The Averaging Method - Lecture 2

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1. Averaging Method for Studying Periodic Solutions of Smooth Differential Equations

Consider regularly perturbed smooth non-autonomous differential equations given in the following **standard form**:

$$x' = \sum_{i=1}^{k} \varepsilon F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon), \ \ (t,x,\varepsilon) \in \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0). \tag{1}$$

Here, D is an open bounded subset of $\mathbb{R}^n$, $\varepsilon_0 > 0$ small, and the functions $F_i\colon \mathbb{R}\times\overline{D} \to \mathbb{R}^n$, for $i = 1, \dots, k$, and $R\colon \mathbb{R}\times\overline{D}\times[0,\varepsilon_0] \to \mathbb{R}^n$ are assumed to be $T-$periodic in the variable t and smooth.

For $i \in \{1, \dots, k\}$, the **averaged function** of order i, $\mathbf{f}_i : D \to \mathbb{R}^n$, is defined by

$$\mathbf{f}_i(z) = \frac{y_i(T,z)}{i!}, \tag{2}$$

where

$$\begin{aligned} y_1(t,z) &= \int_0^t F_1(s,z)\, ds \ \ \text{and} \\ y_i(t,z) &= \int_0^t \left(i! F_i(s,z) + \sum_{j=1}^{i-1} \sum_{m=1}^{j} \frac{i!}{j!} \partial_x^m F_{i-j}(s,z) B_{j,m}\big(y_1, \dots, y_{j-m+1}\big)(s,z) \right) ds, \end{aligned} \tag{3}$$

for $i \in \{2, \dots, k\}$.

Recall that

$$\boxed{\mathbf{f}_1(z) = \int_0^T F_1(t,z) dt} \quad \text{and} \quad \boxed{\mathbf{f}_2(z) = \int_0^T \left(F_2(t,z) + D_x F_1(t,z) \int_0^t F_1(s,z) ds \right) dt}$$

Theorem 1 ([2]). *Denote* $\mathbf{f}_0 = 0$. *Let* $\ell \in \{1, \ldots, k\}$ *satisfying* $\mathbf{f}_0 = \cdots \mathbf{f}_{\ell-1} = 0$ *and* $\mathbf{f}_\ell \neq 0$. *Assume that* $z^* \in D$ *is a simple zero of* $\mathbf{f}_\ell$. *Then, for* $|\varepsilon| \neq 0$ *sufficiently small, the differential equation* (1) *admits a unique* T*-periodic solution* $\varphi(t, \varepsilon)$ *such that* $\varphi(\cdot, \varepsilon) \to z^*$ *as* $\varepsilon \to 0$.

Stability: *In addition, if* z^* *is a hyperbolic singularity of the truncated averaged equation* $z' = \varepsilon^\ell \mathbf{f}_\ell(z)$, *then the stability of the periodic solution* $\varphi(\cdot, \varepsilon)$ *coincides with the stability of the singularity* z^*.

1.1. Example - Smooth Perturbed Harmonic Oscillator

$$x'' + x + b_\varepsilon x' = g_\varepsilon(x, x'), \tag{4}$$

where

$$b_\varepsilon = \varepsilon b_1 + \varepsilon^2 b_2 + \varepsilon^3 O(1) > 0, \quad \text{and } g_\varepsilon(x, y) = \varepsilon g_1(x, y) + \varepsilon^2 g_2(x, y) + \varepsilon^3 O(1)$$

is a sufficiently differentiable function.

Notice that, by taking $y = x'$, the second order differential equation (4) writes

$$\begin{aligned} x' &= y, \\ y' &= -x - b_\varepsilon y + g_\varepsilon(x, y). \end{aligned} \tag{5}$$

Remark: Due to the dumping effect ($b_\varepsilon > 0$), if $g_\varepsilon = 0$ the above system has a globally asymptotically stable equilibrium at $(x, x') = (0, 0)$ (Eigenvalues: $(-b_\varepsilon \pm \sqrt{b_\varepsilon^2 - 4})/2$). Our goal is to find a “forcing term” g_ε in order to this system to have a periodic orbit.

Step 1 (Standard Form): *Write system* (4) *in the standard form* (1) *in order to apply the averaging theorem.*

Writting the vector field (5) in polar coordinates $(x, y) = \big(r\cos(\theta), r\sin(\theta)\big)$, we get

$$r' = \sin(\theta)\big(g_\varepsilon(r\cos(\theta), r\sin(\theta)) - b_\varepsilon r\sin(\theta)\big),$$
$$\theta' = -1 + \frac{\cos(\theta)g_\varepsilon(r\cos(\theta), r\sin(\theta))}{r} - b_\varepsilon\cos(\theta)\sin(\theta).$$

Fixing the domain $D = [r_0, r_1]$, for some $0 < r_0 < r_1$, we notice that there exists $\varepsilon_0 > 0$ small such that $\theta' < 0$ for $(\theta, r, \varepsilon) \in \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0)$. Thus, taking θ as the new time:

$$\begin{aligned} \frac{dr}{d\theta} = \frac{r'}{\theta'} &= \frac{r\sin(\theta)\big(g_\varepsilon(r\cos(\theta), r\sin(\theta)) - b_\varepsilon r\sin(\theta)\big)}{-r + \cos(\theta)g_\varepsilon(r\cos(\theta), r\sin(\theta)) + b_\varepsilon\cos(\theta)\sin(\theta)} \\ &= \varepsilon\sin(\theta)\big(b_1 r\sin(\theta) - g_1(r\cos(\theta), r\sin(\theta))\big) \\ &\quad + \frac{\varepsilon^2}{r}\sin(\theta)\Big(\cos(\theta)\big(b_1 r\sin(\theta) - g_1(r\cos(\theta), r\sin(\theta))\big)^2 \\ &\quad + r\big(b_2 r\sin(\theta) - g_2(r\cos(\theta), r\sin(\theta))\big)\Big) + \varepsilon^3 O(1) \\ &= \varepsilon F_1(\theta, r) + \varepsilon^2 F_2(\theta, r) + \varepsilon^3 R(\theta, r, \varepsilon), \end{aligned} \tag{6}$$

