The Averaging Method - Lecture 3

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1. Averaging Method for Studying Periodic Solutions of Discontinuous Piecewise Smooth Differential Equations

Let $D \subset \mathbb{R}^d$ open bounded subset and $\mathbb{S}^1 = \mathbb{R}/T$ for some $T > 0$. Consider a finite sequence of open disjoints subset $(S_j) \subset \mathbb{S}^1 \times D$, $j = 1, \dots, N$. Assume that the boundaries of S_j are piecewise C^k embedded hypersurfaces. Denote by Σ the union of all boundaries. Notice that the union of Σ with all S_j's cover $\mathbb{S}^1 \times D$.

For $i \in \{1, \dots, k\}$, let

$$F_i(t,x) = \sum_{j=1}^{N} \chi_{\overline{S}_j}(t,x) F_i^j(t,x) \quad \text{and} \quad R(t,x,\varepsilon) = \sum_{j=1}^{N} \chi_{\overline{S_j}}(t,x) R^j(t,x),$$

where, for $A \subset \mathbb{S}^1 \times D$, $\chi_A(t,x)$ denotes the **characteristic function**

$$\chi_A(t,x) = \begin{cases} 1 & \text{if} \quad (t,x) \in A, \\ 0 & \text{if} \quad (t,x) \notin A. \end{cases}$$

Here, the functions $F_i^j : \mathbb{R} \times \overline{D} \to \mathbb{R}^n$, for $i = 1, \dots, k$, and $j = 1, \dots, N$, and $R\colon \mathbb{R} \times \overline{D} \times [0, \varepsilon_0] \to \mathbb{R}^n$ are assumed to be $T-$periodic in the variable t and smooth.

Consider regularly perturbed piecewise smooth non-autonomous differential equations given in the following **standard form**:

$$x' = \sum_{i=1}^{k} \varepsilon F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon), \ (t,x,\varepsilon) \in \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0). \tag{1}$$

Crossing Hypothesis (CH) $\exists$ an open bounded subset $C \subset D$ such that, for $\varepsilon = 0$,
$$\{(t, z) : t \in \mathbb{S}^1\} \cap \Sigma \subset \Sigma^c.$$

1.1. First-Order Averaging Method

$$\mathbf{f}_1(z) = \int_0^T F_1(t, z)dt$$

Under hypothesis **(CH)**, one can see that $\mathbf{f}_1$ is smooth on C. Then, the following result holds.

Theorem 1 ([15]). *In addition to hypothesis* **(CH)**, *assume that* $z^* \in C$ *is a simple zero of* $\mathbf{f}_1$. *Then, for* $|\varepsilon| \neq 0$ *sufficiently small, the differential equation* (1) *admits a unique* T*-periodic solution* $\varphi(t, \varepsilon)$ *such that* $\varphi(\cdot, \varepsilon) \to z^*$ *as* $\varepsilon \to 0$.

1.2. Example - Discontinuous Perturbed Harmonic Oscillator

$$x'' + x + b_\varepsilon x' = g_\varepsilon(x, x'), \tag{2}$$

where $b_\varepsilon = \varepsilon b_1 + \varepsilon^2 O(1) > 0$ and $g_\varepsilon(x, y) = \varepsilon g_1(x, y) + \varepsilon^2 O(1)$.

Standard Form: $\dfrac{dr}{d\theta} = \varepsilon \sin(\theta)\big(b_1 r \sin(\theta) - g_1(r\cos(\theta), r\sin(\theta)\big) + \varepsilon^2 O(1)$

First-Order Averaged Function: $\mathbf{f}_1(r) = b_1 \pi r - \displaystyle\int_0^{2\pi} \sin(\theta) g_1(r\cos(\theta), r\sin(\theta)) d\theta$

Examplo 1.

$$g_1(x, y) = \begin{cases} \beta^+, & y \geq 0 \\ \beta^-, & y \leq 0 \end{cases} = \frac{\beta^+ + \beta^-}{2} + \operatorname{sign}(y)\frac{\beta^+ - \beta^-}{2}.$$

$$\frac{dr}{d\theta} = \varepsilon \begin{cases} \sin(\theta)\big(b_1 r \sin(\theta) - \beta^+\big) + O(\varepsilon), & \theta \in [0, \pi/2] \\ \sin(\theta)\big(b_1 r \sin(\theta) - \beta^-\big) + O(\varepsilon), & \theta \in [\pi/2, 2\pi] \end{cases} \tag{3}$$

$$\begin{aligned} \mathbf{f}_1(r) = &\quad b_1 \pi r - \int_0^{2\pi} \sin(\theta) g_1(r\cos(\theta), r\sin(\theta)) d\theta \\ = &\quad b_1 \pi r - \int_0^{\pi} \beta^+ \sin(\theta) d\theta - \int_\pi^{2\pi} \beta^- \sin(\theta) d\theta \\ = &\quad -2(\beta^+ - \beta^-) + b_1 \pi r \end{aligned}$$

Assuming $b_1(\beta^+ - \beta^-) > 0$, the equation $f_1(r) = 0$ a unique positive solution $r^* = \frac{2(\beta^+ - \beta^-)}{b_1 \pi}$. Hence, the **First-Order Averaging Method** ,Theorem 1, provides the existence of a periodic solution $r(\theta, \varepsilon)$ of differential equation (3) such that $r(\cdot, \varepsilon) \to r^*$ as $\varepsilon \to 0$. Accordingly, one gets the existence of a periodic solution $(x(t, \varepsilon), x'(t, \varepsilon))$ of the differential equation (2) satisfying $|(x(\cdot, \varepsilon), x'(\cdot, \varepsilon))| \to r^*$ as $\varepsilon \to 0$.

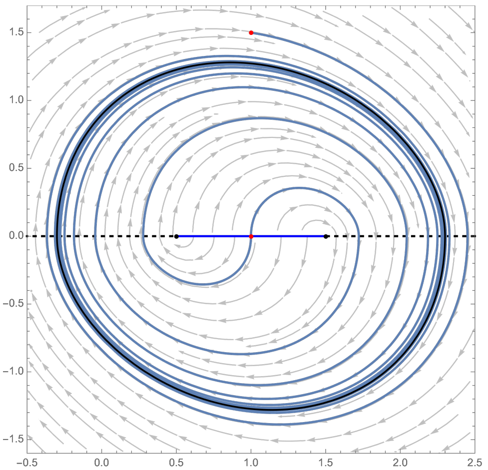


FIGURE 1. Forward trajectories of the differential system $(x', y') = \big(y, -x + \varepsilon\, y + \varepsilon g_1(x, y)\big)$ for $\beta^+ = 3$, $\beta^- = 1$, and $\varepsilon = 0.5$. The red dots indicate the initial conditions $(1, 0)$ and $(1, 1.5)$. The blue segment indicates an escaping region and the dashed black segment a crossing region. The black closed curve indicates a limit cycle.

