The Averaging Method - Lecture 1

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1. Regularly Perturbed Differential Equations

In this course we shall focuses on regularly perturbed differential equations of the following kind:

$$x' = \varepsilon F(t,x,\varepsilon) := \sum_{i=1}^{k} \varepsilon^i F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon), \tag{1}$$

$(t,x,\varepsilon) \in \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0)$, where D is an open bounded subset of $\mathbb{R}^n$, $\varepsilon_0 > 0$ small, and the functions $F_i\colon \mathbb{R} \times \overline{D} \to \mathbb{R}^n$, for $i = 1, \ldots, k$, and $R\colon \mathbb{R} \times \overline{D} \times [0, \varepsilon_0] \to \mathbb{R}^n$ are $T-$periodic in the variable t.

At principle, for the sake of simplicity, we also assume the smoothness of the functions F_i, for $i = 1, \ldots, k$, and R. This assumption with be relaxed latter on.

1.1. Existence, Uniqueness, and Maximality of Solutions

Theorem 1. *There exists $\overline{\varepsilon}$, $0 < \overline{\varepsilon} < \varepsilon_0$, such that for each $z \in D$ and $\varepsilon \in [-\overline{\varepsilon}, \overline{\varepsilon}]$, the differential equation (1) admits a unique maximal solution $x(\cdot, z, \varepsilon) : I(z,\varepsilon) \to D$ satisfying $x(0,z,\varepsilon) = z$. Moreover, $[0,T] \subset I(z,\varepsilon)$ for every $(z,\varepsilon) \in D \times [-\overline{\varepsilon}, \overline{\varepsilon}]$. Here, $I(z,\varepsilon)$ is the maximal interval of definition of solution, which is an open interval.*

1.2. Periodic Solutions

Theorem 2. *Given* $(z,\varepsilon)\in D\times[-\overline{\varepsilon},\overline{\varepsilon}]$, *the solution* $x(t,z,\varepsilon)$ *of the differential equation* (1) *is* T*-periodic in the variable* t *if, and only if,* $x(T,z,\varepsilon)=z$.

1.2.1. Displacement Function Approach.

$\Delta : D\times[-\overline{\varepsilon},\overline{\varepsilon}]\to\mathbb{R}^n$ is defined by

$$\Delta(z,\varepsilon) := x(T,z,\varepsilon)-z. \tag{2}$$

As a consequence of Theorems 1 and 2 we have the following result:

Theorem 3. *Given* $(z^*,\varepsilon^*)\in D\times[-\overline{\varepsilon},\overline{\varepsilon}]$, *the differential equation* (1) *for* $\varepsilon=\varepsilon^*$ *admits a* T*-periodic solution starting at* $z=z^*$ *if, and only if,*

$$\Delta(z^*,\varepsilon^*)=0. \tag{3}$$

1.2.2. Stability.

If one has obtained a smooth branch $z(\varepsilon)$ of zeros of (3), then the stability of the periodic solution $\varphi(t,\varepsilon)=x(t,z(\varepsilon),\varepsilon)$ is characterized by the stability of the fixed point $z(\varepsilon)$ of the Poincaré map

$$z\mapsto\Pi(z;\varepsilon):=x(T,z,\varepsilon)=z+\Delta(z,\varepsilon).$$

1.2.3. Operator Equation Approach. See [1, 7, 8, 9, 12].

Define the real Banach spaces:

$$X = \{x \in C([0,T],\mathbb{R}^n)\colon\ x(0) = x(T)\} \text{ and } Y = \{x \in C([0,T],\mathbb{R}^n)\colon\ x(0) = 0\},$$

where $C([0,T],\mathbb{R}^n)$ denotes the space of continuous functions $x : [0,T] \to \mathbb{R}^n$ endowed with the sup-norm.

Given an open set $V \subset \mathbb{R}^n$ such that $\overline{V} \subset D$, denote

$$\Omega = \{x \in X\colon x(t) \in V,\ \forall\, t \in [0,T]\},$$

which is an open bounded subset of X.

Define the linear map $L\colon X \to Y$ by

$$Lx(t) = x(t) - x(0).$$

Furthermore, for each $\varepsilon \in (0,\varepsilon_0]$, consider the map $N_\varepsilon : \overline{\Omega} \to Y$ given by

$$N_\varepsilon(x)(t) = \int_0^t \varepsilon\, F(s,x(s),\varepsilon)ds.$$

Theorem 4. *A function $x \in X$ can be continuously extended to a $T-$periodic solution of the differential equation (1) in $\overline{V}$ if, and only if, it is a solution of the operator equation*

$$Lx = N_\varepsilon(x),\ \ x \in \overline{\Omega}.$$

2. The Averaging Method

The Averaging Method is mainly concerned in providing asymptotic estimates for solutions of non-autonomous differential equations given in the standard form (1). Such asymptotic estimates are given in terms of solutions of an autonomous *truncated averaged equation*

$$\xi' = \sum_{i=1}^{k} \varepsilon^i \mathbf{g}_i(\xi) \tag{4}$$

as follows:

Theorem 5 (Stroboscopic Averaging Theorem, [14, Theorem 2.9.2]**).** *Let $\xi(t,z,\varepsilon)$ be the solution of* (4) *satisfying $\xi(0,z,\varepsilon) = z$. Then, there exists $\hat{\varepsilon} > 0$ such that*

$$|x(t,z,\varepsilon) - \xi(t,z,\varepsilon)| = \mathcal{O}(\varepsilon^k),$$

for every $t \in [0, L/\varepsilon]$ and $\varepsilon \in (-\hat{\varepsilon}, \hat{\varepsilon})$, for some fixed $L > 0$.

