P}(Q_t^n < z) - N(z)| \leq \frac{\mathbb{E}\left[|X_t^i|^3\right]}{(\Sigma_t^n)^3 \sqrt{n}} < \frac{\mathbb{E}\left[(S_t^i)^3\right] + 1}{(\Sigma_t^n)^3 \sqrt{n}} = t^{3/2} \frac{2 + 3\frac{k_t n}{t} + \frac{(2-\alpha)k_t^2 n^2}{(1-\alpha)t^2}}{k_t^{3/2} n^2} = \frac{2-\alpha}{1-\alpha}\sqrt{\frac{k_t}{t}} + O\left(\frac{1}{n}\right) \quad .$$

The first inequality holds thanks to the Berry-Esseen theorem (see, e.g., Durrett 2019, Th.3.4.17, p.136). The first equality is obtained by substituting the third moment of S_t^i and in the last equality we emphasize the leading term in $1/n$. Thus, $\forall\, \epsilon > 0$ it exists n such that

$$\sup_z |\mathbb{P}(Q_t^n < z) - N(z)| < \frac{2-\alpha}{1-\alpha}\sqrt{\frac{k_t}{t}} + \epsilon \quad .$$

By definition of cumulative distribution function, we get (41).
Equation (41) allow us to prove the limits of the probability.

1.
$$\mathbb{P}(S_t \leq 1) = N(0) + O\left(t^{(\beta-1)/2}\right) = 1/2 + O\left(t^{(\beta-1)/2}\right) \quad ,$$
where the first equality is due to (41) and second term goes to zero because $\beta > 1$. Moreover,
$$P\left(S_t \leq \frac{\varphi_t}{\bar{\sigma}^2 \eta_t}\right) = N\left(\frac{\varphi_t - \bar{\sigma}^2 \eta_t}{\bar{\sigma}^2 \eta_t}\sqrt{\frac{t}{k_t}}\right) + O\left(t^{(\beta-1)/2}\right) \quad ,$$
where $\lim_{t\to 0} N\left(\frac{\varphi_t - \bar{\sigma}^2\eta_t}{\bar{\sigma}^2\eta_t}\sqrt{\frac{t}{k_t}}\right) = \frac{1}{2}$ thanks to **Lemma A.6** point 3 observing that
$$\left(\frac{\varphi_t}{\bar{\sigma}^2 \eta_t} - 1\right)\sqrt{\frac{t}{k_t}} = \sqrt{t}\bar{\sigma}^2 \eta_t \sqrt{k_t} + O\left(t^{2\delta + (3\beta+1)/2}\right) = o(1) \quad ,$$
because, in the additive process boundaries, for $\beta > 1$, $\delta + (\beta+1)/2 > \delta + 1 > 0$.

2.
$$\mathbb{P}(S_t \leq 1 - t^q) = N\left(-t^q \sqrt{\frac{t}{k_t}}\right) + O\left(t^{(\beta-1)/2}\right) \quad ,$$
where $N\left(-t^q\sqrt{\frac{t}{k_t}}\right)$ goes to 1/2 if $q > (\beta-1)/2$, to $N(-1/(\sqrt{\bar{k}})$ if $q = (\beta-1)/2$, and to 0 if $q < (\beta-1)/2$. We emphasize that the second term goes to zero as $O\left(t^{(\beta-1)/2}\right)$.

3.
$$\mathbb{P}(S_t \leq 1 + t^q) = N\left(t^q \sqrt{\frac{t}{k_t}}\right) + O\left(t^{(\beta-1)/2}\right) \quad ,$$
where $N\left(t^q\sqrt{\frac{t}{k_t}}\right)$ goes to 1/2 if $q > (\beta-1)/2$, to $N(1/(\sqrt{\bar{k}})$ if $q = (\beta-1)/2$, and to 1 if $q < (\beta-1)/2$. We emphasize that the second term goes to zero as $O\left(t^{(\beta-1)/2}\right)$.