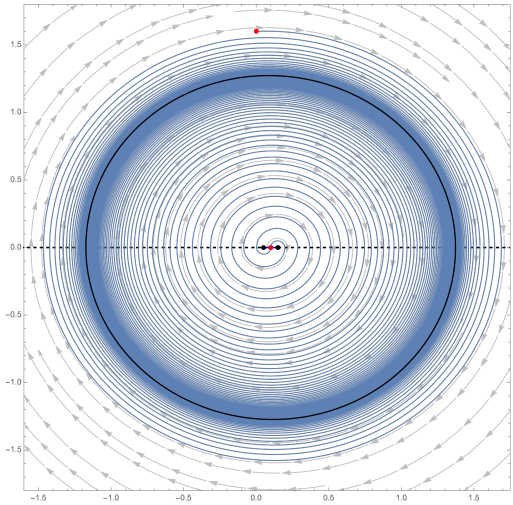


FIGURE 2. Forward trajectories of the differential system $(x', y') = \big(y, -x + \varepsilon\, y + \varepsilon g_1(x, y)\big)$ for $\beta^+ = 3$, $\beta^- = 1$, and $\varepsilon = 0.05$ The red dots indicate the initial conditions $(0.1, 0)$ and $(0.1, 1.5)$. The blue segment indicates an escaping region and the dashed black segment a crossing region. The black closed curve indicates a limit cycle.

1.3. Second-Order Averaging Method

$$\mathbf{f}_2(z) = \int_0^T \big[F_2(t, z) + \partial_x F_1(t, z) y_1(t, z)\big] dt, \text{ where}$$

$$\partial_x F_1(t, z) = \sum_{j=1}^{N} \chi_{\overline{S}_j}(t, z) \partial_x F_1^j(t, z), \text{ and } y_1(t, z) = \int_0^t F_1(s, z) ds.$$

Theorem 2 ([15]). *Suppose that $\mathbf{f}_1 = 0$. In addition to the hypothesis (HC) assume that*

$$\boxed{\textbf{Hb2} \quad \big(0, y_1(t, z)\big) \in T_{(t,z)}\Sigma \quad \textit{for} \quad (t, z) \in \Sigma}$$

Assume that $z^* \in D$ is a simple zero of $\mathbf{f}_2$. Then, for $|\varepsilon| \neq 0$ sufficiently small, the differential equation (1) admits a unique T-periodic solution $\varphi(t, \varepsilon)$ such that $\varphi(\cdot, \varepsilon) \to z^*$ as $\varepsilon \to 0$.

Planar system with rays of discontinuity!

$$\Longrightarrow$$

Discontinuity only in the time variable!

$$\Longrightarrow$$

Hypothesis Hb2 holds!

1.4. Example - Discontinuous Perturbation of Quadratic Homogeneous Center
Consider the following families of quadratic homogeneous center (see [11]):

$$Z_1 : \begin{cases} \dot{x} = -y + x^2 - y^2, \\ \dot{y} = x + 2xy. \end{cases} \quad Z_2 : \begin{cases} \dot{x} = -y + x^2, \\ \dot{y} = x + xy. \end{cases} \quad Z_3 : \begin{cases} \dot{x} = -y - \dfrac{4}{3}x^2, \\ \dot{y} = x - \dfrac{16}{3}xy. \end{cases} \tag{4}$$

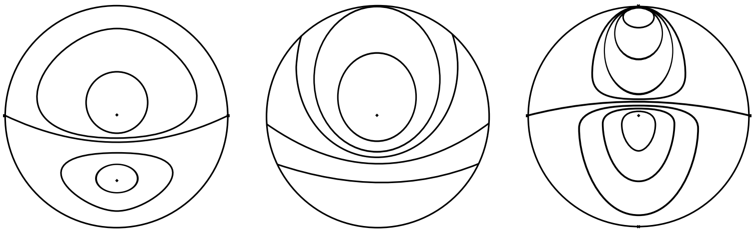


FIGURE 3. Phase portrait of systems Z_1, Z_2, and Z_3 from left to right.

In [12] it was studied bifurcation of limit cycles from such families when they are perturbed by discontinuous piecewise quadratic polynomial systems as follows

$$Z_{i,\varepsilon} = \begin{cases} Z_i(x,y) + \varepsilon\,(P_1^+(x,y), Q_1^+(x,y)) + \varepsilon^2\,(P_2^+(x,y), Q_2^+(x,y)), & \text{if } h(x,y) > 0, \\ Z_i(x,y) + \varepsilon\,(P_1^-(x,y), Q_1^-(x,y)) + \varepsilon^2\,(P_2^-(x,y), Q_2^-(x,y)), & \text{if } h(x,y) < 0, \end{cases} \tag{5}$$

where ε is sufficiently small, $h(x,y) = y - \tan(\alpha)x$, and, for $k = 1, 2$,

$$P_k^\pm(x,y) = \sum_{j=0}^{2}\sum_{i=0}^{j} p^\pm_{k,i,j-i}x^iy^{j-i} \quad \text{and} \quad Q_k^\pm(x,y) = \sum_{j=0}^{2}\sum_{i=0}^{j} q^\pm_{k,i,j-i}x^iy^{j-i}.$$

In what follows we shall study the family S_2 for $\Sigma = \{y + \sqrt{3}x = 0\}$, that is, $\alpha = \pi/3$.
Step 1 (Standard Form): *Write system $Z_{2,\varepsilon}$ in the standard form of the averaging method.*

Assume $P_k^\pm(0,0) = Q_k^\pm(0,0)$. The linearization stated in [11] of Z_2 is given by

$$x = -\frac{u}{v-1} \quad \text{and} \quad y = -\frac{v}{v-1},$$

which has the following rational inverse

$$u = \frac{x}{y+1} \quad \text{and} \quad v = \frac{y}{y+1}.$$

Notice that straight lines passing through the origin are fixed by the transformation. With this change of variables the differential equation Z_2 becomes the linear center $(u', v') = (-v, u)$. Then, composing with the polar coordinates $u = r\cos\theta$ and $v = -r\sin\theta$ and taking θ as the new time variable, $Z_{2,\varepsilon}$ becomes

$$r'(\theta) = \frac{\dot{r}}{\dot{\theta}} = \varepsilon\frac{\mathcal{A}_1(r\cos\theta, r\sin\theta)}{1 + r\cos\theta} + \varepsilon^2\frac{\mathcal{A}_2(r\cos\theta, r\sin\theta)}{1 + r\cos\theta} + \mathcal{O}(\varepsilon^3), \tag{6}$$

where $\mathcal{C}(\theta, r) =$ and $\mathcal{A}_i$, $i = 1, 2$, are piecewise functions

$$\mathcal{A}_i(r\cos\theta, r\sin\theta) = \begin{cases} \mathcal{A}_i^+(r\cos\theta, r\sin\theta) & \text{if} \quad 0 < \theta \leq \pi/3, \\ \mathcal{A}_i^-(r\cos\theta, r\sin\theta) & \text{if} \quad \pi/3 < \theta \leq 4\pi/3, \\ \mathcal{A}_i^+(r\cos\theta, r\sin\theta) & \text{if} \quad 4\pi/3 < \theta \leq 2\pi, \end{cases} \tag{7}$$

being $\mathcal{A}_i^\pm$ polynomials of degree 3.