The functions $\mathbf{g}_i : D \to \mathbb{R}^n$, for $i \in \{1, \dots, k\}$, are obtained by the following result:

Theorem 6 ([14, Lemma 2.9.1]**).** *There exists a smooth T-periodic near-identity transformation*

$$x = U(t,\xi,\varepsilon) = \xi + \sum_{i=1}^{k} \varepsilon^i \, \mathbf{u}_i(t,\xi),$$

satisfying the stroboscopic condition $U(0,\xi,\varepsilon) = \xi$, such that the differential equation (1) *is transformed into*

$$\xi' = \sum_{i=1}^{k} \varepsilon^i \mathbf{g}_i(\xi) + \varepsilon^{k+1} r_k(t,\xi,\varepsilon). \tag{5}$$

Remark 7. *Let $x(t,z,\varepsilon)$ and $\xi(t,z,\varepsilon)$ be, respectively, the solutions of the differential equations* (1) *and* (5) *satisfying $x(0,z,\varepsilon) = \xi(0,z,\varepsilon) = z$. By the conjugation provided by Theorem* 6*, if follows that*

$$x(t,z,\varepsilon) = U(t,\xi(t,z,\varepsilon),\varepsilon).$$

Thus, $x(t,z,\varepsilon)$ is T-periodic if, and only if, the solution $\xi(t,z,\varepsilon)$ is T-periodic.

2.1. First-Order Averaged Function

We know that

$$x' = \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon)$$

is conjugate to a differential equation having the form

$$\xi' = \varepsilon \mathbf{g}_1(\xi) + \varepsilon^2 r_1(t,\xi,\varepsilon)$$

through a near identity transformation

$$x = U(t,\xi,\varepsilon) = \xi + \varepsilon \mathbf{u}_1(t,\xi).$$

Denoting $x(t) = x(t,z,\varepsilon)$ and $\xi(t) = \xi(t,z,\varepsilon)$, Remark 7 implies that

$$x(t) = \xi(t) + \varepsilon \mathbf{u}_1(t,\xi(t)).$$

Computing the derivative of the last relationship, we get

$$\begin{aligned} x'(t) &= \xi'(t) + \varepsilon\left(\frac{\partial \mathbf{u}_1}{\partial t}(t,\xi(t)) + \frac{\partial \mathbf{u}_1}{\partial \xi}(t,\xi(t))\xi'(t)\right) \\ &= \varepsilon \mathbf{g}_1(\xi(t)) + \varepsilon^2 r_1(t,\xi(t),\varepsilon) + \varepsilon\left(\frac{\partial \mathbf{u}_1}{\partial t}(t,\xi(t)) + \frac{\partial \mathbf{u}_1}{\partial \xi}(t,\xi(t))(\varepsilon \mathbf{g}_1(\xi(t)) + \varepsilon^2 r_1(t,\xi(t),\varepsilon))\right) \\ &= \varepsilon\left(\mathbf{g}_1(\xi(t)) + \frac{\partial \mathbf{u}_1}{\partial t}(t,\xi(t))\right) + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{6}$$

On the other hand,

$$\begin{aligned} x'(t) &= \varepsilon F_1(t,x(t)) + \varepsilon^2 R(t,x(t),\varepsilon) \\ &= \varepsilon F_1(t,\xi(t) + \varepsilon \mathbf{u}_1(t,\xi(t))) + \varepsilon^2 R(t,\xi(t) + \varepsilon \mathbf{u}_1(t,\xi(t)),\varepsilon) \\ &= \varepsilon F_1(t,\xi(t)) + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{7}$$

Comparing (6) and (7), we get that $\mathbf{u}_1(t,\xi)$ satisfies the following **Homological Equation**

$$\frac{\partial \mathbf{u}_1}{\partial t}(t,\xi) = F_1(t,\xi) - \mathbf{g}_1(\xi). \tag{8}$$

Thus,

$$\mathbf{u}_1(t,\xi) = \int_0^t \Big(F_1(s,\xi) - \mathbf{g}_1(\xi)\Big)ds + h_1(\xi).$$

