CENTRE DE RECERCA MATEMÀTICA

Intensive Research Program
on Quantitative Finance
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Semi-Analytical method for Barrier Options pricing

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Black-Scholes Model

[F. Black and M. Scholes, 1973]

A **European option** is a contract
which gives the buyer
the right to sell (**put** option) or to buy (**call** option) an **underlying asset**
at a specified **strike** price
on a specified date (**expiry**)

Options can be used, mainly, for insuring against price movements (hedging) or speculating.

(Chicago Board Options Exchange) *According to data compiled by the Options Industry Council, the total volume of options contracts traded on U.S. exchanges in 1999 was about 507 million. By 2007, that number had grown to an all-time record of more than 3 billion.*

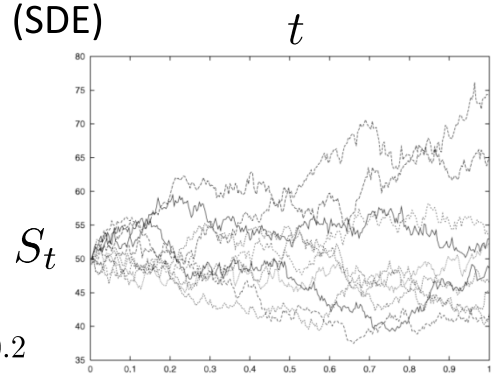
Assumptions:

- No arbitrage opportunities
- Information is instantaneously widespread
- The market is efficient and frictionless: no transaction costs, it is possible to borrow and lend any amount, even fractional, of the stock and of cash at the riskless rate …

- The asset price satisfies the following stochastic differential equation (SDE)

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t$$

- r is the constant risk-free **interest rate**
- σ is the **volatility** constant for the asset price
- W_t is a Wiener process



10 paths of SDE, with $S_0 = 50, r = 0.1, \sigma = 0.2$

following Black-Scholes model

Over the past three decades, the academic literature has highlighted the strong limitations of this model due to the fact that it is based **on restrictive and unrealistic assumptions**.

Therefore other models have been later introduced:

1. models with **time-dependent parameters**;
2. stochastic volatility models, such as the **Heston** model, in which the value of the option depends on time, on the stock price and on the volatility of the underlying asset.

But also, the model has been adapted to other kinds of option contracts:

3. **Asian Options**
4. **Basket Options**

For these more advanced models, the pricing is traditionally based:

• **Monte Carlo methods**: very simple and flexible, but also very slow to converge

• **Finite element methods**: very accurate and fast and capable of handling discontinuous solutions; However, quite difficult to implement, especially if a high-degree polynomial basis are employed and with some troubles in unbounded domains (as Finite Difference methods);

• **Binomial/trinomial lattices**: relatively easy to implement, but not particularly efficient.

• **Finite difference schemes**: easy to implement. However, standard high-order implementations fail to achieve true high-order accuracy, due to the non-smoothness of the options' payoffs

Here a different numerical approach to the European Option Pricing will be presented for the particular case of application of **Barrier Conditions** in the contract (**Semi-Analytical method for Barrier Option**)

The mathematical model problem 1.: time dependent parameters

Under the simple **Black-Scholes paradigm**, still very common in use, with **time dependent** parameters $\sigma(t), r(t), \delta(t)$

European put option differential model problem

- $V(x,t)$ option price $\quad x = \log(S) \in (-\infty, +\infty),\ t \in [0,T]$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + (r - \frac{\sigma^2}{2} - \delta)\frac{\partial V}{\partial x} - rV = 0$$

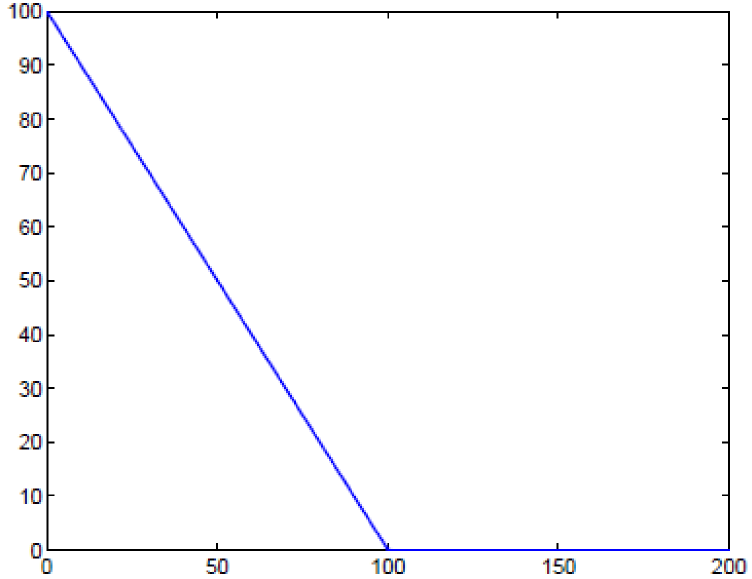
- with **final condition** (payoff)

$$V(x,T) = \max(E - e^x, 0) \qquad x \in (-\infty, +\infty)$$

The option's **payoff** is the amount of money received by the holder at maturity.

S = underlying asset value
r = interest rate
δ = dividend yield
σ = volatility
T = expiry
E = exercise price

For Put Option with **exercise (strike) price *E***:
At maturity *T*, if *S*(*T*) ≤ *E*, the holder can buy the underlying asset at *S*(*T*) and exercise the right to sell it at *E*, and thus the option's payoff is equal to *E* - *S*(*T*).
On the contrary, if *S*(*T*) > *E*, why sell something at a price *E* that is lower than its market price? Thus, if *S*(*T*) > *E*, the option is not exercised and the holder receives a payoff equal to zero.



The payoff is not a differentiable function

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- with **final condition** (payoff)

$$V(x,T) = \max(E - e^x, 0) \qquad x \in (-\infty, +\infty)$$

- with **boundary conditions** on the asset

$$\lim_{x\to-\infty} V(x,t) = Ee^{-\int_t^T r(\tau)d\tau} \qquad \lim_{x\to+\infty} V(x,t) = 0 \qquad t \in [0,T)$$

S = underlying asset value
r = interest rate
δ = dividend yield
σ = volatility
T= expiry
E= exercise price

For this problem the **analytical** solution is known:

$N[q] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{q} e^{-y^2/2}dy$ normal cumulative distribution; $d = -\frac{\log(e^x/E)+\int_t^T (r(\tau)-\delta(\tau)+\sigma^2(\tau)/2)d\tau}{\int_t^T \sigma^2(\tau)d\tau}$

$$V(x,t) = Ee^{-\int_t^T r(\tau)d\tau} N\left[q + \left(\int_t^T \sigma^2(\tau)d\tau\right)^{1/2}\right] - e^{x-\int_t^T \delta(\tau)d\tau} N[q]$$

The mathematical model problem 1.: time dependent parameters

Under the simple **Black-Scholes paradigm**, still very common in use,
with **time dependent** parameters $\sigma(t), r(t), \delta(t)$

European vanilla option differential model problem

- $V(x,t)$ option price $\quad x = \log(S) \in (-\infty, +\infty),\ t \in [0,T)$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + (r - \frac{\sigma^2}{2} - \delta)\frac{\partial V}{\partial x} - rV = 0$$