□

Lemma B.4. *If $\delta = -1/2$, $\exists H > 1$ s.t.*

$$m(z) := N'\left(-l_t^z + \frac{\bar{\sigma}\sqrt{zt}}{2}\right)\bar{\sigma}\sqrt{z} \quad , \quad z > 0 \quad ,$$

is increasing for $z \in [1, H]$ for sufficiently small t, where l_t^z is the quantity defined in equation (6).

Proof. We compute the derivative w.r.t. z of $m(z)$ and study its sign at short time.

$$\frac{\partial m(z)}{\partial z} = \frac{\frac{\partial N'\left(-l_t^z+\frac{\bar{\sigma}\sqrt{zt}}{2}\right)\bar{\sigma}\sqrt{z}}{\partial z}}{N'\left(\bar{\sigma}\bar{\eta}\left(\sqrt{z}-\frac{\varphi_t}{\bar{\sigma}^2\eta_t\sqrt{z}}\right)+\frac{\bar{\sigma}\sqrt{tz}}{2}\right)\bar{\sigma}}$$
$$= \frac{1}{2\sqrt{z}} - 2\left(\bar{\sigma}\bar{\eta}\left(\sqrt{z}-\frac{\varphi_t}{\bar{\sigma}^2\sqrt{z}\eta_t}\right)+\frac{\bar{\sigma}\sqrt{tz}}{2}\right)\left(\bar{\sigma}\bar{\eta}\left(\frac{1}{2}+\frac{\varphi_t}{2\bar{\sigma}^2\eta_t z}\right)+\frac{\bar{\sigma}\sqrt{t}}{4}\right)$$
$$= \frac{1}{z^{3/2}}\left(\frac{z}{2} - z^2\left(\bar{\sigma}\bar{\eta}+\bar{\sigma}\frac{\sqrt{t}}{2}\right)^2 + \bar{\sigma}^2\bar{\eta}^2\frac{\varphi_t^2}{\bar{\sigma}^4\eta_t^2}\right) \quad .$$

The derivative is positive if

$$0 < z < \frac{1/2+\sqrt{1/4+4\left(\bar{\sigma}\bar{\eta}+\bar{\sigma}\frac{\sqrt{t}}{2}\right)^2\bar{\sigma}^2\bar{\eta}^2\frac{\varphi_t}{\bar{\sigma}^4\eta_t^2}}}{2\left(\bar{\sigma}\bar{\eta}+\bar{\sigma}\frac{\sqrt{t}}{2}\right)^2} \quad .$$

Notice that $\exists\ \tau$ and $H > 1$ such that for every $t < \tau$ the derivative is positive if $z < H$ because for sufficiently small time

$$\frac{1/2+\sqrt{1/4+4\left(\bar{\sigma}\bar{\eta}+\bar{\sigma}\frac{\sqrt{t}}{2}\right)^2\bar{\sigma}^2\bar{\eta}^2\frac{\varphi_t^2}{\bar{\sigma}^4\eta_t^2}}}{2\left(\bar{\sigma}\bar{\eta}+\bar{\sigma}\frac{\sqrt{t}}{2}\right)^2} > \frac{1/2}{2(\bar{\sigma}\bar{\eta}+\bar{\sigma}\sqrt{t}/2)} + \frac{\bar{\eta}}{\bar{\eta}+\sqrt{t}/2}\,\frac{\varphi_t}{\bar{\sigma}^2\eta_t} > H > 1 \quad ,$$

where the first inequality is obtained by bounding from below $1/4$ with 0 inside the square root and the second holds because, by **Lemma A.6** point 3,

$$\lim_{t\to 0}\frac{1/2}{2(\bar{\sigma}\bar{\eta}+\bar{\sigma}\sqrt{t}/2)} + \frac{\bar{\eta}}{\bar{\eta}+\sqrt{t}/2}\,\frac{\varphi_t}{\bar{\sigma}^2\eta_t} = \frac{1}{4\bar{\sigma}^4\bar{\eta}^2}+1 \quad .$$

Thus, $m(z)$ is increasing in [1,H] for sufficiently small t □