Since $\mathbf{u}_1(t,\xi)$ is T-periodic, in particular $\mathbf{u}_1(T,\xi) = \mathbf{u}_1(0,\xi) = h_1(\xi)$, we conclude that

$$\int_0^t \Big(F_1(s,\xi) - \mathbf{g}_1(\xi)\Big)ds = 0 \Rightarrow \boxed{\mathbf{g}_1(\xi) = \frac{1}{T}\int_0^T F_1(s,\xi)ds}$$

In addition, by the stroboscopic condition, we get that $h_1(\xi) = 0$. Therefore,

$$\boxed{\mathbf{u}_1(t,\xi) = \int_0^t \left(F_1(s,\xi) - \frac{1}{T}\int_0^T F_1(\tau,\xi)d\tau\right)ds = \int_0^t F_1(s,\xi)ds - \frac{t}{T}\int_0^T F_1(s,\xi)ds}$$

2.1.1. Periodic Solutions Via Firs-Order Averaging. A solution $\xi(t,z,\varepsilon)$ of the differential equation

$$\xi' = \varepsilon \mathbf{g}_1(\xi) + \varepsilon^2 r_1(t,\xi,\varepsilon) \tag{9}$$

satisfies the following integral relation:

$$\xi(t,z,\varepsilon) = z + \int_0^t \Big(\varepsilon \mathbf{g}_1(\xi(s,z,\varepsilon)) + \varepsilon^2 r_1(s,\xi(s,z,\varepsilon),\varepsilon)\Big) ds.$$

Since $\mathbf{g}_1(\xi(s,z,\varepsilon)) = \mathbf{g}_1(z) + \mathcal{O}(\varepsilon)$, we conclude that

$$\begin{aligned}\xi(t,z,\varepsilon) &= z + \varepsilon \int_0^t \mathbf{g}_1(z) ds + \varepsilon^2 \int_0^t \big(\mathcal{O}(1) + r_1(s,\xi(s,z,\varepsilon),\varepsilon)\big) ds \\ &= z + \varepsilon\, t\, \mathbf{g}_1(z) + \varepsilon^2 \int_0^t \big(\mathcal{O}(1) + r_1(s,\xi(s,z,\varepsilon),\varepsilon)\big) ds.\end{aligned}$$

Therefore, the displacement function of differential equation (9) writes

$$\Delta(z,\varepsilon) = \varepsilon T \mathbf{g}_1(z) + \mathcal{O}(\varepsilon^2).$$

Denote

$$\widetilde{\Delta}(z,\varepsilon) := \frac{\Delta(z,\varepsilon)}{\varepsilon} = T\mathbf{g}_1(z) + \mathcal{O}(\varepsilon).$$

Assuming that $z^* \in D$ is a simple zero of $\mathbf{g}_1$, we get that

$$\widetilde{\Delta}(z^*,0) = 0 \quad \text{and} \quad \det\left(\frac{\partial \widetilde{\Delta}}{\partial z}(z^*,0)\right) \neq 0.$$

Hence, from the Implicit Function Theorem, we obtain the existence of a unique function $z(\varepsilon)$, defined in a neighbourhood $I \subset \mathbb{R}$ of $\varepsilon = 0$, satisfying $z(0) = z^*$ and $\widetilde{\Delta}(z(\varepsilon),\varepsilon) = 0$ for every $\varepsilon \in I$.

Therefore, Theorem 3 implies that $t \to \xi(t,z(\varepsilon),\varepsilon)$ is a T-periodic solution of the differential equation (9). Thus, from the comments of Remark 7, we conclude that

$$\varphi(t,\varepsilon) := x(t,z(\varepsilon),\varepsilon)$$

is a T-periodic solution of the differential equation (1) satisfying

$$\varphi(\cdot,\varepsilon) \to z^* \quad \text{as} \quad \varepsilon \to 0.$$

Thus, the following result holds.

Theorem 8. *Assume that $z^* \in D$ is a simple zero of $\mathbf{g}_1$. Then, for $|\varepsilon| \neq 0$ sufficiently small, the differential equation* (5) *and, consequently,* (1) *admits a unique T-periodic solution $\varphi(t,\varepsilon)$ such that $\varphi(\cdot,\varepsilon) \to z^*$ as $\varepsilon \to 0$.*

2.2. Higher-Order Averaged Functions

The functions $\mathbf{g}_i$ and $\mathbf{u}_i$ can be algorithmically computed by solving a recursive sequence of homological equations like (8). Section 3.2 of [14] is devoted to discuss how is the best way to work with such near-identity transformations based on Lie theory (see also [3, 13]).

It is worth mentioning that the so-called *stroboscopic condition* $U(\xi, 0, \varepsilon) = \xi$ does not have to be assumed in order to get (5). However, in the case that it is not assumed, the functions $\mathbf{g}_i$, for $i \geq 2$, are not uniquely determined.

For the stroboscopic averaging, the uniqueness of each $\mathbf{g}_i$ is guaranteed and so it is natural to call it by *averaged function of order* i (or i*th-order averaged function*) of the differential equation (1). Here, we refer these functions by *stroboscopic averaged functions* to indicate that the stroboscopic condition is being assumed.