Step 2 (First Order Analysis): *Compute* $\mathbf{f}_1$ *and its zeros.*

$$\mathbf{f}_1(r) = \int_0^{2\pi} \frac{\mathcal{A}_1(r\cos\theta, r\sin\theta)}{1+r\cos\theta}\,d\theta = \sum_{n=0}^{7} k_n f_n,$$

with

$$\begin{aligned}
f_0(\rho) =& \frac{\rho}{\rho^2+1}, \quad f_1(\rho) = \frac{\rho^2}{(\rho^2+1)^2}, \quad f_2(\rho) = \frac{\rho^3}{(\rho^2+1)^2}, \quad f_4(\rho) = \frac{\rho^5}{(\rho^2+1)^2},\\
f_3(\rho) =& \frac{5(54733\rho^4+94452\rho^2+54733)}{6912(\rho^2+1)^2} + \frac{15(1366\rho^4+1847\rho^2+1366)}{1024(\rho^2+1)\rho}\,\widetilde{L}(\rho)\\
&+ \frac{25\sqrt{3}(236\rho^4-247\rho^2+236)(\rho^2-1)^2}{82944\rho(\rho^2+1)^3}\,\widetilde{\phi}(\rho),\\
f_5(\rho) =& -\frac{35(21835\rho^4+40596\rho^2+21835)}{6912(\rho^2+1)^2} - \frac{105(550\rho^4+797\rho^2+550)}{1024(\rho^2+1)\rho}\,\widetilde{L}(\rho)\\
&- \frac{175\sqrt{3}(176\rho^4-181\rho^2+176)(\rho^2-1)^2}{82944\rho(\rho^2+1)^3}\,\widetilde{\phi}(\rho),\\
f_6(\rho) =& \frac{245(227\rho^4+444\rho^2+227)}{768(\rho^2+1)^2} + \frac{315(122\rho^4+181\rho^2+122)}{1024(\rho^2+1)\rho}\,\widetilde{L}(\rho)\\
&+ \frac{35\sqrt{3}(116\rho^4-115\rho^2+116)(\rho^2-1)^2}{9216\rho(\rho^2+1)^3}\,\widetilde{\phi}(\rho),\\
f_7(\rho) =& -\frac{385(77\rho^4+156\rho^2+77)}{2304(\rho^2+1)^2} - \frac{3465(2\rho^4+3\rho^2+2)}{1024(\rho^2+1)\rho}\,\widetilde{L}(\rho)\\
&- \frac{385\sqrt{3}(8\rho^4-7\rho^2+8)(\rho^2-1)^2}{27648\rho(\rho^2+1)^3}\,\widetilde{\phi}(\rho).
\end{aligned} \tag{8}$$

Here,

$$\widetilde{L}(\rho) = \log\left(\frac{\rho^2-\rho+1}{\rho^2+\rho+1}\right), \quad \widetilde{\phi}(\rho) = \phi\left(\frac{2\rho}{\rho^2+1}, \frac{2\pi}{3}\right) - \phi\left(-\frac{2\rho}{\rho^2+1}, \frac{2\pi}{3}\right).$$

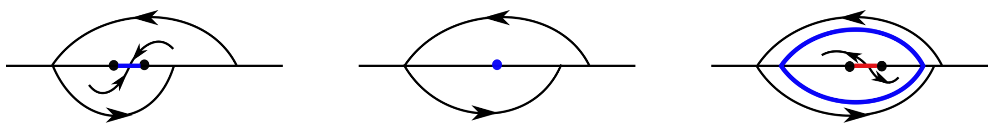
where

$$\phi(r,\theta) = \frac{1}{\sqrt{1-r^2}}\left(\theta - 2\arctan\left(\sqrt{\frac{1-r}{1+r}}\tan\left(\frac{\theta}{2}\right)\right)\right),$$

and the parameters k_n, $n = 0, 1, \dots, 7$, are arbitrary real numbers which depends on the parameters of perturbations. One can see that $f_i(\rho) = \rho^{i+1} + O(\rho^{i+2})$ for $i = 0, 1, \dots, 5$, $f_6(\rho) = \rho^8 + O(\rho^9)$, and $f_7(\rho) = \rho^{10} + O(\rho^{11})$. Therefore, the ordered set of functions $[f_0, f_1, \dots, f_7]$ is an ECT-system in a neighborhood of the origin (see [14, 19]).

Averaging Method $\Rightarrow$ 7 limit cycles.

Pseudo-Hopf-Bifurcation



$b < 0$ $\qquad$ $b = 0$ $\qquad$ $b > 0$

Averaging Method + Pseudo-Hopf Bifurcation $\Rightarrow$ 8 limit cycles.

Step 3 (Second Order Analysis): *Assume minimal conditions in order that* $\mathbf{f}_1 = 0$ *and compute* $\mathbf{f}_2$ *and its zeros.*

Notice that $\mathbf{f}_1 = 0$ if, and only if, $k_i = 0$ for $i \in \{0, \ldots, 7\}$.

$$\mathbf{f}_2(r) = \overbrace{\int_0^{2\pi} \frac{\mathcal{A}_2(r\cos\theta, r\sin\theta)}{1 + r\cos\theta} d\theta}^{I_1} \\ + \underbrace{\int_0^{2\pi} \left[\frac{\partial}{\partial r} \left(\frac{\mathcal{A}_1(r\cos\theta, r\sin\theta)}{1 + r\cos\theta} \right) \int_0^{\theta} \frac{\mathcal{A}_1(r\cos\alpha, r\sin\alpha)}{1 + r\cos\alpha} d\alpha \right] d\theta}_{I_2}$$

An expression for I_1 can be obtained analogously to $\mathbf{f}_1$. However, in order to obtain I_2, one must integrate rational functions with denominators $(1 + r\cos\theta)^2$ and numerators depending on

$$\{r, \theta, \cos\theta, \sin\theta, \lambda(r, -\pi/3 + \theta), \lambda(r, -4\pi/3 + \theta), \phi(r, -\pi/3 + \theta), \phi(r, -4\pi/3 + \theta)\},$$

where ϕ is defined above and $\lambda(r, \theta) = \log(1 + r\cos\theta)$. Unfortunately, I_2 cannot be explicitly computed.

Computing the Taylor series of the integrand around $r = 0$ and then integrating one can see that

$$\mathbf{f}_2(r) = \sum_{i=1}^{n} f_i r^i + O(r^{n+1}).$$

The coefficients f_i's depend linearly on $\{p^{\pm}_{2,i,j}, q^{\pm}_{2,i,j}\}$ and quadratically on $\{p^{\pm}_{1,i,j}, q^{\pm}_{1,i,j}\}$.

Poincaré-Miranda Theorem provides a transformation on the parameters such that

$$\mathbf{f}_2(r) = \sum_{i=1}^{16} d_i r^i + O(r^{n+1}),$$

where $(d_1, \ldots, d_{16})$ depends onto the parameters of perturbation in a surjective way around the origin $0 \in \mathbb{R}^{16}$.

Averaging Method $\Rightarrow$ 15 limit cycles.