- with **final condition** (payoff)

$$V(x,T) = \max(E - e^x, 0) \qquad x \in (-\infty, +\infty)$$

- with **boundary conditions** on the asset

$$\lim_{x \to -\infty} V(x,t) = Ee^{-\int_t^T r(\tau)d\tau} \qquad \lim_{x \to +\infty} V(x,t) = 0 \qquad t \in [0,T)$$

$S =$ underlying asset value
$r =$ interest rate
$\delta =$ dividend yield
$\sigma =$ volatility
$T =$ expiry
$E =$ exercise price

For this problem the **analytical** solution is known:

$$V(x,t) = e^{-\int_t^T r(\tau)d\tau} \int_{-\infty}^{+\infty} V(y,T)G(y,T;x,t)dy$$

where $V(y,T)$ is the payoff and $G(x,y,t,\tau)$ is the **fundamental solution** of the forward PDE

Our financial model problem: European barrier option pricing

But we will take into consideration a slightly different problem whose analytical solution is generally not known in a closed form

A **knock-in barrier option** is an option that comes into existence when the price of the underlying asset reaches a specified **barrier** during the option's life

A **knock-out barrier option** is an option whose price extinguishes when the underlying asset breaches a pre-set **barrier** level

In order to limit profits and losses…

In particular, I will illustrate here the case of a

European put up-and-out barrier option
whose price extinguishes when the underlying asset breaches a pre-set **upper barrier level**

<u>but the method is analogously applicable also to call option, other payoffs and other combinations of barriers too</u>.

The mathematical model problem: European put up-and-out option

Performing these classical **changes of variables** $S \in [0, S_u]$ and $t \in [0, T]$

$$V(S,t) = u(S,t)e^{-\int_t^T r(t')dt'} \qquad S = e^x \qquad \tau = T - t$$

and defining $r(t) = r(T-\tau) =: \overline{r}(\tau), \quad \sigma(t) = \sigma(T-\tau) =: \overline{\sigma}(\tau), \quad \text{and} \quad d(t) = d(T-\tau) =: \overline{d}(\tau)$

European put up-and-out option differential model problem

- $$\frac{\partial u}{\partial \tau} - \frac{\bar{\sigma}^2}{2}\frac{\partial^2 u}{\partial x^2} - (\bar{r} - \frac{\bar{\sigma}^2}{2} - \bar{\delta})\frac{\partial u}{\partial x} = 0 \qquad x \in \Omega = (-\infty, U),\ \tau \in (0, T]$$
- with **initial condition**

  $$u(x,0) = \max(E - e^x, 0) =: u_0(x) \qquad x \in \Omega$$
- with **boundary conditions** on the asset

  $$\lim_{x \to -\infty} u(x,\tau) = E \qquad u(U,\tau) = 0 \qquad \tau \in [0, T]$$

$S =$ underlying asset value
$\bar{r} =$ interest rate
$\bar{\delta} =$ dividend yield
$\bar{\sigma} =$ volatility
$T =$ expiry
$E =$ exercise price
$U = \log(\text{upper barrier})$

Is there a closed form solution?

Semi-Analytical method for the pricing of Barrier Options

SABO: Semi-Analytical method for the pricing of Barrier Options, **under general dynamics**.

Foundations:

- *Analytical* **Integral Representation of PDE solution**
- **Boundary Integral Equation**
- **Numerical Resolution of the Boundary Integral Equation by Collocation Method**
- *Numerical approximation* **of the option price**

Integral Representation Formula of the PDE Solution

following **PDE theory…**

PDE $$\frac{\partial u}{\partial \tau} - \frac{\bar{\sigma}^2}{2}\frac{\partial^2 u}{\partial x^2} - (\bar{r} - \frac{\bar{\sigma}^2}{2} - \bar{\delta})\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \tau}(x,\tau) - \mathcal{L}[u](x,\tau) = 0 \qquad x \in \Omega = (-\infty, U),\ \tau \in (0,T]$$

the related transition probability density (**Green fundamental solution**)

$$G(y,s,x,\tau) = \frac{1}{\sqrt{2\pi\int_s^\tau \bar{\sigma}^2(v)dv}} \exp\left\{ -\frac{[y - x - \int_s^\tau (\bar{r} - \bar{\sigma}^2/2 - \bar{\delta})(v)dv]^2}{2\int_s^\tau \bar{\sigma}^2(v)dv} \right\}, \qquad \tau > s$$

for each $(x,\tau) \in \mathbb{R} \times (0,T]$, $G(y,s,x,\tau)$ solves

$$\begin{cases} -\frac{\partial G}{\partial s}(y,s;x,\tau) - \mathcal{L}^*[G](y,s;x,\tau) = 0 & y \in \mathbb{R}, s < \tau \\ G(y,\tau;x,\tau) = \delta(x,y) & y \in \mathbb{R} \end{cases}$$

Multiplying the **PDE** by G, integrating by parts (**Green's Theorem**)
and using **initial/boundary conditions**

$$u(x,\tau) = \int_\Omega u(y,0)G(y,0,x,\tau)dy$$

RF

for each $x \in \Omega = (-\infty, U),\ \tau \in (0,T]$

Integral Representation Formula of the PDE Solution

following **PDE theory…**

PDE

$$\frac{\partial u}{\partial \tau} - \frac{\bar\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - (\bar r - \frac{\bar\sigma^2}{2} - \bar\delta)\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \tau}(x,\tau) - \mathcal{L}[u](x,\tau) = 0 \qquad x \in \Omega = (-\infty, U),\ \tau \in (0,T]$$

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for each $(x,\tau) \in \mathbb{R} \times (0,T]$, $\quad G(y,s,x,\tau)$ solves

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Multiplying the **PDE** by G, integrating by parts (**Green's Theorem**)
and using **initial/boundary conditions**

$$u(x,\tau) \quad = \quad \int_\Omega u(y,0)G(y,0,x,\tau)dy + \int_0^\tau \int_{\partial\Omega} \frac{\bar\sigma^2}{2}\frac{\partial u}{\partial y}(y,s)G(y,s,x,\tau)dy\,ds$$

RF

for each $x \in \Omega = (-\infty, U),\ \tau \in (0,T]$

Integral Representation Formula of the PDE Solution

following **PDE theory…**

PDE $$\frac{\partial u}{\partial \tau}-\frac{\bar{\sigma}^2}{2}\frac{\partial^2 u}{\partial x^2}-(\bar{r}-\frac{\bar{\sigma}^2}{2}-\bar{\delta})\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \tau}(x,\tau)-\mathcal{L}[u](x,\tau)=0 \qquad x\in\Omega=(-\infty,U),\ \tau\in(0,T]$$

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for each $(x,\tau)\in\mathbb{R}\times(0,T]$, $G(y,s,x,\tau)$ solves

$$\begin{cases} -\frac{\partial G}{\partial s}(y,s;x,\tau)-\mathcal{L}^*[G](y,s;x,\tau)=0 & y\in\mathbb{R}, s<\tau \\ G(y,\tau;x,\tau)=\delta(x,y) & y\in\mathbb{R} \end{cases}$$

Multiplying the **PDE** by G, integrating by parts (**Green's Theorem**)
and using **initial/boundary conditions**