2.2.1. Periodic Solutions Via Higher Order Averaging.

Theorem 9. *Denote* $\mathbf{g}_0 = 0$. *Let* $\ell \in \{1, \ldots, k\}$ *satisfying* $\mathbf{g}_0 = \cdots = \mathbf{g}_{\ell-1} = 0$ *and* $\mathbf{g}_\ell \neq 0$. *Assume that* $z^* \in D$ *is a simple zero of* $\mathbf{g}_\ell$. *Then, for* $|\varepsilon| \neq 0$ *sufficiently small, the differential equation* (5) *and, consequently,* (1) *admits a unique* T*-periodic solution* $\varphi(t, \varepsilon)$ *such that* $\varphi(0, \varepsilon) \to z^*$ *as* $\varepsilon \to 0$.

Proof. A solution $\xi(t, z, \varepsilon)$ of the differential equation

$$\xi' = \sum_{i=1}^{k} \varepsilon^i \mathbf{g}_i(\xi) + \varepsilon^{k+1} r_k(t, \xi, \varepsilon) \tag{10}$$

satisfies the following integral relation:

$$\xi(t, z, \varepsilon) = z + \int_0^t \left[\sum_{i=1}^{k} \varepsilon^i \mathbf{g}_i(\xi(s, z, \varepsilon)) + \varepsilon^{k+1} r_k(s, \xi(s, z, \varepsilon), \varepsilon) \right] ds.$$

Assuming that $\mathbf{g}_0 = \cdots = \mathbf{g}_{\ell-1} = 0$, we have that

$$\xi(t, z, \varepsilon) = z + \varepsilon^\ell t \mathbf{g}_\ell(z) + \varepsilon^{\ell+1} \int_0^t \Big(\sum_{i=\ell+1}^{k} \varepsilon^{i-\ell-1} \mathbf{g}_i(\xi(s, z, \varepsilon)) + \varepsilon^{k-\ell} r_k(s, \xi(s, z, \varepsilon), \varepsilon) \Big) ds.$$

Therefore, the displacement function of the differential equation (10) writes

$$\Delta(z, \varepsilon) = \varepsilon^\ell T \mathbf{g}_\ell(z) + \mathcal{O}(\varepsilon^{\ell+1}).$$

Denote

$$\widetilde{\Delta}(z, \varepsilon) = \frac{\Delta(z, \varepsilon)}{\varepsilon^\ell} = T \mathbf{g}_\ell(z) + \mathcal{O}(\varepsilon).$$

Assuming that $z^* \in D$ is a simple zero of $\mathbf{g}_\ell$, we get, from the Implicit Function Theorem, the existence of a unique function $z(\varepsilon)$, defined in a neighbourhood $I \subset \mathbb{R}$ of $\varepsilon = 0$, satisfying $z(0) = z^*$ and $\widetilde{\Delta}(z(\varepsilon), \varepsilon) = 0$ for every $\varepsilon \in I$.

Therefore, Theorem 3 implies that $t \to \xi(t, z(\varepsilon), \varepsilon)$ is a T-periodic solution of the differential equation (10). Thus, from the comments of Remark 7, we conclude that

$$\varphi(t, \varepsilon) := x(t, z(\varepsilon), \varepsilon)$$

is a T-periodic solution of the differential equation (1) satisfying

$$\varphi(\cdot, \varepsilon) \to z^* \quad \text{as} \quad \varepsilon \to 0.$$

□

3. Melnikov Method

The *Melnikov Method* consists in expanding the displacement function (2) around $\varepsilon = 0$ as follows

$$\Delta(z,\varepsilon) = \sum_{i=1}^{k} \varepsilon^i \mathbf{f}_i(z) + \mathcal{O}(\varepsilon^{k+1}).$$

A recursive formula for the functions $\mathbf{f}_i : D \to \mathbb{R}^n$, $i \in \{1,\dots,k\}$, was obtained in [6], which was simplified in [10] by means of Bell functions. See also [4].

Just as performed in the previous section, the following result concerning bifurcation of periodic solutions follows as a simple application of the Implicit Function Theorem:

Theorem 10 ([6]). *Denote $\mathbf{f}_0 = 0$. Let $\ell \in \{1,\dots,k\}$ satisfying $\mathbf{f}_0 = \cdots \mathbf{f}_{\ell-1} = 0$ and $\mathbf{f}_\ell \neq 0$. Assume that $z^* \in D$ is a simple zero of $\mathbf{f}_\ell$. Then, for $|\varepsilon| \neq 0$ sufficiently small, the differential equation* (1) *admits a unique T-periodic solution $\varphi(t,\varepsilon)$ such that $\varphi(\cdot,\varepsilon) \to z^*$ as $\varepsilon \to 0$.*

Remark 11. *Under the assumption $\mathbf{f}_0 = \cdots \mathbf{f}_{\ell-1} = 0$, if $\mathbf{f}_\ell(z) \neq 0$ for some $z \in D$, then the differential equation* (1) *does not admite periodic solutions converging to z.*