where

$$\begin{aligned} F_1(\theta, r) &= \sin(\theta)\big(b_1 r\sin(\theta) - g_1(r\cos(\theta), r\sin(\theta))\big), \\ F_2(\theta, r) &= \frac{1}{r}\sin(\theta)\Big(\cos(\theta)\big(b_1 r\sin(\theta) - g_1(r\cos(\theta), r\sin(\theta))\big)^2 \\ &\quad + r\big(b_2 r\sin(\theta) - g_2(r\cos(\theta), r\sin(\theta))\big)\Big). \end{aligned}$$

Attention! $\theta' < 0$, then the above time rescaling change the direction of the flow and, consequently, the stability of the periodic solutions, if any.

Examplo 1 (Quadratic Example).

$$g_1(x,y) = a_{20}x^2 + a_{11}xy + a_{02}y^2$$
$$g_2(x,y) = \alpha_{20}x^2 + \alpha_{11}xy + \alpha_{02}y^2$$

Step 2 (First Order Analysis): *Compute the first-order averaged function and its zeros.*

In this case, $\mathbf{f}_1(r) = \pi b_1 r$. The equation $\mathbf{f}_1(r) = 0$ has a unique solution $r = r^* = 0$. Since r^* is not in the domain D, then the **First-Order Averaging Method**, that is, Theorem 1 for $\ell = 1$, does not provide the existence of a periodic solutions.

Step 3 (Second Order Analysis): *Assume minimal conditions on the parameters of perturbation in order to ensure that* $\mathbf{f}_1 = 0$ *and compute the second-order averaged function and its zeros.*

Notice that $\mathbf{f}_1 = 0$ if, and only if, $b_1 = 0$. Under this condition,

$$\mathbf{f}_2(r) = \pi\, r\left(b_2 - \frac{\pi a_{11}(a_{02}+a_{20})}{4} r^2\right).$$

The equation $\mathbf{f}_2(r) = 0$ has 3 solutions, namely

$$r_1 = 0,\ r_2 = -2\sqrt{\frac{b_2}{a_{11}(a_{02}+a_{20})}},\quad \text{and}\quad r_3 = 2\sqrt{\frac{b_2}{a_{11}(a_{02}+a_{20})}}.$$

Assuming $b_2 a_{11}(a_{02}+a_{20}) > 0$, $r^* = r_3$ is contained in the domain D. Hence, the **Second-Order Averaging Method**, that is, Theorem 1 for $\ell = 2$, provides the existence of a periodic solution $r(\theta,\varepsilon)$ of differential equation (6) such that $r(\cdot,\varepsilon) \to r^*$ as $\varepsilon \to 0$. Accordingly, going back to the changes of variables, one gets the existence of a periodic solution $(x(t,\varepsilon), x'(t,\varepsilon))$ of the system (4) satisfying $|(x(\cdot,\varepsilon), x'(\cdot,\varepsilon))| \to r^*$ as $\varepsilon \to 0$.

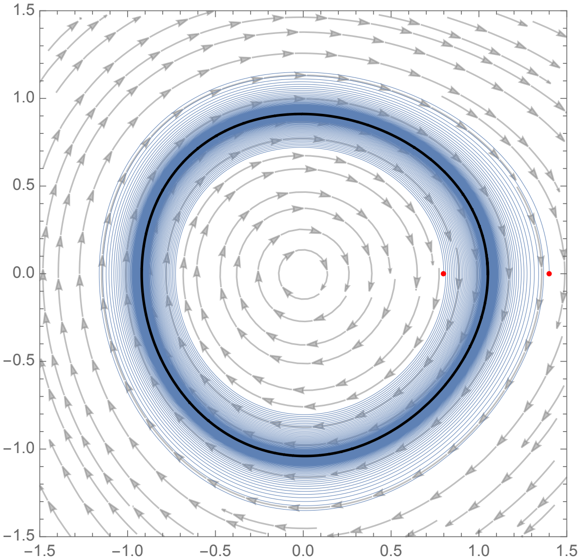


FIGURE 1. Mathematica simulation of some backward trajectories of the differential system $(x', y') = \left(y, -x + \varepsilon\,(2x^2 + 2xy) - \varepsilon^2 y\right)$ for $\varepsilon = 0.1$ The red dots indicate the initial conditions $(0.8, 0)$ and $(1.4, 0)$. and $(2.4, 0)$. The black closed curve indicates a limit cycle.

Examplo 2 (Cubic Example).

$$g_1(x,y) = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3$$
$$g_2(x,y) = \alpha_{30}x^3 + \alpha_{21}x^2y + \alpha_{12}xy^2 + \alpha_{03}y^3$$

Step 2 (First Order Analysis): *Compute the first-order averaged function and its zeros.*

$$\mathbf{f}_1(r) = \pi b_1 r - \frac{\pi(3a_{03} + a_{21})}{4} r^2.$$

The equation $\mathbf{f}_1(r) = 0$ has 3 solutions, namely

$$r_1 = 0,\ r_2 = -2\sqrt{\frac{b_1}{3a_{03} + a_{21}}}, \quad \text{and} \quad r_3 = 2\sqrt{\frac{b_1}{3a_{03} + a_{21}}}.$$

Assuming $b_1(3a_{03} + a_{21}) > 0$, $r^* = r_3$ is contained in the domain D. Hence, the **First-Order Averaging Method** provides the existence of a periodic solution $r(\theta, \varepsilon)$ of differential equation (6) such that $r(\cdot, \varepsilon) \to r^*$ as $\varepsilon \to 0$. Accordingly, one gets the existence of a periodic solution $(x(t,\varepsilon), x'(t,\varepsilon))$ of the differential equation (4) satisfying $|(x(\cdot,\varepsilon), x'(\cdot,\varepsilon))| \to r^*$ as $\varepsilon \to 0$.