RF

$$\begin{aligned} u(x,\tau) &= \int_\Omega u(y,0)G(y,0,x,\tau)dy+\int_0^\tau\int_{\partial\Omega}\frac{\bar{\sigma}^2}{2}\frac{\partial u}{\partial y}(y,s)G(y,s,x,\tau)dy\,ds \\ &= \int_{-\infty}^U u_0(y)G(y,0,x,\tau)dy+\int_0^\tau\frac{\bar{\sigma}^2}{2}\frac{\partial u}{\partial y}(U,s)G(U,s,x,\tau)ds \end{aligned}$$

for each $x\in\Omega=(-\infty,U),\ \tau\in(0,T]$

Boundary Integral Equation

analytical **INTEGRAL REPRESENTATION FORMULA**

RF

$$u(x,\tau) = \int_{-\infty}^{U} u_0(y) G(y,0,x,\tau) dy + \int_0^{\tau} \frac{\bar{\sigma}^2(s)}{2} \underbrace{\frac{\partial u}{\partial y}(U,s)}_{\text{unknown density}} G(U,s,x,\tau) ds$$

for each $x \in \Omega = (-\infty, U)$, $\tau \in (0,T]$

unknown density

Boundary Integral Equation

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for each $x \in \Omega = (-\infty, U)$, $\tau \in (0,T]$

unknown density

but on the boundary, letting $x \to U$ **BOUNDARY INTEGRAL EQUATION**

BIE

$$0 = u(U,\tau) := \int_{-\infty}^{U} u_0(y) G(y,0,U,\tau)\,dy + \int_0^{\tau} \frac{\bar{\sigma}^2(s)}{2} \frac{\partial u}{\partial y}(U,s) G(U,s,U,\tau)\,ds$$

for each $\tau \in (0,T]$

solve the equation… *numerically*

Boundary Integral Equation

analytical **INTEGRAL REPRESENTATION FORMULA**

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$$u(x,\tau) = \int_{-\infty}^{U} u_0(y)G(y,0,x,\tau)dy + \int_0^{\tau} \frac{\bar{\sigma}^2(s)}{2}\frac{\partial u}{\partial y}(U,s)G(U,s,x,\tau)ds$$

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for each $\tau \in (0,T]$

solve the equation… *numerically*

Note!: when $U \to +\infty$
the method reduces to the evaluation of the payoff expected value

Numerical Resolution of the Boundary Integral Equation

by **COLLOCATION METHOD**:

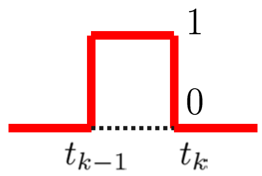
- uniform decomposition of the time interval $[0, T]$ with time step

$$\Delta t = T/N_{\Delta t}: \qquad t_k = k\Delta t \quad k = 0, \ldots, N_{\Delta t}$$

- approximation of the BIE unknown

$$\frac{\partial u}{\partial y}(U, s) \approx \phi(s) := \sum_{k=1}^{N_{\Delta t}} \alpha_k \varphi_k(s)$$

with $\varphi_k(s) := H[s - t_{k-1}] - H[s - t_k]$ for $k = 1, \ldots, N_{\Delta t}$



- evaluation of BIE at the **collocation nodes**: $\bar{t}_j = \frac{t_j + t_{j-1}}{2} \qquad j = 1, \ldots, N_{\Delta t}$

BIE

$$0 = u(U, \tau) := \int_{-\infty}^{U} u_0(y) G(y, 0, U, \tau) dy + \int_0^{\tau} \frac{\bar{\sigma}^2(s)}{2} \frac{\partial u}{\partial y}(U, s) G(U, s, U, \tau) ds$$

Numerical Resolution of the Boundary Integral Equation

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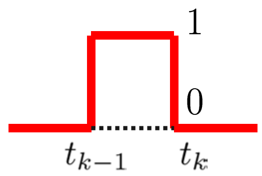

- evaluation of BIE in the **collocation nodes**: $\bar{t}_j = \dfrac{t_j + t_{j-1}}{2} \qquad j = 1, \ldots, N_{\Delta t}$

BIE

$$0 = u(U, \bar{t}_j) := \int_{-\infty}^{U} u_0(y) G(y, 0, U, \bar{t}_j) dy + \int_0^{\bar{t}_j} \frac{\bar{\sigma}^2(s)}{2} \sum_{k=1}^{N_{\Delta t}} \alpha_k \varphi_k(s) G(U, s, U, \bar{t}_j) ds$$

Numerical Resolution of the Boundary Integral Equation

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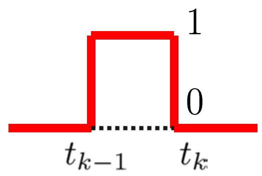

- evaluation of BIE in the **collocation nodes**: $\bar{t}_j = \dfrac{t_j + t_{j-1}}{2} \qquad j = 1, \ldots, N_{\Delta t}$

$$\sum_{k=1}^{N_{\Delta}} \alpha_k \underbrace{\int_0^{\bar{t}_j} \varphi_k(s) \frac{\bar{\sigma}^2(s)}{2} G(U, s, U, \bar{t}_j) ds}_{\mathcal{A}_{jk}} = \underbrace{\int_{-\infty}^{U} u_0(y) G(y, 0, U, \bar{t}_j) dy}_{\mathcal{F}_j}$$

Numerical Resolution of the Boundary Integral Equation

$$\mathcal{A}\alpha = \mathcal{F}$$

$$\mathcal{A} = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ A_{31} & A_{32} & A_{33} & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ A_{N_{\Delta t}1} & A_{N_{\Delta t}2} & \cdots & A_{N_{\Delta t}N_{\Delta t}-1} & A_{N_{\Delta t}N_{\Delta t}} \end{pmatrix}$$

as the Green's function is defined for $\tau > s$

N.B.: if τ, σ, r are constant then

$$\mathcal{A} = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ A_2 & A_1 & 0 & \cdots & 0 \\ A_3 & A_2 & A_1 & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ A_{N_{\Delta t}} & A_{N_{\Delta t-1}} & \cdots & A_{N_2} & A_{N_1} \end{pmatrix}$$

Toeplitz structure

Numerical Resolution of the Boundary Integral Equation

$$\mathcal{A}\alpha = \mathcal{F}$$

$$\mathcal{A} = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ A_{31} & A_{32} & A_{33} & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ A_{N_{\Delta t}1} & A_{N_{\Delta t}2} & \cdots & A_{N_{\Delta t}N_{\Delta t}-1} & A_{N_{\Delta t}N_{\Delta t}} \end{pmatrix}$$

as the Green's function is defined for $\tau > s$

$$\mathcal{A}\alpha = \mathcal{F} \longrightarrow \alpha$$

Boundary Integral Equation

analytical **INTEGRAL REPRESENTATION FORMULA**

RF

$$u(x,\tau) = \int_{-\infty}^{U} u_0(y) G(y,0,x,\tau)dy + \int_0^{\tau} \frac{\bar{\sigma}^2(s)}{2} \frac{\partial u}{\partial y}(U,s) G(U,s,x,\tau)ds$$

for each $x \in \Omega = (-\infty, U)$, $\tau \in (0,T]$

unknown density

but on the boundary, letting $x \to U$ **BOUNDARY INTEGRAL EQUATION**

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for each $\tau \in (0,T]$

solve the equation… *numerically*

Numerical Approximation of the option price

approximated **INTEGRAL REPRESENTATION FORMULA**

RF

$$u(x,\tau) \approx \int_{-\infty}^{U} u_0(y) G(y,0,x,\tau) dy + \int_0^{\tau} \frac{\bar{\sigma}^2(s)}{2} \sum_{k=1}^{N_{\Delta t}} \alpha_k \varphi_k(s) G(U,s,x,\tau) ds$$

for each $x \in \Omega = (-\infty, U)$, $\tau \in (0,T]$

approximation

$$\forall S \in (0, S_u),\ \forall t \in [t_0, T) \qquad V(S,t) = u(\log(S), T-t) e^{-\int_t^T r(t')dt'}$$