The bifurcation functions $\mathbf{f}_i$, $i \in \{1,\dots,k\}$, are defined by

$$\mathbf{f}_i(z) = \frac{y_i(T,z)}{i!}, \tag{11}$$

where $y_i(t,z)$ are obtained by as follows:

Lemma 12 ([6, 10]). *Let $x(t,z,\varepsilon)$ be the solution* (1) *such that $x(0,z,\varepsilon) = z$. Then,*

$$x(t,z,\varepsilon) = z + \sum_{i=1}^{k} \varepsilon^i \frac{y_i(t,z)}{i!} + \mathcal{O}(\varepsilon^{k+1}),$$

where

$$\begin{aligned} y_1(t,z) &= \int_0^t F_1(s,z)\,ds \quad \text{and} \\ y_i(t,z) &= \int_0^t \Big(i!F_i(s,z) + \sum_{j=1}^{i-1}\sum_{m=1}^{j} \frac{i!}{j!}\partial_x^m F_{i-j}(s,z) B_{j,m}\big(y_1,\dots,y_{j-m+1}\big)(s,z)\Big)\,ds, \end{aligned} \tag{12}$$

for $i \in \{2,\dots,k\}$.

Here, for p and q positive integers, $B_{p,q}$ denotes the *partial Bell polynomials*:

$$B_{p,q}(x_1,\dots,x_{p-q+1}) = \sum \frac{p!}{b_1!\,b_2!\cdots b_{p-q+1}!} \prod_{j=1}^{p-q+1} \left(\frac{x_j}{j!}\right)^{b_j}.$$

The sum above is taken over all the $(p-q+1)$-tuple of nonnegative integers $(b_1, b_2, \cdots, b_{p-q+1})$ satisfying $b_1 + 2b_2 + \cdots + (p-q+1)b_{p-q+1} = p$, and $b_1 + b_2 + \cdots + b_{p-q+1} = q$.

It is worth mentioning that $\partial_x^m F_{i-j}(s,z)$ denotes the Frechet's derivative of F_{i-j} with respect to the variable x evaluated at $x = z$, which is a symmetric m-multilinear map that is applied to combinations of "products" of m vectors in $\mathbb{R}^n$, in our case $B_{j,m}\big(y_1,\dots,y_{j-m+1}\big)$.

Remark 13. *Notice that*

$$\mathbf{f}_1(z) = \int_0^T F_1(s,z)ds = T\mathbf{g}_1(z).$$

Usually, $\mathbf{f}_i$ *is likewise called by averaged function of order* i *(or* i*th-order averaged function) of the differential equation* (1). *It is worth mentioning that the bifurcation functions* $\mathbf{f}_i$*'s also receive the name of* **Poincaré-Pontryagin-Melnikov functions** *or just* **Melnikov functions**. *Such functions can be formally easily computed, for instance*

$$\begin{aligned}\mathbf{f}_2(z) &= \int_0^T \Big(F_2(t,z) + \partial_x F_1(t,z) y_1(t,z)\Big)dt \\ &= \int_0^T \Big(F_2(t,z) + \partial_x F_1(t,z)\int_0^t F_1(s,z)ds\Big)dt.\end{aligned} \tag{13}$$

3.1. Idea of the Proof of Lemma 12

We recall the Faá di Bruno's Formula about the l^{th} derivative of a composite function.

Faá di Bruno's Formula *If* g *and* f *are functions with a sufficient number of derivatives, then*

$$\frac{d^l}{d\alpha^l} g(h(\alpha)) = \sum_{m=1}^{l} g^{(m)}(h(\alpha))B_{l,m}\big(h'(\alpha), h''(\alpha), \ldots, h^{(l-m+1)}(\alpha)\big),$$

where S_l *is the set of all* l*–tuples of non–negative integers* $(b_1, b_2, \cdots, b_l)$ *which are solutions of the equation* $b_1 + 2b_2 + \cdots + lb_l = l$ *and* $L = b_1 + b_2 + \cdots + b_l$.

First of all, notice that

$$x(t,z,\varepsilon) = z + \sum_{i=0}^{k} \varepsilon^i \int_0^t F_i(s, x(s,z,\varepsilon))ds + \mathcal{O}(\varepsilon^{k+1}). \tag{14}$$

Result about differentiable dependence on parameters implies that the map $\varepsilon \mapsto x(t,z,\varepsilon)$ is smooth in all variables. So we can use the Faá di Bruno's Formula a follows.