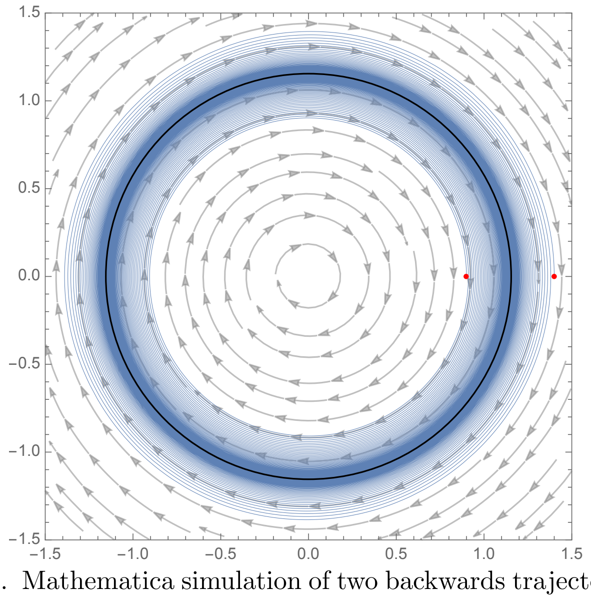


FIGURE 2. Mathematica simulation of two backwards trajectories of the differential system $(x', y') = \big(y, -x - \varepsilon\, y + \varepsilon\, y^3\big)$ for $\varepsilon = 0.1$. The red dots indicate the initial conditions $(0.9, 0)$ and $(1.4, 0)$. The black closed curve indicates the limit cycle.

Step 3 (Second Order Analysis): *Assume minimal conditions on the parameters of perturbation in order to ensure that* $\mathbf{f}_1 = 0$ *and compute the second-order averaged function and its zeros.*

Notice that $\mathbf{f}_1 = 0$ if, and only if, $b_1 = 0$ and $a_{21} = -3a_{03}$. Under this condition,

$$\mathbf{f}_2(r) = \frac{\pi r}{8}\big(a_{03}(a_{12} + 3a_{30})r^4 - 2(3\alpha_{03} + \alpha_{21})r^2 + 8b_2\big).$$

Conditions can be assumed in order that the equation $\mathbf{f}_2(r) = 0$ has zero, one or two positive simple solutions.

For instance, by taking $g_1(x,y) = y^3 + 2xy^2 - 3x^2y$, $g_2(x,y) = 5x^2y + 2xy^2$, and $b_2 = 1$, we get that

$$\mathbf{f}_2(r) = \frac{\pi r}{4}\big(r^4 - 5r^2 + 4\big).$$

The equation $\mathbf{f}_2(r) = 0$ has two positive solutions, namely $r_1^* = 1$ and $r_2^* = 2$. Hence, the **Second-Order Averaging Method** provides the existence of two periodic solutions $r_1(\theta,\varepsilon)$ and $r_2(\theta,\varepsilon)$ of differential equation (6) such that $r_1(\cdot,\varepsilon) \to r_1^*$ and $r_2(\cdot,\varepsilon) \to r_2^*$ as $\varepsilon \to 0$. Accordingly, one gets the existence of periodic solutions $(x_1(t,\varepsilon), x_1'(t,\varepsilon))$ and $(x_2(t,\varepsilon), x_2'(t,\varepsilon))$ of the differential equation (4) satisfying $|(x_1(\cdot,\varepsilon), x_1'(\cdot,\varepsilon))| \to r_1^*$ and $|(x_2(\cdot,\varepsilon), x_2'(\cdot,\varepsilon))| \to r_2^*$ as $\varepsilon \to 0$.

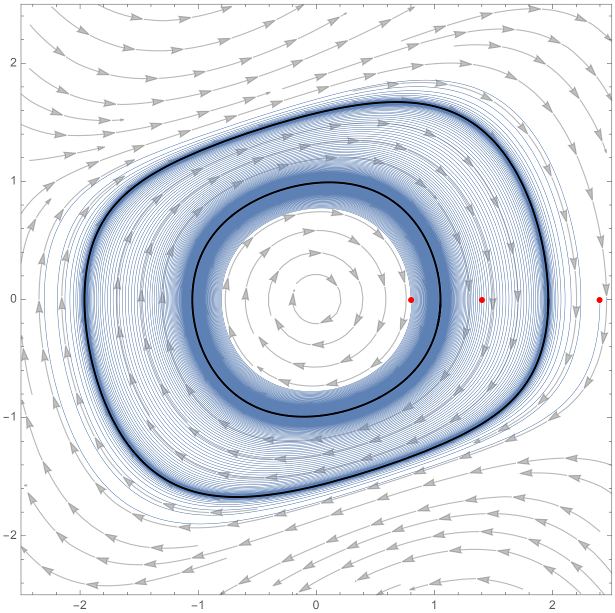


FIGURE 3. Mathematica simulation of some trajectories of the differential system $(x', y') = \big(y, -x+\varepsilon\,(y^3+2xy^2-3x^2y)+\varepsilon^2(-y+5x^2y+2xy^2)\big)$ for $\varepsilon = 0.1$ The red dots indicate the initial conditions $(0.8, 0)$, $(1.4, 0)$, and $(2.4, 0)$. The blue continuous lines indicate: the backward trajectory for initial condition $(0.8, 0)$; the orbit for initial condition $(1.4, 0)$; and the forward trajectory for the initial condition $(2.4, 0)$. The black closed curves indicate limit cycles.