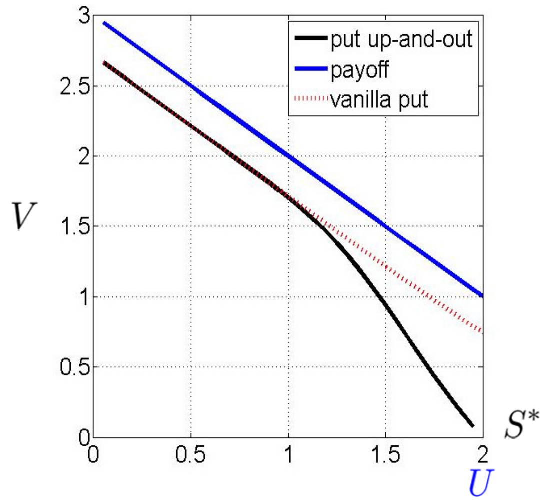
Numerical Example: test with constant parameters

$\sigma = 0.25$ constant volatility
$r = 0.1$ interest rate
$\delta = 0$ dividend yield
$T = 1$ maturity

$E = 3$ exercise price
$e^{x^*} = S^* = [0 : 0.05 : 2]$
current underlying asset values

$U = 2$ upper barrier $< E$

$V(S^*, 0)$



closed-form solution **[J.C. Hull, 2011]**

$$\begin{cases} \text{if } U \le E\,, & V(x,t) = P(x,t) + e^{x-\delta(T-t)}(U/e^x)^{2\lambda}\mathcal{N}[-y_1] \\ & \quad -Ee^{-r(T-t)}(U/e^x)^{2\lambda-2}\mathcal{N}[-y_1+\sigma\sqrt{(T-t)}]\;; \\ \text{if } U \ge E\,, & V(x,t) = P + e^{x-\delta(T-t)}(U/e^x)^{2\lambda}\mathcal{N}[-y] \\ & \quad -Ee^{-r(T-t)}(U/e^x)^{2\lambda-2}\mathcal{N}[-y+\sigma\sqrt{(T-t)}]\;; \end{cases}$$

$$\lambda = \frac{r-\delta+\sigma^2/2}{\sigma^2}\,; \quad y_1 = \frac{\log(U/e^x)}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t}\,; \quad y = \frac{\log\left(U^2/(Ee^x)\right)}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t}\,;$$

$P(x,t)$ is the value of the European put option without barriers

Numerical Example: test with constant parameters

SABO

	Max Abs Err	*Max Rel Err*	*CPU time*
0.1	$7.4\ 10^{-4}$	$9.6\ 10^{-3}$	$7.8\ 10^{-1}$ s
0.05	$2.0\ 10^{-4}$	$2.6\ 10^{-3}$	$1.4\ 10^{+0}$ s
0.025	$5.2\ 10^{-5}$	$6.8\ 10^{-4}$	$2.5\ 10^{+0}$ s
0.0125	$1.5\ 10^{-5}$	$1.9\ 10^{-4}$	$4.9\ 10^{+0}$ s
0.00625	$5.3\ 10^{-6}$	$6.4\ 10^{-5}$	$9.7\ 10^{+0}$ s

FINITE DIFFERENCES

$\Delta t = \Delta x^2$ (implicit in time and centered in space)

	Max Abs Err	*Max Rel Err*	CPU *time*
0.0125	$8.1\ 10^{-4}$	$2.0\ 10^{-3}$	$2.4\ 10^{+0}$ s
0.00625	$2.0\ 10^{-4}$	$4.9\ 10^{-4}$	$6.1\ 10^{+1}$ s

MONTE CARLO

$M = 50\,000$ is the initial sampling

$N_{\Delta t} = 100$ is the number of initial time interval decomposition

$(M, N_{\Delta t}) \cdot k$	*Max Abs Err*	*Max Rel Err*	CPU *time*
k=1	$5.0\ 10^{-2}$	$5.7\ 10^{-1}$	$5.1\ 10^{+0}$ s
k=2	$3.4\ 10^{-2}$	$4.4\ 10^{-1}$	$2.7\ 10^{+1}$ s
k=3	$2.7\ 10^{-2}$	$3.2\ 10^{-1}$	$7.2\ 10^{+1}$ s

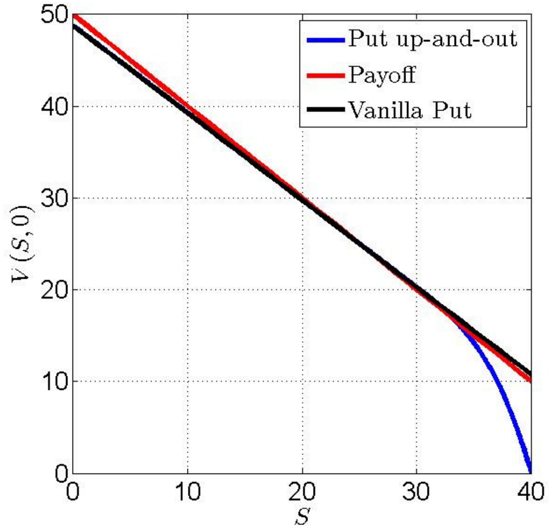
[L.V. Ballestra – G. Pacelli, 2014]

[C. Guardasoni - S. Sanfelici, *A boundary element approach to barrier option pricing in Black–Scholes framework*, International Journal of Computer Mathematics, 2016]

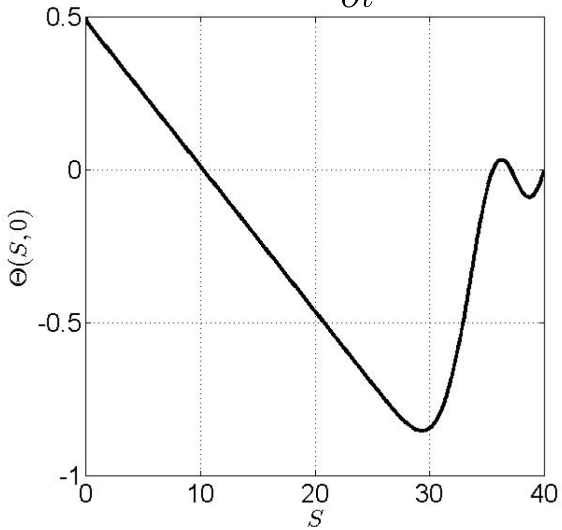
Numerical Example: time varying interest rate

[Guardasoni. Semi-Analytical method for the pricing of barrier options in case of time-dependent parameters (with Matlab codes), CAIM, 2018]

B	T	S	E	$r(t)$	d	σ
40	1	35	50	$\begin{cases} r_1 = 0.01 & t < 0.25 \\ r_2 = 0.03 & 0.25 \le t \le T \end{cases}$	0.05	0.105



$$\Theta(S,t) := \frac{\partial V}{\partial t}(S,t)$$



n	$V_{SABO}(35,0)$	$\Theta_{SABO}(35,0)$	CPU time	n	$V_{FD}(35,0)$	$\Theta_{FD}(35,0)$	CPU time
4	14.13354	-0.06753	$1.7 \cdot 10^{+0}$	0	14.24680	-0.03076	$3.1 \cdot 10^{-2}$
8	14.13236	-0.07581	$3.1 \cdot 10^{+0}$	1	14.16124	-0.06392	$5.0 \cdot 10^{-1}$
16	14.13188	-0.07724	$7.1 \cdot 10^{+0}$	2	14.13902	-0.07417	$1.3 \cdot 10^{+1}$
32	14.13168	-0.07760	$1.9 \cdot 10^{+1}$	3	14.13341	-0.07686	$2.0 \cdot 10^{+2}$
64	14.13160	-0.07771	$6.0 \cdot 10^{+1}$	4	14.13201	-0.07754	$2.8 \cdot 10^{+3}$

Results obtained by SABO with $\Delta t = T/2^n$ (on the left)
and by FD with $\Delta t = \Delta x^2$ and $\Delta x = 0.25/2^n$ (on the right).