Since $x(t,z,0) = z$, the Taylor expansion of $F_i(t, x(t,z,\varepsilon))$ around $\varepsilon = 0$, for $i = 0, 1, \ldots, k-1$, is given by

$$F_i(t, x(t,z,\varepsilon)) = F_i\,(t,z) + \sum_{j=1}^{k-i} \frac{\varepsilon^j}{j!}\left(\frac{\partial^j}{\partial\varepsilon^j} F_i(t, x(t,z,\varepsilon))\right)\Bigg|_{\varepsilon=0} + \mathcal{O}(\varepsilon^{k-i+1}). \tag{15}$$

The Faá di Bruno's formula allows to compute the j–derivatives of $F_i(t, x(t,z,\varepsilon))$ in ε, for $i = 0, 1, \ldots, k-1$:

$$\frac{\partial^j}{\partial\varepsilon^j} F_i(t, x(t,z,\varepsilon))\Bigg|_{\varepsilon=0} = \sum_{m=1}^{j} \partial_x^m F_i(s,z) B_{j,m}\big(y_1, \ldots, y_{j-m+1}\big)(s,z) \tag{16}$$

where

$$y_j(t,z) = \left(\frac{\partial^j}{\partial\varepsilon^j} x(t,z,\varepsilon)\right)\Bigg|_{\varepsilon=0}. \tag{17}$$

Substituting (16) in (15) the Taylor expansion at $\varepsilon=0$ of $F_i(s,x(t,z,\varepsilon))$ becomes

$$\begin{aligned}F_i(s,x(s,z,\varepsilon)) =&F_i\,(s,z)\\ &+\sum_{j=1}^{k-i}\varepsilon^j\sum_{m=1}^{j}\frac{1}{j!}\partial_x^m F_i(s,z)B_{j,m}\big(y_1,\ldots,y_{j-m+1}\big)(s,z)\\ &+\mathcal{O}(\varepsilon^{k-i+1}),\end{aligned}\tag{18}$$

for $i=0,1,\ldots,k-1$. Moreover, for $i=k$ we have that

$$F_k(s,x(s,z,\varepsilon))=F_k\,(s,z)+\mathcal{O}(\varepsilon).\tag{19}$$

Now, from (14), (18), and (19) the following equation holds

$$x(t,z,\varepsilon)=z+Q(t,z,\varepsilon)+\sum_{i=0}^{k}\varepsilon^i\int_0^t F_i(s,z)ds+\mathcal{O}(\varepsilon^{k+1}),\tag{20}$$

where

$$Q(t,z,\varepsilon)=\sum_{i=0}^{k-1}\sum_{j=1}^{k-i}\varepsilon^{j+i}\sum_{m=1}^{j}\int_0^t\frac{1}{j!}\partial_x^m F_i(s,z)B_{j,m}\big(y_1,\ldots,y_{j-m+1}\big)(s,z)ds.$$

We may write

$$\begin{aligned}Q(t,z,\varepsilon)=&\sum_{j=1}^{k}\sum_{i=j}^{k}\varepsilon^i\sum_{m=1}^{j}\int_0^t\frac{1}{j!}\partial_x^m F_{i-j}(s,z)B_{j,m}\big(y_1,\ldots,y_{j-m+1}\big)(s,z)ds\\ =&\sum_{i=1}^{k}\varepsilon^i\sum_{j=1}^{i}\sum_{m=1}^{j}\int_0^t\frac{1}{j!}\partial_x^m F_{i-j}(s,z)B_{j,m}\big(y_1,\ldots,y_{j-m+1}\big)(s,z)ds.\end{aligned}\tag{21}$$

Finally, from (20) and (21), we get

$$\begin{aligned}x(t,z,\varepsilon)=&\\ =&z+\sum_{i=1}^{k}\varepsilon^i\left(\int_0^t F_i(s,z)+\sum_{j=1}^{i}\sum_{m=1}^{j}\int_0^t\frac{1}{j!}\partial_x^m F_{i-j}(s,z)B_{j,m}\big(y_1,\ldots,y_{j-m+1}\big)(s,z)\,ds\right)\\ &+\mathcal{O}(\varepsilon^{k+1}).\end{aligned}$$

Now, using this last expression of $x(t,z,\varepsilon)$ we conclude that functions $y_i(t,z)$ defined in (17), for $i=1,2,\ldots,k-1$, can be computed recurrently from the following integral equation

$$\boxed{\begin{aligned}y_1(t,z)=&\frac{\partial x}{\partial\varepsilon}(t,z,\varepsilon)\Big|_{\varepsilon=0}=\int_0^t F_1(s,z)ds\quad\text{and}\\ y_i(t,z)=&\frac{\partial^i x}{\partial\varepsilon^i}(t,z,\varepsilon)\Big|_{\varepsilon=0}\\ &=i!\int_0^t\left(F_i(s,z)+\sum_{j=1}^{i}\sum_{m=1}^{j}\int_0^t\frac{1}{j!}\partial_x^m F_{i-j}(s,z)B_{j,m}\big(y_1,\ldots,y_{j-m+1}\big)(s,z)\right)ds\end{aligned}}$$

Finally,

$$\boxed{x(t,z,\varepsilon)=z+\sum_{i=1}^{k}\varepsilon^i\frac{y_i(t,z)}{i!}+\mathcal{O}(\varepsilon^{k+1})}$$

which completes the proof of Lemma 12.