2. Averaging Method for Studying Periodic Solutions of Lipschitz Continuous Differential Equations

If, instead of the smoothness of differential system (1), we assume

- $F_i(t,\cdot) \in C^{k-i}$, $D_x^{k-i} F_i$ is locally Lipschitz in the second variable, and
- R is a continuous function, locally Lipschitz in the second variable,

Theorem 1 remains valid as follows:

Theorem 2 ([2])**.** *Denote* $\mathbf{f}_0 = 0$. *Let* $\ell \in \{1, \dots, k\}$ *satisfying* $\mathbf{f}_0 = \cdots \mathbf{f}_{\ell-1} = 0$ *and* $\mathbf{f}_\ell \neq 0$. *Assume that for* $z^* \in D$, *such that* $\mathbf{f}_\ell(z^*) = 0$, *there exists a neighborhood* $V \subset D$ *of* z^* *satisfying* $\mathbf{f}_\ell(z) \neq 0$ *for every* $z \in \overline{V} \setminus \{z^*\}$ *and* $d_B(\mathbf{f}_\ell, V, 0) \neq 0$. *Then, for* $|\varepsilon| \neq 0$ *sufficiently small, the differential equation* (1) *admits a* T*-periodic solution* $\varphi(t,\varepsilon)$ *such that* $\varphi(\cdot,\varepsilon) \to z^*$ *as* $\varepsilon \to 0$.

2.1. Brouwer degree

Let $X = \mathbb{R}^n = Y$ and $V \subset X$ be an open bounded subset. Consider a continuou map $f : \overline{V} \to Y$, and a point y_0 in Y such that $y_0 \notin f(\partial V)$. Then each triple (f, V, y_0) corresponds to an integer $d_B(f, V, y_0)$ having the following three properties.

(i) If $d_B(f, V, y_0) \neq 0$, then $y_0 \in f(V)$, and $d_B\left(Id\big|_V, V, y_0\right) = 1$.

(ii) If V_1 and V_2 are disjoint open subsets of V such that $y_0 \notin f(\overline{V} \backslash (V_1 \cup V_2))$, then

$$d_B\left(f, V, y_0\right) = d_B\left(f, V_1, y_0\right) + d_B\left(f, V_1, y_0\right).$$

(iii) (Invariance under homotopy) Let $\{f_t : 0 \leq t \leq 1\}$ be a continuous homotopy of maps of $\overline{V}$ into Y. Let $\{y_t : 0 \leq t \leq 1\}$ be a continuous curve in Y such that $y_t \notin f_t(\partial V)$ for any $t \in [0,1]$. Then $d_B(f_t, V, y_t)$ is constant in $t \in [0,1]$.

Moreover the degree function $d_B(f, V, y_0)$ is uniquely determined by the three above conditions.

When $f : \overline{V} \subset X \to Y$ is a C^1 function and $\det(Df(x)) \neq 0$ for every $x \in f^{-1}(y_0)$, the Brouwer degree may be computed as follows:

$$d_B(f, V, y_0) = \sum_{x \in f^{-1}(y_0)} \operatorname{sign}\left(\det(Df(x))\right). \tag{7}$$

Remark 3. *The above formula implies that if* z^* *is a point of* X *such that* $f(z^*) = 0$ *and* $\det(Df(z^*)) \neq 0$, *then there exists a neighborhood* $V \subset X$ *of* z^* *such that* $f(z) \neq 0$ *for every* $z \in V \setminus \{z^*\}$ *and* $d_B(f, V, 0) \neq 0$.

2.2. Example - Lipschitz Continuous Perturbed Harmonic Oscillator

$$x'' + x + b_\varepsilon x' = g_\varepsilon(x, x'), \tag{8}$$

where $b_\varepsilon = \varepsilon b_1 + \varepsilon^2 O(1) > 0$ and $g_\varepsilon(x, y) = \varepsilon g_1(x, y) + \varepsilon^2 O(1)$ is a **continuous function**.

Standard Form: $\dfrac{dr}{d\theta} = \varepsilon \sin(\theta)\big(b_1 r \sin(\theta) - g_1(r\cos(\theta), r\sin(\theta)\big) + \varepsilon^2 O(1)$

First-Order Averaged Function: $f_1(r) = b_1 \pi r - \displaystyle\int_0^{2\pi} \sin(\theta) g_1(r\cos(\theta), r\sin(\theta)) d\theta$

Examplo 3. $g_1(x, y) = \begin{cases} \beta^+ y^2, & y \geq 0 \\ \beta^- y^2, & y \leq 0 \end{cases} = \dfrac{\beta^+}{2}(y + |y|)y + \dfrac{\beta_2}{2}(y - |y|)y.$

$$\begin{aligned} f_1(r) = &\quad b_1 \pi r - \int_0^{2\pi} \sin(\theta) g_1(r\cos(\theta), r\sin(\theta)) d\theta \\ = &\quad b_1 \pi r - \int_0^{\pi} \beta^+ r \sin^3(\theta) d\theta - \int_\pi^{2\pi} \beta^- r \sin^3(\theta) d\theta = \frac{r}{3}\big(3b_1\pi + 4r(\beta^- - \beta^+)\big) \end{aligned}$$

Assuming $b_1(\beta^+ - \beta^-) > 0$, the equation $f_1(r) = 0$ a unique positive solution $r^* = \frac{3b_1\pi}{4(\beta^+ - \beta^-)}$. Hence, the **First-Order Averaging Method**, that is, Theorem 2 for $\ell = 1$, provides the existence of a periodic solution $r(\theta, \varepsilon)$ of differential equation (8) such that $r(\cdot, \varepsilon) \to r^*$ as $\varepsilon \to 0$. Accordingly, one gets the existence of a periodic solution $(x(t, \varepsilon), x'(t, \varepsilon))$ of the differential equation (4) satisfying $|(x(\cdot, \varepsilon), x'(\cdot, \varepsilon))| \to r^*$ as $\varepsilon \to 0$.