Hedging

This numerical strategy is very useful and efficient for **hedging** that needs computing **Greeks**

- $\Delta := \dfrac{\partial V}{\partial S}$
- $\Gamma := \dfrac{\partial^2 V}{\partial S^2}$
- $\Theta := \dfrac{\partial V}{\partial t}$
- $\rho := \dfrac{\partial V}{\partial r}$
- $\text{Vega} := \dfrac{\partial V}{\partial \sigma}$

because it is sufficient to evaluate the derivative of the **Representation Formula**

$$V(S,t) \approx \int_0^U u(s,T)G(s,T,S,t)ds + \int_t^T \varphi_U(\tau)G(U,\tau,S,t)d\tau$$

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without computing the primary unknown V

The mathematical model problem 2.: Heston model

$V(x, v, t)$ option price: V depends also on v (the square of volatility)

$x \in \Omega_x = (-\infty, +\infty)$, $v \in \Omega_v = (0, +\infty)$, $t \in [0, T)$

- **Stochastic differential equations:** **[S.L. Heston (1993)]**

$$\begin{aligned} dx_t &= \left(r - d - \tfrac{1}{2} v_t dt + \sqrt{v_t} dW_t^1\right) \\ dv_t &= -\lambda(v_t - \bar{v})dt + \eta\sqrt{v_t}dW_t^2 \end{aligned}$$

W_t^1, W_t^2 are correlated Brownian motions with instantaneous correlation ρ

$x_t = \log(S_t)$
$v_t =$ asset return variance
$r =$ constant risk free interest rate
$d =$ dividend yield

$\eta =$ volatility of volatility
$\lambda =$ speed of mean reversion
$\bar{v} =$ mean level of variance
$\theta =$ market price of volatility risk

The Feller condition, $2\lambda\bar{v} \geq \eta^2$, guarantees that v_t stays positive; otherwise, it may reach zero.

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$$\frac{\partial V}{\partial t} + \frac{1}{2} v \frac{\partial^2 V}{\partial x^2} + \rho \eta v \frac{\partial^2 V}{\partial x \partial v} + \frac{1}{2} \eta^2 v \frac{\partial^2 V}{\partial v^2} + \left(r - d - \frac{1}{2} v \right) \frac{\partial V}{\partial x} - \big(\lambda(v - \bar{v}) - \theta v\big) \frac{\partial V}{\partial v} - rV = 0$$

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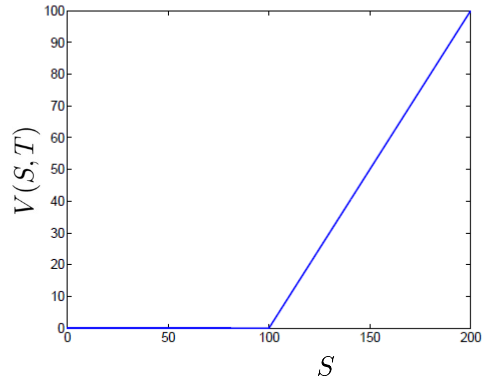
$$V(x,v,T)=\max(e^x-E,0) \qquad x \in (-\infty,+\infty) \quad v \in (0,+\infty)$$



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closed-form solution **[S.L. Heston (1993)]** *to be numerically evaluated* **[P.Carr-D.B.Madan (1999)]**

$$V(x,v,t) = e^{-r(T-t)} \int_{\Omega_x} \int_{\Omega_v} V(y,w,T) G(y,w,T;x,v,t) dw\, dy$$

$G(y,w,\tau;x,v,t)$ is the joint transition probability density (or fundamental solution) that expresses the probability to move from (x,v) at time t to (y,w) at time τ

The mathematical model problem 2.: Heston model

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$V(y,w,T)$ is the payoff

$G(y,w,\tau;x,v,t)$ is the joint transition probability density (or fundamental solution) that expresses the probability to move from (x,v) at time t to (y,w) at time τ

$$G(y,w,\tau;x,v,t) = p_{t\to\tau}(x\to y, v\to w) = p_{t\to\tau}(y-x, w|v) = p_{t\to\tau}(y-x|w,v)\widetilde{p}_{t\to\tau}(v,w)$$

- $\widetilde{p}_{t\to\tau}(v,w)$ is the transition density of the variance

$$\widetilde{p}_{t\to\tau}(v,w) = \gamma e^{-\gamma\left(ve^{-\lambda(\tau-t)}+w\right)} \left(\frac{w}{ve^{-\lambda(\tau-t)}}\right)^{\frac{\alpha-1}{2}} I_{\alpha-1}\left(2\sqrt{\gamma^2 vwe^{-\lambda(\tau-t)}}\right)$$

$$\gamma = \frac{2\lambda}{\left(1-e^{-\lambda(\tau-t)}\right)\eta^2} \qquad \alpha = \frac{2\lambda\bar{v}}{\eta^2};$$

- with an inverse Fourier transform: $p_{t\to\tau}(y-x|w,v) = \mathcal{F}_\omega^{-1}[\widehat{p}](y-x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} \widehat{p}(\omega,v,w,t,\tau) e^{-\mathbf{i}\omega(y-x)} d\omega$

$$\widehat{p}(\omega,v,w,t,\tau) = e^{\mathbf{i}\omega\left\{(r-d)(\tau-t)+\frac{\rho}{\eta}\left(w-v-\lambda\bar{v}(\tau-t)\right)\right\}} \phi\left[\omega\left(\frac{\lambda\rho}{\eta}-\frac{1}{2}\right)+\frac{1}{2}\mathbf{i}\omega^2(1-\rho^2)\right]$$

$\phi[\cdot] = \ldots$ is the characteristic function of the integrated variance $\displaystyle\int_t^\tau v(s)ds$ given v_t and v_τ

[C. Guardasoni, S. Sanfelici, SIAM journal on Applied Mathematics 2016]

The mathematical model problem 2.: Heston model

European DOWN-and-OUT CALL option differential model problem

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  $V(x,v,T) = \max(e^x - E, 0)$ $\quad x \in (L, +\infty) \quad v \in (0, +\infty)$

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  on the **asset lower barrier**

  $V(L,v,t) = 0$ $\quad t \in [0, T) \quad v \in (0, +\infty)$

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RF

$$V(x,v,t) \;=\; e^{-r(T-t)}\int_{-\infty}^{+\infty}\int_{\Omega_v} V(y,w,T)G(y,w,T;x,v,t)\,dw\,dy$$