Averaging Method + Pseudo-Hopf Bifurcation $\Rightarrow$ 16 limit cycles.

I THINK THAT IT IS THE BEST LOWER BOUND SO FAR
for the number of limit cycles in piecewise quadratic polynomial differential systems
with two zones separated by a straight line.

1.5. Higher Order Averaging Method

Let $0=\alpha_0<\alpha_1<\cdots<\alpha_{N-1}<\alpha_N=T$. Consider the differential equation

$$x'(\theta)=\sum_{i=1}^{k}\varepsilon^i F_i(\theta,x)+\varepsilon^{k+1}R(\theta,x,\varepsilon), \tag{9}$$

where

$$\begin{aligned} F_i(\theta,x)&=\sum_{j=0}^{N}\chi_{[\alpha_j,\alpha_{j+1}]}(\theta)F_i^j(\theta,x),\ i=0,1,...,k, \quad \text{and}\\ R(\theta,x,\varepsilon)&=\sum_{j=0}^{N}\chi_{[\alpha_j,\alpha_{j+1}]}(\theta)R^j(\theta,x,\varepsilon).\end{aligned} \tag{10}$$

Here, the functions $F_i^j:\mathbb{R}\times\overline{D}\to\mathbb{R}^n$, for $i=1,\dots,k$, and $j=1,\dots,N$, and $R\colon\mathbb{R}\times\overline{D}\times[0,\varepsilon_0]\to\mathbb{R}^n$ are assumed to be $T-$periodic in the variable t and smooth. Notice that $\Sigma=(\{\theta=0\}\cup\{\theta=\alpha_1\}\cup\cdots\cup\{\theta=\alpha_{N-1}\})\cap\mathbb{S}^1\times D$ is the discontinuity set of (9).

Theorem 3 **([18]).** *Denote* $\mathbf{f}_0=0$. *Let* $\ell\in\{1,\dots,k\}$ *satisfying* $\mathbf{f}_0=\cdots\mathbf{f}_{\ell-1}=0$ *and* $\mathbf{f}_\ell\neq 0$. *Assume that* $z^*\in D$ *is a simple zero of* $\mathbf{f}_\ell$. *Then, for* $|\varepsilon|\neq 0$ *sufficiently small, the differential equation* (9) *admits a unique* T*-periodic solution* $\varphi(t,\varepsilon)$ *such that* $\varphi(\cdot,\varepsilon)\to z^*$ *as* $\varepsilon\to 0$.

2. Bifurcation Functions in a More General Case

Consider, an open subset $D\subset\mathbb{R}^d$, $\mathbb{S}^1=\mathbb{R}/T$ for some period $T>0$, and k a positive integer. Let $\theta_j:D\to\mathbb{S}^1$, $j\in\{1,\dots,N\}$, be C^{k-1} functions such that $\theta_0(x)\equiv 0<\theta_1(x)<\cdots<\theta_N(x)<T\equiv\theta_{N+1}(x)$, for all $x\in D$. Under the assumptions above, we consider the following piecewise smooth differential system

$$\dot{x}=\sum_{i=1}^{k}\varepsilon^i F_i(t,x)+\varepsilon^{k+1}R(t,x,\varepsilon), \tag{11}$$

where where

$$F_i(t,x)=\sum_{j=0}^{N}\chi_{[\theta_j(x),\theta_{j+1}(x)]}(t)F_i^j(t,x)=\begin{cases} F_i^0(t,x), & 0<t<\theta_1(x),\\ F_i^1(t,x), & \theta_1(x)<t<\theta_2(x),\\ \vdots & \\ F_i^N(t,x), & \theta_N(x)<t<T,\end{cases}$$

$$R(\theta,x,\varepsilon)=\sum_{j=0}^{N}\chi_{[\theta_j(x),\theta_{j+1}(x)]}(t)R^j(t,x,\varepsilon)=\begin{cases} R^0(t,x,\varepsilon), & 0<t<\theta_1(x),\\ R^1(t,x,\varepsilon), & \theta_1(x)<t<\theta_2(x),\\ \vdots & \\ R^N(t,x,\varepsilon), & \theta_N(x)<t<T,\end{cases}$$

with $F_i^j:\mathbb{S}^1\times D\to\mathbb{R}^d$, $R^j:\mathbb{S}^1\times D\times(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}^d$, for $i\in\{1,\dots,k\}$ and $j\in\{1,\dots,N\}$, being smooth functions and $T-$periodic in the variable t. In this case, the switching manifold is provided by $\Sigma=\{(\theta_i(x),x);\ x\in D,\ i\in\{0,1,\dots,N\}\}$.

2.1. Second-Order Bifurcation Function

Define second order bifurcation (Melnikov) function $\mathbf{m}_2 : D \to \mathbb{R}^n$ by

$$\mathbf{m}_2(x) = \mathbf{f}_2(x) + \mathbf{f}_2^*(x),$$

where the increment $\mathbf{f}_2^*$ is given by

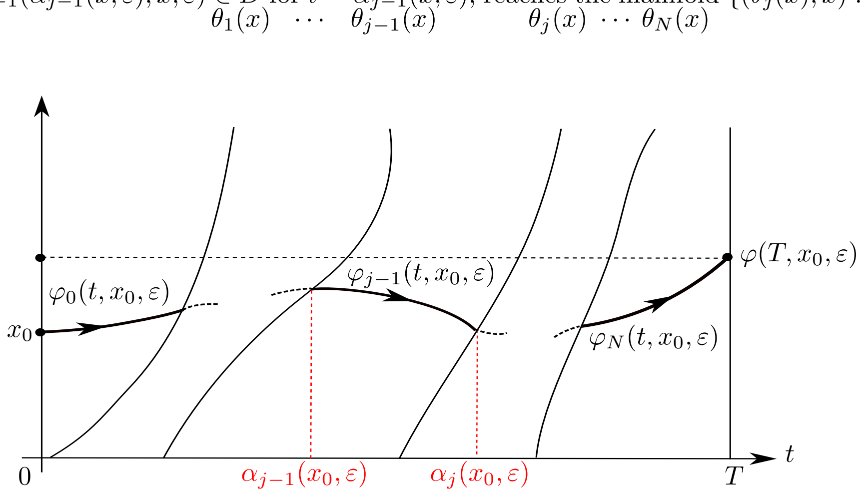
$$\mathbf{f}_2^*(x) = \sum_{j=1}^{N} \Big(F_1^{j-1}(\theta_j(x), x) - F_1^{j}(\theta_j(x), x) \Big) \partial_x \theta_j(x) \int_0^{\theta_j(x)} F_1(s,x) ds.$$

Theorem 4 ([2]). *Suppose that* $\mathbf{f}_1 = 0$. *Assume that* $z^* \in D$ *is a simple zero of* $\mathbf{m}_2$. *Then, for* $|\varepsilon| \neq 0$ *sufficiently small, the differential equation* (11) *admits a unique* T*-periodic solution* $\varphi(t,\varepsilon)$ *such that* $\varphi(\cdot,\varepsilon) \to z^*$ *as* $\varepsilon \to 0$.