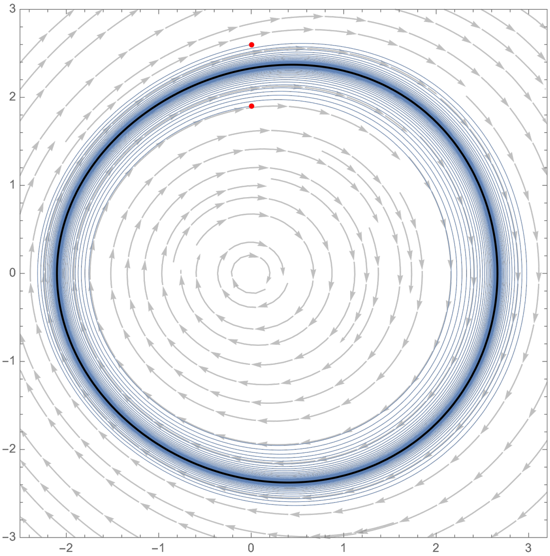


FIGURE 4. Mathematica simulation of some backward trajectories of the differential system $(x', y') = \big(y, -x + \varepsilon(-y + g_1(x, y))\big)$ for $\beta_1 = 1$, $\beta_2 = 2$, and $\varepsilon = 0.1$ The red dots indicate the initial conditions $(0, 1.9)$ and $(0, 2.6)$. The black closed curves indicate limit cycles.

3. Averaging Method for Studying Periodic Solutions of non-Lipschitz Continuous Differential Equations

Consider the regularly perturbed continuous non-autonomous differential equation:

$$x' = \varepsilon F(t, x, \varepsilon). \tag{9}$$

Again, D is an open bounded subset of $\mathbb{R}^n$, $\varepsilon_0 > 0$ small, and $F\colon \mathbb{R} \times \overline{D} \times [0, \varepsilon_0] \to \mathbb{R}^n$ is assumed to be $T-$periodic in the variable t and continuous.

3.1. First-Order Averaging Method

Define the *first-order averaged function* $\mathbf{h}_1 : D \to \mathbb{R}^n$ by

$$\mathbf{h}_1(z) = \frac{1}{T}\int_0^T F(s, z, 0)ds.$$

Theorem 4 ([1, 4])**.** *Let $z^* \in D$ be an isolated zero of $\mathbf{h}_1$ and assume that there exists a neighbourhood $V \subset \mathbb{R}^n$ of z^*, with $\overline{V} \subset D$ and $\mathbf{h}_1(z) \neq 0$ for every $z \in \overline{V} \setminus \{z^*\}$, such that $d_B(\mathbf{h}_1, V, 0) \neq 0$. Then, for $|\varepsilon|$ sufficiently small, the differential equation* (9) *has a $T-$periodic solution $\varphi(t, \varepsilon)$ satisfying $\varphi(t, \varepsilon) \in \overline{V}$, for every $t \in [0, T]$, and $\varphi(\cdot, \varepsilon) \to z^*$ uniformly as $\varepsilon \to 0$.*

3.2. Higher Order Averaging Method

Now, assume that

$$x' = F(t, x, \varepsilon) := \sum_{i=1}^{k} \varepsilon^i F_i(t, x) + \varepsilon^{k+1} R(t, x, \varepsilon). \tag{10}$$

For $i = 1, \ldots, k$, define the *ith-order averaged function* function $\mathbf{h}_i\colon D \to \mathbb{R}^n$ by

$$\mathbf{h}_i(z) = \frac{1}{T}\int_0^T F_i(s, z)ds.$$

Assume that, for a given open bounded subset $V \subset \mathbb{R}^n$, with $\overline{V} \subset D$,

H. there exists $\varepsilon_1 \in (0, \varepsilon_0]$ such that, for each $\lambda \in (0, 1)$ and $\varepsilon \in (0, \varepsilon_1]$, any T-periodic solution of the differential equation

$$x' = \varepsilon\lambda F(t, x, \varepsilon),\ x \in \overline{V}, \tag{11}$$

is entirely contained in V.

Theorem 5 ([4])**.** *Denote $\mathbf{h}_0 = 0$. Let $\ell \in \{1, 2, \ldots, k\}$ satisfying $\mathbf{h}_0 = \ldots = \mathbf{h}_{\ell-1} = 0$, and $\mathbf{h}_\ell \neq 0$. Assume that for $z^* \in D$, such that $\mathbf{h}_\ell(z^*) = 0$, there exists a neighbourhood $V \subset \mathbb{R}^n$ of z^*, with $\overline{V} \subset D$, satisfying $\mathbf{h}_\ell(z) \neq 0$ for $z \in \overline{V} \setminus \{z^*\}$, such that hypothesis* **H** *holds and $d_B(\mathbf{h}_\ell, V, 0) \neq 0$. Then, for $|\varepsilon|$ sufficiently small, the differential equation* (10) *has a $T-$periodic solution $\varphi(t, \varepsilon)$ satisfying $\varphi(t, \varepsilon) \in \overline{V}$, for every $t \in [0, T]$, and $\varphi(\cdot, \varepsilon) \to z^*$ uniformly as $\varepsilon \to 0$.*