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$$\begin{aligned} V(x, v, t) &= e^{-r(T-t)} \left\{ \int_L^{+\infty} \int_{\Omega_v} V(y, w, T) G(y, w, T; x, v, t) \, dw \, dy \right. \\ &\quad \left. - \int_t^T \int_{\Omega_v} \frac{\partial V}{\partial y}(L, w, \tau) \frac{w}{2} G(L, w, \tau; x, v, t) \, dw \, d\tau \right\} \end{aligned}$$

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Boundary Integral Equation

analytical **INTEGRAL REPRESENTATION FORMULA**

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RF

$$
\begin{aligned}
V(x,v,t) &= e^{-r(T-t)}\left\{\int_L^{+\infty}\int_{\Omega_v} V(y,w,T)G(y,w,T;x,v,t)\,dw\,dy\right.\\
&- \left.\int_t^T\int_{\Omega_v}\frac{\partial V}{\partial y}(L,w,\tau)\frac{w}{2}G(L,w,\tau;x,v,t)\,dw\,d\tau\right\}
\end{aligned}
$$

unknown density

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unknown density

but on the boundary, letting $x \to L$ **BOUNDARY INTEGRAL EQUATION**

for each $v \in \Omega_v = (0, +\infty),\ t \in [0, T)$

BIE

$$\begin{aligned} 0 = V(L,v,t) &= e^{-r(T-t)} \left\{ \int_L^{+\infty} \int_{\Omega_v} V(y,w,T) G(y,w,T) G(y,w,T;L,v,t)\, dw\, dy \right. \\ &\quad \left. - \int_t^T \int_{\Omega_v} \frac{\partial V}{\partial y}(L,w,\tau) \frac{w}{2} G(L,w,\tau;L,v,t)\, dw\, d\tau \right\} \end{aligned}$$

solve the equation... *numerically*

Boundary Integral Equation

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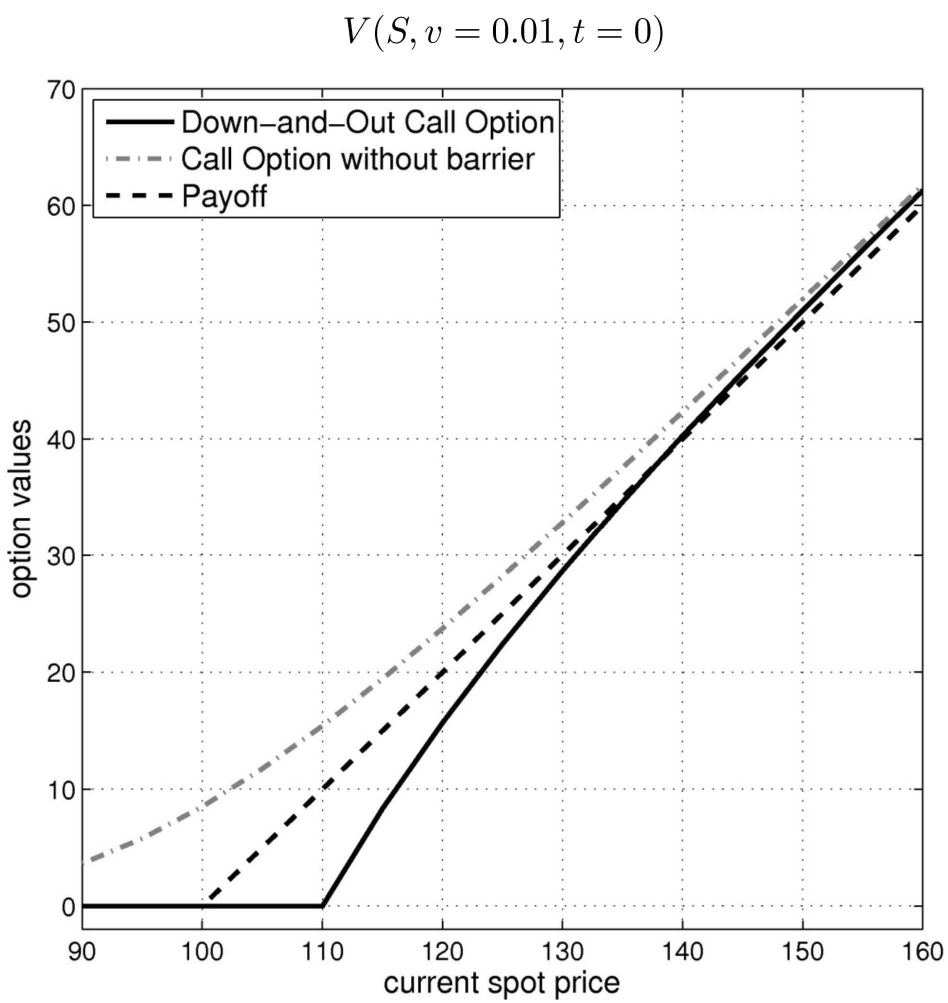
solve the equation… *numerically*

Numerical example

[C. Guardasoni, S. Sanfelici. Fast numerical pricing of barrier options under stochastic volatility and jumps, SIAM Journal on Applied Mathematics, 2016]

speed of mean reversion	$\lambda = 4\,;$
long run mean level of variance	$\overline{v} = 0.04\,;$
correlation	$\rho = -0.5\,;$
volatility of volatility	$\eta = 0.1\,;$
free risk interest rate	$r = 0.05\,;$
dividend yield	$\delta = 0.02\,;$
strike price	$E = 100\,;$
down barrier value	$L = 110\,;$



• The mathematical model problem 3.: Asian options

An **Asian option** is an option whose payoff depends the average of the stock price S_t over a time interval

arithmetic Asian option

average A_t/t

$$A_t := \int_0^t S_t dt$$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial A} - rV = 0$$

[A.Aimi, C.Guardasoni, L.A.Terranova, in preparation]

[A.Aimi, L.Diazzi, C.Guardasoni, Mathematical Methods in the Applied Sciences 2018]

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[A.Aimi, C.Guardasoni, L.A.Terranova, in preparation]

geometric Asian option

average $\exp(A_t/t)$

$$A_t := \int_0^t \log(S_t) dt$$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + \log(S) \frac{\partial V}{\partial A} - rV = 0$$

geometric Asian fixed strike call up-and-out option
whose price extinguishes when the underlying asset breaches a pre-set **upper barrier level**

[A.Aimi, L.Diazzi, C.Guardasoni, Mathematical Methods in the Applied Sciences 2018]

• The mathematical model problem 3.: Asian options

If the stochastic process S_t is modeled by the usual geometric Brownian motion, A_t is a geometric Brownian process too

$S =$ underlying asset value
$r =$ interest rate
$\sigma =$ volatility
$T =$ expiry
$E =$ exercise price
$B =$ **UPPER BARRIER**

geometric Asian option differential model problem

- $V(S, A, t)$ option price $\quad S \in (0, B), A \in \mathbb{R}, t \in [0, T)$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + \log(S) \frac{\partial V}{\partial A} - rV = 0$$

- with **final condition** (payoff) $\quad S \in (0, B),\ A \in \mathbb{R}$

- with **boundary conditions** deduced from payoff and put-call parity
  - on the asset $\quad A \in \mathbb{R},\ t \in [0, T)$
  - on A - variable $\quad S \in (0, B),\ t \in [0, T)$

[A.Aimi, L.Diazzi, C.Guardasoni, Mathematical Methods in the Applied Sciences 2018]

• The mathematical model problem 3.: Asian options

If the stochastic process S_t is modeled by the usual geometric Brownian motion, A_t is a geometric Brownian process too