Proof. We denote the solution of (11) by

$$\varphi(t,x,\varepsilon) = \begin{cases} \varphi_0(t,x,\varepsilon), \ 0 \le t \le \alpha_1(x,\varepsilon); \\ \varphi_1(t,x,\varepsilon), \ \alpha_1(x,\varepsilon) \le t \le \alpha_2(x,\varepsilon); \\ \cdots \\ \varphi_N(t,x,\varepsilon), \ \alpha_N(x,\varepsilon) \le t \le T. \end{cases} \tag{12}$$

In the above expression, $\alpha_j(x,\varepsilon)$ is the flight-time that the trajectory $\varphi_{j-1}(\cdot,x,\varepsilon)$, starting at $\varphi_{j-1}(\alpha_{j-1}(x,\varepsilon),x,\varepsilon) \in D$ for $t = \alpha_{j-1}(x,\varepsilon)$, reaches the manifold $\{(\theta_j(x),x) : x \in D\} \subset \Sigma$:



Notice that

$$\alpha_j(x,\varepsilon) = \theta_j(\varphi_{j-1}(\alpha_j(x,\varepsilon),x,\varepsilon)).$$

for $j = 1,2,\ldots,N$, $\alpha_0(x,\varepsilon) = 0$, and $\alpha_{N+1}(x,\varepsilon) = T$. Moreover,

$$\begin{aligned} &\frac{\partial \varphi_j}{\partial t}(t,x,\varepsilon) = F^j(t,\varphi_j(t,x,\varepsilon),\varepsilon), \ \text{for } j = 0,1,\ldots,N, \ \text{where} \\ &\begin{cases} \varphi_0(0,x,\varepsilon) = x, \\ \varphi_j(\alpha_j(x,\varepsilon),x,\varepsilon) = \varphi_{j-1}(\alpha_j(x,\varepsilon),x,\varepsilon), \ \text{for } j = 1,2,\ldots,N. \end{cases} \end{aligned} \tag{13}$$

From (13) and from the differential dependence of the solutions on the initial conditions and on the parameter, we can see inductively that $\alpha_j(x,\varepsilon)$ and $\varphi_j(t,x,\varepsilon)$ are C^r functions, for $j=0,1,\ldots,N$.

We notice that the recurrence (13) describes initial value problems. Therefore, it is equivalent to the following recurrence:

$$\varphi_0(t,x,\varepsilon)=x+\int_0^t F^0(s,\varphi_0(s,x,\varepsilon),\varepsilon)ds,$$

$$\varphi_j(t,x,\varepsilon)=\varphi_{j-1}(\alpha_j(x,\varepsilon),x,\varepsilon)+\int_{\alpha_j(x,\varepsilon)}^t F^j(s,\varphi_j(s,x,\varepsilon),\varepsilon)ds,$$

for $j=1,2,\ldots,N$. So, we may compute

$$\begin{aligned}\varphi_N(t,x,\varepsilon)=x&+\sum_{j=1}^N\int_{\alpha_{j-1}(x,\varepsilon)}^{\alpha_j(x,\varepsilon)} F^{j-1}(s,\varphi_{j-1}(s,x,\varepsilon),\varepsilon)ds\\&+\int_{\alpha_N(x,\varepsilon)}^t F^N(s,\varphi_N(s,x,\varepsilon),\varepsilon)ds.\end{aligned}$$

The displacement function is given by

$$\Delta(x,\varepsilon)=\varphi(T,x,\varepsilon)-x=\varphi_N(T,x_0,\varepsilon)-x. \tag{14}$$

Notice that, from the above comments, $\Delta(x,\varepsilon)$ is a C^r function. Since $\Delta(x,0)=0$, then, expanding (14) around $\varepsilon=0$, we have

$$\Delta(x,\varepsilon)=\varepsilon\mathbf{m}_1(x)+\varepsilon^2\mathbf{m}_2(x)+\mathcal{O}(\varepsilon^3).$$

On the other hand, the displacement function writes

$$\Delta(x,\varepsilon)=\varphi_N(T,x,\varepsilon)-x=\sum_{j=1}^{N+1}\int_{\alpha_{j-1}(x,\varepsilon)}^{\alpha_j(x,\varepsilon)} F^{j-1}(s,\varphi_{j-1}(s,x,\varepsilon),\varepsilon)ds. \tag{15}$$

In what follows, we compute the expansion, around $\varepsilon=0$, of the $N+1$ summands of (15). For the first one we obtain

$$\begin{aligned}&\int_0^{\alpha_1(x,\varepsilon)} F^0(s,\varphi_0(s,x,\varepsilon),\varepsilon)ds\\&=\int_0^{\alpha_1(x,\varepsilon)}[\varepsilon F_1^0(s,\varphi_0(s,x,\varepsilon))+\varepsilon^2F_2^0(s,\varphi_0(s,x,\varepsilon))+\mathcal{O}(\varepsilon^3)]ds\\&=\varepsilon\left(\int_0^{\theta_1(x)}F_1^0(s,x)ds\right)+\varepsilon^2\left(\int_0^{\theta_1(x)}\left[DF_1^0(s,x)\frac{\partial\varphi_0}{\partial\varepsilon}(s,x,0)\right.\right.\\&\left.\left.+F_2^0(s,x)\right]ds+F_1^0(\theta_1(x),x)\frac{\partial\alpha_1}{\partial\varepsilon}(x,0)\right)+\mathcal{O}(\varepsilon^3).\end{aligned} \tag{16}$$

In order to finish the computation of the second summand we have to compute the terms $\frac{\partial\varphi_0}{\partial\varepsilon}(s,x,0)$ and $\frac{\partial\alpha_1}{\partial\varepsilon}(x,0)$. Expanding the equation

$$\varphi_0(t,x,\varepsilon)=x+\int_0^t F^0(s,\varphi_0(s,x,\varepsilon),\varepsilon)ds$$

around $\varepsilon=0$ we obtain

$$x+\varepsilon\frac{\partial\varphi_0}{\partial\varepsilon}(t,x,0)+\mathcal{O}(\varepsilon^2)=x+\varepsilon\int_0^t F_1^0(s,x)ds+\mathcal{O}(\varepsilon^2).$$

So, we get

$$\frac{\partial \varphi_0}{\partial \varepsilon}(t,x,0) = \int_0^t F_1^0(s,x)ds. \tag{17}$$

In the same way, expanding the equation $\alpha_1(x,\varepsilon) = \theta_1(\varphi_0(\alpha_1(x,\varepsilon),x,\varepsilon))$ around $\varepsilon = 0$ we obtain

$$\begin{aligned}
&\alpha_1(x,0) + \varepsilon \frac{\partial \alpha_1}{\partial \varepsilon}(x,0) + \mathcal{O}(\varepsilon^2) = \\
&\theta_1(x) + \varepsilon D_x\theta_1(x)\left(\frac{\partial \varphi_0}{\partial t}(\theta_1(x),x,0)\frac{\partial \alpha_1}{\partial \varepsilon}(x,0) + \frac{\partial \varphi_0}{\partial \varepsilon}(\theta_1(x),x,0)\right) + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Since $\varphi_0(t,x,0) = x$ for all t we get

$$\frac{\partial \alpha_1}{\partial \varepsilon}(x,0) = D_x\theta_1(x)\int_0^{\theta_1(x)} F_1^0(s,x)ds. \tag{18}$$