Remark 6. *The negation of hypothesis* **H** *provides:*

- *numerical convergent sequences* $(\varepsilon_m)_{m\in\mathbb{N}} \subset (0,\varepsilon_0)$ *and* $(\lambda_m)_{m\in\mathbb{N}} \subset (0,1)$, *such that* $\varepsilon_m \to 0$ *as* $m \to \infty$,
- *and a sequence of* T*-periodic solutions* $x_m(t) \in \overline{V}$ *of*
$$x' = \varepsilon_m \lambda_m F(t,x,\varepsilon_m)$$
for which there exists $t_m \in [0,T]$ *such that* $x_m(t_m) \in \partial V$ *for each* $m \in \mathbb{N}$. *In particular,*
$$x_m(t) = x_m(0) + \varepsilon_m \lambda_m \int_0^t F(s,x_m(s),\varepsilon_m)ds \ \text{ and}$$
$$\int_0^T F(t,x_m(t),\varepsilon_m)dt = 0,$$
for each $m \in \mathbb{N}$.
- *Furthermore, as an application of Arzelá-Ascoli's Theorem, the sequence of functions* $(x_m)_{m\in\mathbb{N}}$ *can be considered uniformly convergent to a constant function in* ∂V.

Remark 7. *When the boundary of* V, ∂V, *is a smooth manifold, hypothesis* **H** *holds provided that:*

"there exists $\varepsilon_1 \in (0,\varepsilon_0]$ *such that, for each* $z \in \partial V$, $F(t,z,\varepsilon)$ *is transversal to* ∂V *at* z, *for every* $t \in [0,T]$ *and* $\varepsilon \in (0,\varepsilon_1]$*".*

3.3. General Averaging Method and Idea of the Proof

Define the *full averaged function* $\mathbf{h} : D \times [0,\varepsilon_0] \to \mathbb{R}^n$ as the average of the right-hand side of (9), that is,
$$\mathbf{h}(z,\varepsilon) = \frac{1}{T}\int_0^T \varepsilon F(s,z,\varepsilon)ds.$$
Theorems 4 and 5 are consequences of the following result:

Theorem 8 ([4])**.** *Assume that for a given open subset* $V \subset \mathbb{R}^n$, *with* $\overline{V} \subset D$, *hypothesis* **H** *holds,*
$$\mathbf{h}(z,\varepsilon) \neq 0, \qquad \text{for all} \qquad z \in \partial V \qquad \text{and} \qquad \varepsilon \in (0,\varepsilon_1], \tag{12}$$
and $d_B(\mathbf{h}(\cdot,\varepsilon^*),V,0) \neq 0$, *for some* $\varepsilon^* \in (0,\varepsilon_1]$. *Then, for each* $\varepsilon \in (0,\varepsilon_1]$, *there exists a* $T-$*periodic solution* $\varphi(t,\varepsilon)$ *of the differential equation* (10) *satisfying* $\varphi(t,\varepsilon) \in \overline{V}$, *for every* $t \in [0,T]$.

Proof. Define the following real Banach spaces

$$X = \{x \in C([0,T],\mathbb{R}^n)\colon\ x(0) = x(T)\} \text{ and } Y = \{x \in C([0,T],\mathbb{R}^n)\colon\ x(0) = 0\},$$

where $C([0,T],\mathbb{R}^n)$ denotes the space of continuous functions $x : [0,T] \to \mathbb{R}^n$ endowed with the sup-norm.

Denote

$$\Omega = \{x \in X\colon x(t) \in V,\, \forall\, t \in [0,T]\},$$

which is an open bounded subset of X.

Define the linear map $L\colon X \to Y$ by

$$Lx(t) = x(t) - x(0).$$

Notice that $\mathrm{Im}L = X \cap Y$, which is closed in Y. In addition,

$$\ker L = \big\{x \in X : x(t) = z,\, z \in \mathbb{R}^n\big\},$$

which can be identified with $\mathbb{R}^n$.

Notice that a function $x \in X$ can be continuously extended to a $T-$periodic solution of the differential equation (10) in $\overline{V}$ if, and only if, it is a solution of the operator equation

$$Lx = N_\varepsilon(x),\ \ x \in \overline{\Omega}. \tag{13}$$

Finally, define the projection $Q : Y \to Y$ given by

$$Qy(t) = \frac{y(T)}{T}, \quad \text{for} \ \ t \in [0,T].$$

The next result is a consequence of [3, Theorem IV.13] (abstract continuation result for operator equation) and will be used for proving Theorem 8.

Proposition 9. *Let L, N_ε, and Q be like above. Assume that the following conditions are verified for every $\varepsilon \in (0,\varepsilon_1]$:*

H.1 $Lx \neq \lambda N_\varepsilon(x)$, *for every* $x \in (\mathrm{dom}\ L \setminus \ker L) \cap \partial\Omega$ *and* $\lambda \in (0,1)$;
H.2 $QN_\varepsilon(x) \neq 0$, *for every* $x \in \ker L \cap \partial\Omega$; *and*
H.3 $d_B(QN_\varepsilon\big|_{\Omega\cap\ker L}, \Omega \cap \ker L, 0) \neq 0$.

Then, for every $\varepsilon \in (0,\varepsilon_1]$, the operator equation (13) *admits a solution in* $\overline{\Omega} \cap \mathrm{dom}\ L$.

Hypothesis **H** and condition (12) imply, respectively, that hypothesis **H.1** and **H.2** hold for operation equation (13).