4. General Relationship Between Stroboscopic Averaging Functions and Melnikov Functions

A general relationship between the functions $\mathbf{g}_i$ and $\mathbf{f}_i$ defined above is provided by the following result. See also [2, 5].

Theorem 14 ([11]). *For $i \in \{1, \dots, k\}$, the following recursive relationship between $\mathbf{g}_i$ and $\mathbf{f}_i$ holds:*

$$\begin{aligned}\mathbf{g}_1(z) =& \frac{1}{T}\mathbf{f}_1(z),\\ \mathbf{g}_i(z) =& \frac{1}{T}\left(\mathbf{f}_i(z) - \sum_{j=1}^{i-1}\sum_{m=1}^{j}\frac{1}{j!}d^m\mathbf{g}_{i-j}(z)\int_0^T B_{j,m}\big(\tilde{y}_1, \dots, \tilde{y}_{j-m+1}\big)(s,z)ds\right),\end{aligned} \tag{22}$$

where $\tilde{y}_i(t,z)$, for $i \in \{1, \dots, k\}$, are polynomial in the variable t recursively defined as follows

$$\begin{aligned}\tilde{y}_1(t,z) =& t\,\mathbf{g}_1(z)\\ \tilde{y}_i(t,z) =& i!t\,\mathbf{g}_i(z) + \sum_{j=1}^{i-1}\sum_{m=1}^{j}\frac{i!}{j!}d^m\mathbf{g}_{i-j}(z)\int_0^t B_{j,m}\big(\tilde{y}_1, \dots, \tilde{y}_{j-m+1}\big)(s,z)ds.\end{aligned} \tag{23}$$

Remark 15. *By applying the formula above for $i = 2$, we get*

$$\mathbf{g}_2(z) = \frac{1}{T}\left(\mathbf{f}_2(z) - \frac{1}{2}d\mathbf{f}_1(z)\mathbf{f}_1(z)\right),$$

where $\mathbf{f}_2$ is explicitly given by (13). Thus,

$$\mathbf{g}_2(z) = \frac{1}{T}\int_0^T \left(F_2(t,z) + \partial_x F_1(t,z)\int_0^t \Big(F_1(s,z) - \frac{1}{2}\mathbf{g}_1(z)\Big)ds\right)dt,$$

which coincides with the expression provided by [14, Section 2.9.1].

4.1. Some consequences

Corollary 16. *Denote $\mathbf{f}_0 = \mathbf{g}_0 = 0$ and let $\ell \in \{1, \dots, k\}$. If either*

$$\mathbf{f}_1 = \cdots = \mathbf{f}_{\ell-1} = 0 \quad or \quad \mathbf{g}_1 = \cdots = \mathbf{g}_{\ell-1} = 0,$$

then $\mathbf{f}_i = T\,\mathbf{g}_i$ for $i \in \{1, \dots, \ell\}$.

Now, as a direct consequence of Corollary 16, the first non-vanishing stroboscopic averaged function can be computed in relatively simple way. In particular, we get the following result:

Corollary 17. *Let $\ell \in \{1, \dots, k\}$ satisfying $\mathbf{f}_1 = \cdots \mathbf{f}_{\ell-1} = 0$. Then, there exists a T-periodic near-identity transformation $x = U(t, \xi, \varepsilon)$ satisfying $U(\xi, 0, \varepsilon) = \xi$, such that the differential equation (1) is transformed into*

$$\xi' = \varepsilon^\ell \frac{1}{T}\mathbf{f}_\ell(\xi) + \varepsilon^{\ell+1} r_\ell(t, \xi, \varepsilon).$$

5. Algorithm

In what follows, $F_i(t,x)$ is denoted by F[i,t,x], $y_i(t,x)$ is denoted by y0[i,t], $\tilde{y}_i(t,x)$ is denoted by y1[i,t], $\mathbf{f}_i(z)$ is denoted by f[i,z], and $\mathbf{g}_i(z)$ is denoted by g[i,z]. The order of perturbation k must be specified in order to run the code.

Listing 1. Mathematica's algorithm for computing $\mathbf{f}_i$

```
y0[1, t_] = Integrate[F[1, s, z], {s, 0, t}];
Y0[1] = {y0[1, t]};
For[i = 2, i <= k, i++,
 y0[i, t_] := Integrate[i! F[i, s, z] +
 Sum[Sum[i!/j! D[F[i - j, t, z], {z, m}] BellY[j, m,
 Y0[j - m + 1]], {m, 1, j}], {j, 1, i - 1}], {s, 0, t}];
 Y0[i] = Join[Y0[i - 1], {y0[i, t]}];
 f[i, z_] = y0[i, T]/i!];
```