Substituting (17) and (18) in (16) we have the expression for the first summand of (15)

$$\begin{aligned}
\int_0^{\alpha_1(x,\varepsilon)} F^0(s,\varphi_0(s,x,\varepsilon),\varepsilon)ds &= \varepsilon\left(\int_0^{\theta_1(x)} F_1^0(s,x)ds\right) \\
&+ \varepsilon^2\left(\int_0^{\theta_1(x)}\left[D_xF_1^0(s,x)\int_0^s F_1^0(t,x)dt + F_2^0(s,x)\right]ds\right. \\
&\left. + F_1^0(\theta_1(x),x)D_x\theta_1(x)\int_0^{\theta_1(x)} F_1^0(s,x)ds\right) + \mathcal{O}(\varepsilon^3).
\end{aligned} \tag{19}$$

The other summands of (15) can be computed in a similar way and they are given by

$$\begin{aligned}
\int_{\alpha_{j-1}(x,\varepsilon)}^{\alpha_j(x,\varepsilon)} F^{j-1}(s,\varphi_{j-1}(s,x,\varepsilon),\varepsilon)ds &= \varepsilon\left(\int_{\theta_{j-1}(x)}^{\theta_j(x)} F_1^{j-1}(s,x)ds\right) \\
&+ \varepsilon^2\left(\int_{\theta_{j-1}(x)}^{\theta_j(x)}\left[D_xF_1^{j-1}(s,x)\int_0^s F_1(t,x)dt + F_2^{j-1}(s,x)\right]ds\right. \\
&+ F_1^{j-1}(\theta_j(x),x)D_x\theta_j(x)\int_0^{\theta_j(x)} F_1(s,x)ds \\
&\left. - F_1^{j-1}(\theta_{j-1}(x),x)D_x\theta_{j-1}(x)\int_0^{\theta_{j-1}(x)} F_1(s,x)ds\right) + \mathcal{O}(\varepsilon^3).
\end{aligned} \tag{20}$$

From (19) and (20), we have that

$$\mathbf{m}_1(x) = \int_0^T F_1(s,x)ds = \mathbf{f}_1(x)$$

and

$$\mathbf{m}_2(x) = \mathbf{f}_2(x) + \mathbf{f}_2^*(x),$$

where

$$\mathbf{f}_2(x) = \int_0^T\left[D_xF_1(s,x)\int_0^s F_1(t,x)dt + F_2(s,x)\right]ds$$

and

$$\mathbf{f}_2^*(x) = \sum_{j=1}^N\left(F_1^{j-1}(\theta_j(x),x) - F_1^j(\theta_j(x),x)\right)D_x\theta_j(x)\int_0^{\theta_j(x)} F_1(s,x)ds.$$

From here, the proof follows by applying the Implicit Function Theorem. □

2.2. Higher Order Bifurcation Functions

In [1] it was obtained a sequence of function $\mathbf{m}_i$, $i \in \{1, \dots, k\}$, satisfying the following result:

Theorem 5 ([1]). *Denote* $\mathbf{m}_0 = 0$. *Let* $\ell \in \{1, \dots, k\}$ *satisfying* $\mathbf{m}_0 = \cdots \mathbf{m}_{\ell-1} = 0$ *and* $\mathbf{f}_\ell \neq 0$. *Assume that* $z^* \in D$ *is a simple zero of* $\mathbf{m}_\ell$. *Then, for* $|\varepsilon| \neq 0$ *sufficiently small, the differential equation* (11) *admits a unique* T*-periodic solution* $\varphi(t, \varepsilon)$ *such that* $\varphi(\cdot, \varepsilon) \to z^*$ *as* $\varepsilon \to 0$.

2.3. Example - Piecewise Linear Differential Systems

Given a positive integer n, let $H(n)$ denote the maximum number of limit cycles that planar piecewise linear systems with two zones separated by the curve $y = x^n$ can have. In order to provide lower bounds for $H(n)$ consider the following class of piecewise linear differential system:

$$Z(x,y) = \begin{cases} X(x,y) = \begin{pmatrix} y + \displaystyle\sum_{i=1}^{k} \varepsilon^i P_i^+(x,y) \\ -x + \displaystyle\sum_{i=1}^{k} \varepsilon^i Q_i^+(x,y) \end{pmatrix}, & y - x^n > 0, \\ \\ Y(x,y) = \begin{pmatrix} y + \displaystyle\sum_{i=1}^{k} \varepsilon^i P_i^-(x,y) \\ -x + \displaystyle\sum_{i=1}^{k} \varepsilon^i Q_i^-(x,y) \end{pmatrix}, & y - x^n < 0, \end{cases} \tag{21}$$

where n is a positive integer, and $P_i^\pm$ and $Q_i^\pm$ are affine functions provided by

$$\begin{aligned} P_i^+(x,y) &= a_{0i} + a_{1i}x + a_{2i}y, \\ P_i^-(x,y) &= \alpha_{0i} + \alpha_{1i}x + \alpha_{2i}y, \\ Q_i^+(x,y) &= b_{0i} + b_{1i}x + b_{2i}y, \\ Q_i^-(x,y) &= \beta_{0i} + \beta_{1i}x + \beta_{2i}y, \end{aligned}$$

with $a_{ji}, \alpha_{ji}, b_{ji}, \beta_{ji} \in \mathbb{R}$, for $i \in \{1, \dots, k\}$ and $j \in \{0, 1, 2\}$. The switching curve of system (21) is provided by $\Sigma = \{(x,y) \in \mathbb{R}^2 : y = x^n\}$.

Denote $\mathbf{m}_0 = 0$ and let $\mathbf{m}_\ell$, for some $\ell \in \{1, 2, \dots, k\}$, be the first non-vanishing bifurcation function, that is $\mathbf{m}_i = 0$ for $i \in \{0, \dots, \ell-1\}$ and $\mathbf{m}_\ell \neq 0$. Accordingly, denote by $m_\ell(n)$ the maximum number of simple zeros that the first non-vanishing Melnikov function $\mathbf{m}_\ell$ can have for any choice of parameters $a_{ji}, \alpha_{ji}, b_{ji}, \beta_{ji} \in \mathbb{R}$, for $i \in \{1, \dots, \ell\}$ and $j \in \{0, 1, 2\}$.

Notice that the values $m_\ell(n)$, for $\ell \in \{1, \dots, k\}$, provide lower bounds for $H(n)$, indeed $H(n) \geq m_\ell(n)$ for every $\ell \in \{1, \dots, k\}$.