In addition, for $x \in \ker L \cap \partial\Omega$, that is, $x(t) \equiv z \in \partial V$,

$$QN_\varepsilon(x)(t) = \frac{1}{T}\int_0^T \varepsilon F(s,z,\varepsilon)ds = \mathbf{h}(z,\varepsilon). \tag{14}$$

Thus,

$$d_B(QN_\varepsilon\big|_{\ker L\cap\Omega}, \ker L \cap \Omega, 0) = d_B(\mathbf{h}(\cdot,\varepsilon), V, 0),$$

for $\varepsilon \neq 0$. By hypothesis, there exists $\varepsilon^* \in (0,\varepsilon_1]$ such that $d_B(\mathbf{h}(\cdot,\varepsilon^*), V, 0) \neq 0$. Using that the Brouwer degree is piecewise constant, then by a connectedness and compactness topological argument we can show that $d_B(\mathbf{h}(\cdot,\varepsilon), V, 0) \neq 0$ for every $\varepsilon \in (0,\varepsilon_1]$. Hence, hypothesis **H.3** holds for operation equation (13).

Therefore, from Proposition 9 we get the existence of a solution of the operator equation (13) and, consequently, a $T-$periodic solution $\varphi(t,\varepsilon)$ of the differential equation (10), for each $\varepsilon \in (0,\varepsilon_1]$, such that $\varphi(t,\varepsilon) \in \overline{V}$ for every $t \in [0,T]$. $\square$

3.4. Example - non-Lipschitz Continuous Perturbed Harmonic Oscillator

- Consider the continuous higher order perturbation of a harmonic oscillator:

$$\ddot{x} = -x + \varepsilon(x^2 + \dot{x}^2) + \varepsilon^k \dot{x}\sqrt[3]{x^2 + \dot{x}^2 - 1} + \varepsilon^{k+1} E(x, \dot{x}, \varepsilon). \tag{15}$$

- Notice that (15) is not Lipschitz in any neighborhood of $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + \dot{x}^2 = 1\}$.

Proposition 10. *For any positive integer k and $|\varepsilon| \neq 0$ sufficiently small, the differential equation* (15) *admits a periodic solution $x(t;\varepsilon)$ satisfying $(x(t;\varepsilon), \dot{x}(t;\varepsilon)) \to \mathbb{S}^1$ uniformly as $\varepsilon \to 0$.*

Proof. Changing to polar coordinates, we obtain $\dfrac{dr}{d\theta} = \varepsilon F(\theta, r, \varepsilon)$, where

$$\begin{aligned} F(\theta, r, \varepsilon) = & -\sum_{i=1}^{k-1} \varepsilon^{i-1} r^{i+1} \cos^{i-1}\theta \sin\theta \\ & + \varepsilon^{k-1} r\left(\sqrt[3]{r^2 - 1}\sin\theta - r^k \cos^{k-1}\theta\right)\sin\theta + \varepsilon^k R(\theta, r, \varepsilon), \end{aligned}$$

which is not Lipschitz in any neighbourhood of $r = 1$.

Notice that

$$\mathbf{h}_i = 0, \text{ for } i \in \{1, 2, \ldots, k-1\}, \text{ and } \mathbf{h}_k(r) = \frac{r\sqrt[3]{r^2 - 1}}{2}.$$

Moreover, $\mathbf{h}_k$ has a unique positive zero $r^* = 1$ and is homotopic to the mapping $r \mapsto r - 1$ in $V = (1 - \alpha, 1 + \alpha)$, $0 < \alpha < 1$. Therefore, $d_B(\mathbf{h}_k, V, 0) \neq 0$.

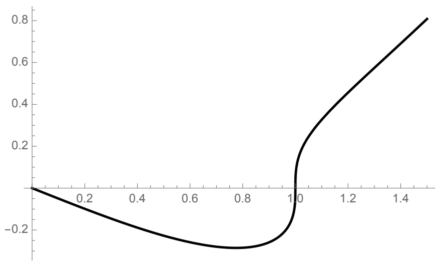


FIGURE 5. Graph of $\mathbf{h}_k$.

In order to apply Theorem 5, it remains to check hypothesis **H**.

The negation of hypothesis **H** provides:

- numerical convergent sequences $(\varepsilon_m)_{m\in\mathbb{N}} \subset (0, \varepsilon_0]$ and $(\lambda_m)_{m\in\mathbb{N}} \subset (0, 1)$, such that $\varepsilon_m \to 0$ as $m \to \infty$,
- and a sequence $(r_m)_{m\in\mathbb{N}}$ of 2π-periodic solutions of

$$r' = \varepsilon_m \lambda_m F(\theta, r, \varepsilon_m), \quad r \in \overline{V}, \tag{16}$$

for which there exists $\theta_m \in [0, 2\pi]$ such that $r_m(\theta_m) \in \partial V$ for each $m \in \mathbb{N}$.

As an application of Arzelá-Ascoli's Theorem, the sequence of functions $(r_m)_{m\in\mathbb{N}}$ can be taken uniformly convergent to a constant function $r_0 \in \partial V = \{1 \pm \alpha\}$. In particular,

$$\int_0^{2\pi} F(\theta, r_m(\theta), \varepsilon_m) d\theta = 0.$$

Therefore,

$$\int_0^{2\pi} r_m(\theta)\sqrt[3]{r_m(\theta)^2-1}\sin^2\theta d\theta = \frac{1}{\varepsilon_m^{k-1}}\sum_{i=1}^{k}\varepsilon_m^{i-1}G_m^{i+1,i-1} + \mathcal{O}(\varepsilon_m),$$

where

$$G_m^{i,j} = \int_0^{2\pi} r_m(\theta)^i \cos^j(\theta)\sin(\theta)\ d\theta.$$

By applying integration by parts and using that $r_m(\theta)$ is $2\pi-$periodic, we obtain

$$G_m^{i,j} = \frac{i}{j+1}\int_0^{2\pi} r_m(\theta)^{i-1}\cos^{j+1}(\theta)r_m'(\theta)\ d\theta.$$