Listing 2. Mathematica's algorithm for computing $\mathbf{g}_i$

```
g[1, z_] = f[1, z]/T;
y1[1, t] = t g[1, z];
Y1[1, t_] = {y1[1, t]};
For[i = 2, i <= k, i++,
 g[i, z_] = 1/T (f[i, z] -
 Sum[Sum[1/j! D[g[i - j,z], {z, m}]
 Integrate[BellY[j, m, Y1[j - m + 1, s]], {s, 0, T}],
  {m, 1, j}], {j, 1, i - 1}]);
 y1[i, t_] =i! t g[i,z] +
 Sum[Sum[i!/j! D[g[i - j,z], {z, m}]
 Integrate[BellY[j, m, Y1[j - m + 1, s]], {s, 0, T}],
  {m, 1, j}], {j, 1, i - 1}];
 Y1[i, t_] = Join[Y1[i - 1, t], {y1[i, t]}]];
```

6. Some Formulae

$$
\begin{aligned}
y_1(t,z) = & \int_0^t F_1(s,z)ds, \\
y_2(t,z) = & \int_0^t \Big(2F_2(s,z) + 2\partial_z F_1(s,z)y_1(s,z)\Big)ds, \\
y_3(t,z) = & \int_0^t \Big(6F_3(s,z) + 6\partial_z F_2(s,z)y_1(s,z) + 3\partial_z^2 F_1(s,z)y_1(s,z)^2 + 3\partial_z F_1(s,z)\, y_2(s,z)\Big)ds, \\
y_4(t,z) = & \int_0^t \Big(24F_4(s,z) + 24\partial_z F_3(s,z)y_1(s,z) + 12\partial_z^2 F_2(s,z)y_1(s,z)^2 + 12\partial_z F_2(s,z)y_2(s,z) \\
& +12\partial_z^2 F_1(s,z)y_1(s,z)y_2(s,z) + 4\partial_z^3 F_1(s,z)y_1(s,z)^3 + 4\partial_z F_1(s,z)y_3(s,z)\Big)ds, \\
y_5(t,z) = & \int_0^t \Big(120F_5(s,z) + 120\partial_z F_4(s,z)y_1(s,z) \\
& +60\partial_z^2 F_3(s,z)y_1(s,z)^2 + 60\partial_z F_3(s,z)y_2(s,z) + 60\partial_z^2 F_2(s,z)y_1(s,z)y_2(s,z) \\
& +20\partial_z^3 F_2(s,z)y_1(s,z)^3 + 20\partial_z F_2(s,z)y_3(s,z) + 20\partial_z^2 F_1(s,z)y_1(s,z)y_3(s,z) \\
& +15\partial_z^2 F_1(s,z)y_2(s,z)^2 + 30\partial_z^3 F_1(s,z)y_1(s,z)^2 y_2(s,z) \\
& +5\partial_z^4 F_1(s,z)y_1(s,z)^4 + 5\partial_z F_1(s,z)y_4(s,z)\Big)ds.
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{f}_1(z) = & \int_0^T F_1(t,z)dt, \\
\mathbf{f}_2(z) = & \int_0^T \Big(F_2(t,z)ds + \partial_z F_1(t,z)y_1(t,z)\Big)dt, \\
\mathbf{f}_3(z) = & \int_0^T \Big(F_3(t,z) + \partial_z F_2(t,z)y_1(t,z) + \frac{1}{2}\partial_z^2 F_1(t,z)y_1(t,z)^2 + \frac{1}{2}\partial_z F_1(t,z)y_2(t,z)\Big)dt, \\
\mathbf{f}_4(z) = & \int_0^T \Big(F_4(t,z) + \partial_z F_3(t,z)y_1(t,z) + \frac{1}{2}\partial_z^2 F_2(t,z)y_1(t,z)^2 + \frac{1}{2}\partial_z F_2(t,z)y_2(t,z) \\
& +\frac{1}{2}\partial_z^2 F_1(t,z)y_1(t,z)y_2(t,z)dt + \frac{1}{6}\partial_z^3 F_1(t,z)y_1(t,z)^3 + \frac{1}{6}\partial_z F_1(t,z)y_3(t,z)\Big)dt, \\
\mathbf{f}_5(z) = & \int_0^T \Big(F_5(t,z) + \partial_z F_4(t,z)y_1(t,z) \\
& +\frac{1}{2}\partial_z^2 F_3(t,z)y_1(t,z)^2 + \frac{1}{2}\partial_z F_3(t,z)y_2(t,z) + \frac{1}{2}\partial_z^2 F_2(t,z)y_1(t,z)y_2(t,z) \\
& +\frac{1}{6}\partial_z^3 F_2(t,z)y_1(t,z)^3 + \frac{1}{6}\partial_z F_2(t,z)y_3(t,z) + \frac{1}{6}\partial_z^2 F_1(t,z)y_1(t,z)y_3(t,z) \\
& +\frac{1}{8}\partial_z^2 F_1(t,z)y_2(t,z)^2 + \frac{1}{4}\partial_z^3 F_1(t,z)y_1(t,z)^2 y_2(t,z) \\
& +\frac{1}{24}\partial_z^4 F_1(t,z)y_1(t,z)^4 + \frac{1}{24}\partial_z F_1(t,z)y_4(t,z)\Big)dt.
\end{aligned}
$$

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