In [7], a higher order analysis of system (21) was performed assuming a straight line as the switching curve, that is $n = 1$. It was shown that $m_1(1) = m_2(1) = 1$, $m_3(1) = 2$, and $m_\ell(1) = 3$ for $\ell \in \{4, \dots, 7\}$. The nonlinear case of switching curves was firstly addressed in [15] by means of *Averaging Theory.* In particular, it was shown that $m_1(3) = 3$ and $m_2(3) = 7$. The known values (before [1]) in research literature for $m_\ell(n)$, for $\ell \in \{1, \dots, 6\}$, are summarized in Table 1. In particular, these previous studies provided $H(1) \geq 3$, $H(2) \geq 2$, and $H(3) \geq 7$.

Known results for $m_\ell(n)$

Degree n \ Order ℓ	**1**	**2**	**3**	**$4 \le \ell \le 6$**
1	1	1	2	3
2	3	–	–	–
3	3	7	–	–
$n \ge 4$	–	–	–	–

Table 1. Known values (before [1]) in the research literature. In particular, $H(1) \geq 3$, $H(2) \geq 2$, and $H(3) \geq 7$.

In [1], using the Theorem 5 up to $\ell = 6$, we were able to complete table 1 as follows

Our contribution

Degree n \ Order k	**1**	**2**	**3**	**4**	**5**	**6**
1	1	1	2	3	3	3
2	3	4	4	4	4	4
3	3	7	7	7	7	$8 \le m_6 \le 10$
$n \ge 4$ even	4	7	7	7	7	7
$n \ge 5$ odd	3	7	7	7	7	$9 \le m_6 \le 14$

Table 2. Our main result competes Table 1. In particular, $H(2) \geq 4$, $H(3) \geq 8$, $H(n) \geq 7$, for $n \geq 4$ even, and $H(n) \geq 9$, for $n \geq 5$ odd

3. Other problems that can be approached by averaging method

3.1. Bifurcation of Periodic Solutions from Families of Periodic Solutions

$$x' = F_0(t,x) + \sum_{i=1}^{k} \varepsilon^i F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon), \tag{22}$$

For $m \leq n$, let V be an open bounded subset of $\mathbb{R}^m$ and $\beta : \overline{V} \to \mathbb{R}^{n-m}$ a $\mathcal{C}^{k+1}$ function. Define

$$\mathcal{Z} = \{z_\alpha = (\alpha, \beta(\alpha)) : \alpha \in \overline{V}\}. \tag{23}$$

Denote by $x(t,z,\varepsilon)$ the solution of (22) such that $x(0,z,\varepsilon) = z$. As the main hypothesis we assume that:

$$\boxed{\mathcal{Z} \subset D \ \text{ and, for each } \ \alpha \in \overline{V}, \ x(t, z_\alpha, 0) \ \text{ is } T\text{-periodic}}$$

Smooth systems: first order [3]; second order [4, 5]; third order [16]; any order [8, 13].

Lipschitz continuous systems: first order [6].

Discontinuous systems any order [17].

Example - Limit cycle in a 3D polynomial system:

Consider the following 3D autonomous polynomial differential system

$$\begin{aligned} \dot u =& -v+\varepsilon\left(u^3-u^2-uv^2-\pi v^3\right),\\ \dot v =& u+\varepsilon\left(\pi u^3-1\right),\\ \dot w =& w-\varepsilon u. \end{aligned} \tag{24}$$

Writing the differential system (24) in the cylindrical coordinates

$$(u,v,w)=(r\cos\theta, r\sin\theta, w),$$

we get

$$\begin{aligned} \dot r =& \frac{\varepsilon}{4}\left(r^3+r^2(r(\pi\sin(4\theta)+2\cos(2\theta)+\cos(4\theta))-3\cos\theta-\cos(3\theta))-4\sin\theta\right),\\ \dot\theta =& 1+\frac{\varepsilon}{4r}\left(r^2(\sin\theta+\sin(3\theta)-r\sin(4\theta)+\pi r\cos(4\theta)+3\pi r)-4\cos\theta\right),\\ \dot w =& w-\varepsilon r\cos\theta. \end{aligned}$$

Since $\dot\theta\neq 0$ for $|\varepsilon|\neq 0$ sufficiently small, we can take θ as the new independent variable. So

$$\begin{aligned} \frac{dr}{d\theta} =& \varepsilon F_{11}(\theta,z)+\mathcal{O}(\varepsilon^2),\\ \frac{dz}{d\theta} =& z+\varepsilon F_{12}(\theta,z)+\mathcal{O}(\varepsilon^2), \end{aligned} \tag{25}$$

where $z=(r,w)\in\mathbb{R}^2$ and

$$\begin{aligned} F_{11}(\theta,z) =& \frac{1}{4}\big(r^3+r^2(r(\pi\sin(4\theta)+2\cos(2\theta)+\cos(4\theta))-3\cos\theta-\cos(3\theta))\\ &-4\sin\theta\big),\\ F_{12}(\theta,z) =& \frac{-1}{4}\big(4\cos\theta\left(r^2-z\right)+r^2z(\sin\theta+\sin(3\theta)-r\sin(4\theta)+\pi r\cos(4\theta)\\ &+3\pi r)\big). \end{aligned}$$

The differential system (25) is 2π-periodic in the variable θ and it is written in the standard form form (9) with $F_0(\theta,z)=\big(0,z\big)$ and $F_1(\theta,z)=\big(F_{11}(\theta,z),F_{12}(\theta,z)\big)$. Moreover the solution of the unperturbed differential system for a initial condition $z_0=(r_0,w_0)$ is given by

$$\Phi(\theta,z_0)=\big(r_0,w_0e^{\theta}\big).$$

Consider the set $\mathcal{Z}\subset\mathbb{R}^2$ such that $\mathcal{Z}=\{(\alpha,0):\alpha>0\}$. Clearly for each $z_\alpha\in\mathcal{Z}$, the solution $\Phi(\theta,z_\alpha)$ is 2π–periodic, and therefore the differential system (25) satisfies the hypothesis above.

Proposition 6 **([8]).** *For $|\varepsilon|>0$ sufficiently small system (24) has a periodic solution $\varphi(t,\varepsilon)=\big(u(t,\varepsilon),v(t,\varepsilon),w(t,\varepsilon)\big)$, such that*

$$\begin{aligned} u(t,\varepsilon) =& \sqrt{8\,\varepsilon}\,\cos t+\mathcal{O}(\varepsilon),\\ v(t,\varepsilon) =& \sqrt{8\,\varepsilon}\,\sin t+\mathcal{O}(\varepsilon),\ \textit{and}\\ w(t,\varepsilon) =& \mathcal{O}(\varepsilon). \end{aligned} \tag{26}$$

3.2. Bifurcation of Tori

Consider two-parameter families of non-autonomous differential equations given by

$$\mathbf{x}'(t)=\sum_{i=1}^{k}\varepsilon^i F_i(t,\mathbf{x};\mu)+\varepsilon^{k+1}R(t,\mathbf{x};\mu,\varepsilon). \tag{27}$$

Here, F_i, $i=1,2,\dots,k$, and R are smooth functions and T-periodic in the variable $t\in\mathbb{R}$, $\mathbf{x}=(x,y)\in D$ with D an open bounded subset of $\mathbb{R}^2$, $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$ for some $\varepsilon_0>0$ small, and $\mu\in\mathbb{R}$.