Since $r_m(\theta)$ satisfies (16), we conclude that

$$\begin{aligned} G_m^{i,j} &= -\frac{i}{j+1}\sum_{l=1}^{k-1}\lambda_m\varepsilon_m^l\int_0^{2\pi} r_m(\theta)^{j+l}\cos^{j+l}(\theta)\sin\theta\ d\theta + \mathcal{O}(\varepsilon_m^k) \\ &= -\frac{i}{j+1}\sum_{l=1}^{k-1}\lambda_m\varepsilon_m^l G_{i+l,j+l}(r_m) + \mathcal{O}(\varepsilon_m^k) = \mathcal{O}(\varepsilon_m). \end{aligned}$$

Applying the last procedure recursively, we get $G_m^{i,j} = \mathcal{O}(\varepsilon_m^k)$. Thus,

$$\int_0^{2\pi} r_m(\theta)\sqrt[3]{r_m(\theta)^2-1}\sin^2\theta d\theta = \mathcal{O}(\varepsilon_m).$$

Since $r_m \to r_0 \in \{1\pm\alpha\}$ uniformly, we compute the limit of the integral above as

$$\int_0^{2\pi} r_0\sqrt[3]{r_0^2-1}\sin^2\theta d\theta = 0,$$

which is an absurd, because

$$\int_0^{2\pi} r_0\sqrt[3]{r_0^2-1}\sin^2\theta d\theta = \pi r_0\sqrt[3]{r_0^2-1} \neq 0,$$

for $r_0 \notin \{-1,0,1\}$. Thus, we obtain that hypothesis **H** holds and Theorem 5 can be applied in order to conclude this proof. $\square$

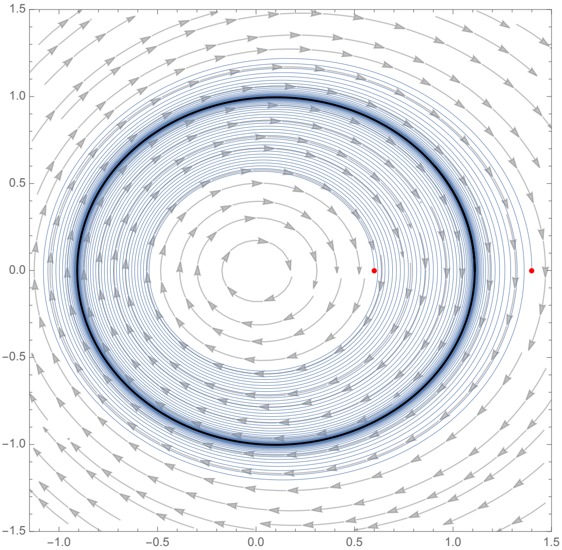
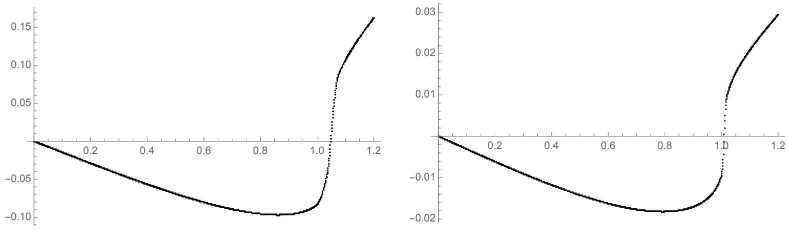
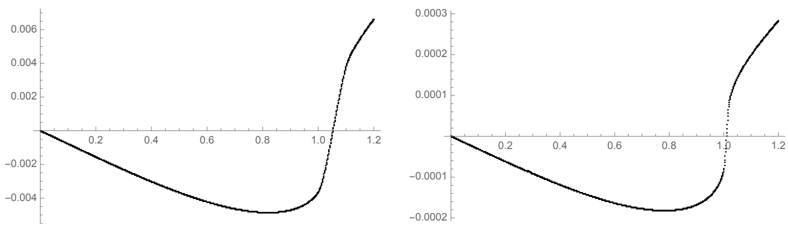


FIGURE 6. Mathematica simulation of some backwards trajectories of the differential system $(x', y') = \left(y, -x + \varepsilon(x^2 + y^2) + \varepsilon^k y \sqrt[3]{x^2 + y^2 - 1}\right)$ for $k = 2$, $\varepsilon = 0.1$ The red dots indicate the initial conditions $(0.6, 0)$ and $(1.4, 0)$. The black closed curve indicates a limit cycle.

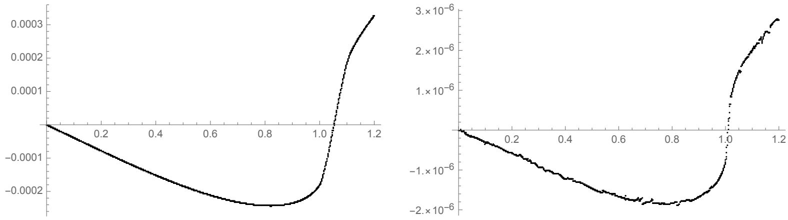
Displacement function of
$(x', y') = \left(y, -x + \varepsilon(x^2 + y^2) + \varepsilon^k y \sqrt[3]{x^2 + y^2 - 1}\right)$
for $k = 1$ and $\varepsilon = 1/20$ (left) and $\varepsilon = 1/100$ (right).



Displacement function of
$(x', y') = \left(y, -x + \varepsilon(x^2 + y^2) + \varepsilon^k y \sqrt[3]{x^2 + y^2 - 1}\right)$
for $k = 2$ and $\varepsilon = 1/20$ (left) and $\varepsilon = 1/100$ (right).



Displacement function of
$(x', y') = \left(y, -x + \varepsilon(x^2 + y^2) + \varepsilon^k y \sqrt[3]{x^2 + y^2 - 1}\right)$
for $k = 3$ and $\varepsilon = 1/20$ (left) and $\varepsilon = 1/100$ (right).



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