Let ℓ be the index of the first non-vanishing averaged function. Consider the *truncated averaged system*

$$\dot{\mathbf{x}}=\varepsilon^\ell \mathbf{g}_\ell(\mathbf{x};\mu). \tag{28}$$

Theorem 7 ([9])**.** *If the truncated averaged system* (28) *undergoes a Hopf bifurcation (as μ crosses a critical value μ^*), then, for $|\varepsilon|\neq 0$ sufficiently small, the differential equation* (27) *undergoes a Torus Bifurcation (as μ crosses the critical value).*

Example - Rössler System:

$$\begin{aligned}\dot{x}&=-y-z,\\ \dot{y}&=x+ay,\\ \dot{z}&=bx-cz+xz.\end{aligned} \tag{29}$$

Consider the parameter vector (a,b,c) of the Rössler system (29) ε-close to $(\overline{a},1,\overline{a})$, with $\overline{a}\in(-\sqrt{2},\sqrt{2})$. More specifically, we assume that

$$(a,b,c)=\left(\overline{a}+\varepsilon\alpha_1+\varepsilon^2\alpha_2,1+\varepsilon\beta_1+\varepsilon^2\beta_2,\overline{a}+\varepsilon\gamma_1+\varepsilon^2\gamma_2\right)+\mathcal{O}(\varepsilon^3), \tag{30}$$

with $\overline{a}\in(-\sqrt{2},\sqrt{2})\setminus\{0\}$ and $\varepsilon,\alpha_i,\beta_i,\gamma_i\in\mathbb{R}$, for $i=1,2$. Also, define

$$\begin{aligned}d_0=&\quad\left(\alpha_1-\gamma_1+\beta_1\overline{a}\left(\overline{a}^2-1\right)\right)\left(\alpha_1\left(\overline{a}^2-1\right)+\overline{a}(\beta_1-\overline{a}\gamma_1)+\gamma_1\right),\\ d_1=&\quad\gamma_1-\alpha_1+\beta_1\overline{a},\ \text{and}\\ \ell_1=&\quad 12\pi\overline{a}^4\left(\overline{a}^4-16\right)-\overline{a}\left(4\overline{a}^8-12\overline{a}^6+193\overline{a}^4-640\overline{a}^2-144\right)\sqrt{2-\overline{a}^2}.\end{aligned} \tag{31}$$

Notice that the parameters above, d_0,d_1, and ℓ_1, do not depend on ε.

Theorem 8 ([10])**.** *Let (a,b,c) be given by* (30)*.*

(i) *If $d_0>0$, then for $|\varepsilon|\neq 0$ sufficiently small the Rössler System* (29) *admits a periodic solution $\varphi(t,\varepsilon)$ satisfying $\varphi(t,\varepsilon)\to(0,0,0)$ when $\varepsilon\to 0$. Moreover, for $\varepsilon>0$, such a periodic solution is asymptotically stable (resp. unstable) provided that $d_1>0$ (resp. $d_1<0$). Denote $\varphi(t,\gamma_1,\varepsilon)=\varphi(t,\varepsilon)$.*

(ii) *In addition, if $\ell_1\neq 0$, then there exist a smooth curve $\gamma(\varepsilon)$, defined for $\varepsilon>0$ sufficiently small and satisfying $\gamma(\varepsilon)=\overline{\gamma}_1+\mathcal{O}(\varepsilon)$ with $\overline{\gamma}_1=\alpha_1-\overline{a}\beta_1$, and intervals J_ε containing $\gamma(\varepsilon)$ such that a unique invariant torus bifurcates from the periodic solution $\varphi(t,\gamma(\varepsilon),\varepsilon)$ as γ_1 passes through $\gamma(\varepsilon)$. Such a torus exists whenever $\gamma_1\in J_\varepsilon$ and $\ell_1(\gamma_1-\gamma(\varepsilon))>0$, and surrounds the periodic solution $\mathbf{f}(t,\gamma_1,\varepsilon)$. In addition, if $\ell_1>0$ (resp. $\ell_1<0$) the torus is unstable (resp. asymptotically stable), whereas the periodic solution $\varphi(t,\gamma_1,\varepsilon)$ is asymptotically stable (resp. unstable).*

Examplo 2. $\overline{a} = \dfrac{1}{2},\ \alpha_1 = 2,\ \beta_1 = 2,\ \alpha_2 = -2,\ \beta_2 = -1\ \ \textit{and}\ \ \gamma_2 = -1$

We compute $\overline{\gamma}_1 = 1$ and $\ell_1 \simeq 155.66 > 0$. Theorem 8 predicts, for $\varepsilon > 0$ small enough, the existence of an asymptotically stable periodic solution of (29) surrounded by an unstable invariant torus:

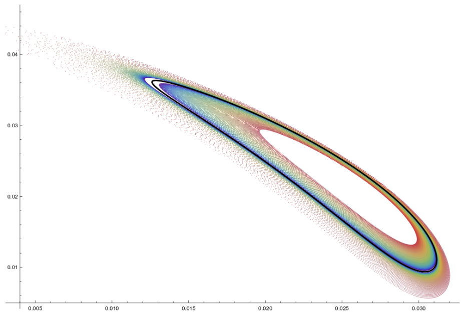


FIGURE 4. Poincaré section $\{z = 0 \text{ and } x > 0\}$ of the Rössler System (29) for $\varepsilon = 10^{-2}$. Trajectories starting at $(2982 \times 10^{-5}, 9656 \times 10^{-6}, 0)$ and $(3020 \times 10^{-5}, 9260 \times 10^{-6}, 0)$. The unstable invariant torus corresponds to an unstable invariant closed curve of the Poincaré map.

Examplo 3. $\overline{a} = -\dfrac{39}{32},\ \alpha_1 = 1,\ \beta_1 = 2,\ \alpha_2 = -2,\ \beta_2 = -1,\ \ \textit{and}\ \ \gamma_2 = -1.$

We compute $\overline{\gamma}_1 = \dfrac{55}{16}$ and $\ell_1 \simeq -1122.13 < 0$. Theorem 8 predicts, for $\varepsilon > 0$ small enough, the existence of an unstable periodic solution of (29) surrounded by an asymptotically stable invariant torus:

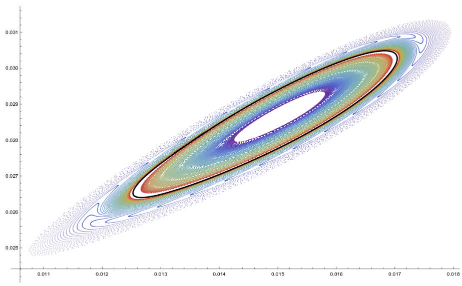


FIGURE 5. Poincaré section $\{z = 0 \text{ and } x > 0\}$ of the Rössler System (29) for $\varepsilon = 10^{-2}$. Trajectories starting at $(1504 \times 10^{-5}, 2852 \times 10^{-5}, 0)$ and $(1523 \times 10^{-5}, 2695 \times 10^{-5}, 0)$. The asymptotically stable invariant torus corresponds to an asymptotically stable invariant closed curve of the Poincaré map.

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