S = underlying asset value
r = interest rate
σ = volatility
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$V(S, A, T) = \max\left(\exp\left(A/T\right) - E, 0\right)$ fixed strike call
$V(S, A, T) = \max\left(E - \exp\left(A/T\right), 0\right)$ fixed strike put
$V(S, A, T) = \max\left(S - \exp\left(A/T\right), 0\right)$ floating strike call
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fixed strike call

- with **boundary conditions** deduced from payoff and put-call parity
  - on the asset

$$\begin{aligned} \lim_{S \to 0} V(S,A,t) &= e^{-r(T-t)} \lim_{S \to 0} V(S,A,T) \\ V(B,A,t) &= 0 \end{aligned}$$

  - on A - variable

$$\begin{aligned} \lim_{A \to -\infty} V(S,A,t) &= 0 \\ \lim_{A \to +\infty} V(S,A,t) &= S^{\frac{T-t}{T}} \exp\left(\frac{A}{T} + \left[\left(r - q - \frac{\sigma^2}{2}\right)\frac{T-t}{2T} + \frac{\sigma^2}{6}\frac{(T-t)^2}{T^2} - r\right](T-t)\right) - Ee^{-r(T-t)} \end{aligned}$$

[J. Hugger , Wellposedness of the boundary value formulation of a fixed strike Asian option, J. Comput. Appl. Math. 2006]

[A.Aimi, L.Diazzi, C.Guardasoni, Mathematical Methods in the Applied Sciences 2018]

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CLOSED-FORM SOLUTION?

geometric Asian option differential model problem

- $V(S,A,t)$ option price $\quad S \in (0,B),\ A \in \mathbb{R},\ t \in [0,T)$

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Integral representation formula for up-out call

Transition probability density

$\widetilde{S} \in \mathbb{R}^+,\ \widetilde{A} \in \mathbb{R},\ \widetilde{t} \in (t, T)$

$$\begin{aligned} G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}) &= \frac{\sqrt{3}H[\widetilde{t}-t]}{\pi\sigma^2(\widetilde{t}-t)^2}\exp\left\{-\frac{2}{\sigma^2(\widetilde{t}-t)}\log^2\left(\frac{S}{\widetilde{S}}\right)+\frac{6}{\sigma^2(\widetilde{t}-t)^2}\log\left(\frac{S}{\widetilde{S}}\right)\left(A-\widetilde{A}+(\widetilde{t}-t)\log(S)\right)\right. \\ &\left. - \frac{6}{\sigma^2(\widetilde{t}-t)^3}\left(A-\widetilde{A}+(\widetilde{t}-t)\log(S)\right)^2-\left(\frac{2r+\sigma^2}{2\sqrt{2}\sigma}\right)^2(\widetilde{t}-t)\right\}\left(\frac{\widetilde{S}}{S}\right)^{\frac{2r-\sigma^2}{2\sigma^2}}\frac{1}{\widetilde{S}} \end{aligned}$$

Integral representation formula

$$V(S,A,t) = \int_\Omega V(\widetilde{S},\widetilde{A},T)G(S,A,t;\widetilde{S},\widetilde{A},T)\,d\widetilde{S}\,d\widetilde{A}$$

for each $S \in (0,B),\ A \in \mathbb{R},\ t \in [0,T)$

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Integral representation formula

$$V(S,A,t) = \int_\Omega V(\widetilde{S},\widetilde{A},T)G(S,A,t;\widetilde{S},\widetilde{A},T)d\widetilde{S}\,d\widetilde{A}+\int_t^T\int_{\partial\Omega}\frac{\sigma^2}{2}B^2\frac{\partial V}{\partial\widetilde{S}}(B,\widetilde{A},\widetilde{t})G(S,A,t;B,\widetilde{A},\widetilde{t})d\widetilde{A}\,d\widetilde{t}$$

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$\widetilde{S} \in \mathbb{R}^+,\ \widetilde{A} \in \mathbb{R},\ \widetilde{t} \in (t,T)$

$$
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G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}) &= \frac{\sqrt{3}H[\widetilde{t}-t]}{\pi\sigma^2(\widetilde{t}-t)^2}\exp\left\{-\frac{2}{\sigma^2(\widetilde{t}-t)}\log^2\left(\frac{S}{\widetilde{S}}\right)+\frac{6}{\sigma^2(\widetilde{t}-t)^2}\log\left(\frac{S}{\widetilde{S}}\right)(A-\widetilde{A}+(\widetilde{t}-t)\log(S))\right.\\
&\left. - \frac{6}{\sigma^2(\widetilde{t}-t)^3}(A-\widetilde{A}+(\widetilde{t}-t)\log(S))^2-\left(\frac{2r+\sigma^2}{2\sqrt{2}\sigma}\right)^2(\widetilde{t}-t)\right\}\left(\frac{\widetilde{S}}{S}\right)^{\frac{2r-\sigma^2}{2\sigma^2}}\frac{1}{\widetilde{S}}
\end{aligned}
$$

Integral representation formula

$$
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V(S,A,t) &= \int_{\Omega} V(\widetilde{S},\widetilde{A},T)G(S,A,t;\widetilde{S},\widetilde{A},T)d\widetilde{S}\,d\widetilde{A} + \int_t^T\int_{\partial\Omega}\frac{\sigma^2}{2}B^2\frac{\partial V}{\partial\widetilde{S}}(B,\widetilde{A},\widetilde{t})G(S,A,t;B,\widetilde{A},\widetilde{t})d\widetilde{A}\,d\widetilde{t}\\
&= \int_{-\infty}^{+\infty}\int_0^B V(\widetilde{S},\widetilde{A},T)G(S,A,t;\widetilde{S},\widetilde{A},T)d\widetilde{S}\,d\widetilde{A} + \int_t^T\int_{-\infty}^{+\infty}\frac{\sigma^2}{2}B^2\frac{\partial V}{\partial\widetilde{S}}(B,\widetilde{A},\widetilde{t})G(S,A,t;B,\widetilde{A},\widetilde{t})d\widetilde{A}\,d\widetilde{t}
\end{aligned}
$$

for each $\quad S \in (0,B),\ A \in \mathbb{R},\ t \in [0,T)$

Boundary Integral Equation

analytical **INTEGRAL REPRESENTATION FORMULA**

RF $$V(S,A,t) = \int_{-\infty}^{+\infty}\int_0^B V(\widetilde{S},\widetilde{A},T)G(S,A,t;\widetilde{S},\widetilde{A},T)d\widetilde{S}\,d\widetilde{A} + \int_t^T\int_{-\infty}^{+\infty}\frac{\sigma^2}{2}B^2\frac{\partial V}{\partial \widetilde{S}}(B,\widetilde{A},\widetilde{t})G(S,A,t;B,\widetilde{A},\widetilde{t})d\widetilde{A}\,d\widetilde{t}$$

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unknown density

but on the boundary, letting $S \to B$ **BOUNDARY INTEGRAL EQUATION**

BIE $$0 = V(B,A,t) = \int_{-\infty}^{+\infty}\int_0^B V(\widetilde{S},\widetilde{A},T)G(B,A,t;\widetilde{S},\widetilde{A},T)d\widetilde{S}\,d\widetilde{A} + \int_t^T\int_{-\infty}^{+\infty}\frac{\sigma^2}{2}B^2\frac{\partial V}{\partial \widetilde{S}}(B,\widetilde{A},\widetilde{t})G(B,A,t;B,\widetilde{A},\widetilde{t})d\widetilde{A}\,d\widetilde{t}$$

for each $A \in \mathbb{R}$, $t \in [0,T)$

solve the equation… *numerically*

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for each $A \in \mathbb{R},\ t \in [0,T)$

solve the equation… *numerically*

Numerical Resolution of the Boundary Integral Equation

by **COLLOCATION METHOD**:

- uniform decomposition of the time interval $[0,T]$ with time step

$$\Delta t := T/N_{\Delta t}, \qquad t_k := k\Delta t \quad k = 0,\dots,N_{\Delta t}$$

- uniform decomposition of the A - domain $[A_{\min}, A_{\max}] = [0, T\log(E)]$ with time step

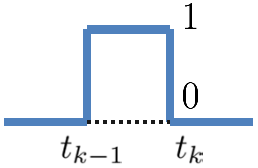
$$\Delta A := \frac{A_{\max} - A_{\min}}{N_A}, \qquad A_h := A_{\min} + h\Delta A, \quad h = 0,\dots,N_A$$

- approximation of the BIE unknown

$$\frac{\partial V}{\partial \widetilde{S}}(B,\widetilde{A},\widetilde{t}) \approx \sum_{k=1}^{N_t}\sum_{h=1}^{N_A} \alpha_h^{(k)} \psi_h(\widetilde{A})\varphi_k(\widetilde{t})$$

with $\varphi_k(\widetilde{t}) := H[\widetilde{t} - t_{k-1}] - H[\widetilde{t} - t_k],$ for $k = 1,\dots,N_t$

$\psi_h(\widetilde{A}) := H[\widetilde{A} - A_{h-1}] - H[\widetilde{A} - A_h],$ $h = 1,\dots,N_A$



- evaluation of BIE at the **collocation nodes**:

$$\overline{A}_i = \frac{A_i + A_{i-1}}{2}, \quad i = 1,\dots,N_A$$
$$\bar{t}_j = \frac{t_j + t_{j-1}}{2}, \quad j = 1,\dots,N_t$$

Numerical Resolution of the Boundary Integral Equation

BIE

$$0 = V(B,A,t) = \int_{-\infty}^{+\infty}\int_0^B V(\widetilde{S},\widetilde{A},T)G(B,A,t;\widetilde{S},\widetilde{A},T)d\widetilde{S}\,d\widetilde{A} + \int_t^T\int_{-\infty}^{+\infty}\frac{\sigma^2}{2}B^2\frac{\partial V}{\partial \widetilde{S}}(B,\widetilde{A},\widetilde{t})G(B,A,t;B,\widetilde{A},\widetilde{t})d\widetilde{A}\,d\widetilde{t}$$

↓

$$0 = V(B,\overline{A}_i,\overline{t}_j) \approx \int_{-\infty}^{+\infty}\int_0^B V(\widetilde{S},\widetilde{A},T)G(B,\overline{A}_i,\overline{t}_j;\widetilde{S},\widetilde{A},T)\,d\widetilde{S}\,d\widetilde{A} + \int_{\overline{t}_j}^T\int_{-\infty}^{+\infty}\frac{\sigma^2}{2}B^2\sum_{k=1}^{N_t}\sum_{h=1}^{N_A}\alpha_h^{(k)}\psi_h(\widetilde{A})\varphi_k(\widetilde{t})G(B,\overline{A}_i,\overline{t}_j;B,\widetilde{A},\widetilde{t})\,d\widetilde{A}\,d\widetilde{t}$$

↓

$$\frac{\sigma^2}{2}B^2\sum_{k=1}^{N_t}\sum_{h=1}^{N_A}\alpha_h^{(k)}\underbrace{\frac{\sigma^2}{2}B^2\int_{\max(t_{k-1},\overline{t}_j)}^{t_k}\int_{A_{h-1}}^{A_h}G(B,\overline{A}_i,\overline{t}_j;B,\widetilde{A},\widetilde{t})\,d\widetilde{A}\,d\widetilde{t}}_{\mathcal{A}_{ih}^{(jk)}} = \underbrace{-\int_{-\infty}^{+\infty}\int_0^B V(\widetilde{S},\widetilde{A},T)G(B,\overline{A}_i,\overline{t}_j;\widetilde{S},\widetilde{A},T)\,d\widetilde{S}\,d\widetilde{A}}_{\mathcal{F}_i^{(j)}}$$

Numerical Resolution of the Boundary Integral Equation

$$\frac{\sigma^2}{2}B^2\sum_{k=1}^{N_t}\sum_{h=1}^{N_A}\alpha_h^{(k)}\underbrace{\frac{\sigma^2}{2}B^2\int_{\max(t_{k-1},\bar{t}_j)}^{t_k}\int_{A_{h-1}}^{A_h}G(B,\overline{A}_i,\bar{t}_j;B,\widetilde{A},\widetilde{t})\,d\widetilde{A}\,d\widetilde{t}}_{\mathcal{A}_{ih}^{(jk)}}=\underbrace{-\int_{-\infty}^{+\infty}\int_0^B V(\widetilde{S},\widetilde{A},T)G(B,\overline{A}_i,\bar{t}_j;\widetilde{S},\widetilde{A},T)\,d\widetilde{S}\,d\widetilde{A}}_{\mathcal{F}_i^{(j)}}$$

$$\mathcal{A}=\begin{bmatrix} A^{(0)} & A^{(1)} & A^{(2)} & \cdots & A^{(N_t-1)} \\ 0 & A^{(0)} & A^{(1)} & \cdots & A^{(N_t-2)} \\ 0 & 0 & A^{(0)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & A^{(1)} \\ 0 & 0 & \cdots & 0 & A^{(0)} \end{bmatrix}$$

as the Green's function is defined only for $\widetilde{t}>t$ and depends on $\widetilde{t}-t$

$$\mathcal{A}^{(\ell)}=\begin{bmatrix} A_0^{(\ell)} & A_{-1}^{(\ell)} & A_{-2}^{(\ell)} & \cdots & A_{-N_A+1}^{(\ell)} \\ A_1^{(\ell)} & A_0^{(\ell)} & A_{-1}^{(\ell)} & \cdots & A_{-N_A+2}^{(\ell)} \\ A_2^{(\ell)} & A_1^{(\ell)} & A_0^{(\ell)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & A_{-1}^{(\ell)} \\ A_{N_A-1}^{(\ell)} & A_{N_A-2}^{(\ell)} & \cdots & A_1^{(\ell)} & A_0^{(\ell)} \end{bmatrix} \qquad \ell=0,\ldots,N_t-1$$

as the Green's function depends on $A-\widetilde{A}$

$$\boxed{\mathcal{A}\alpha=\mathcal{F}} \longrightarrow \alpha$$

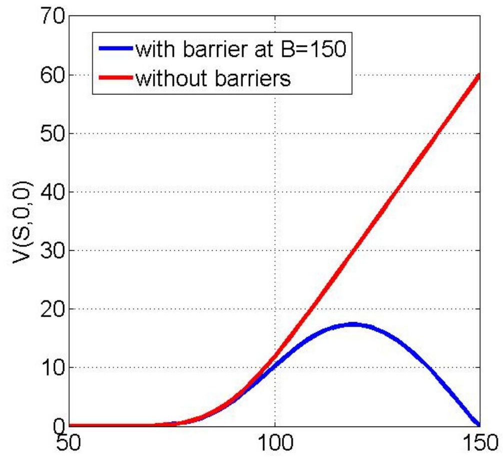
Numerical example

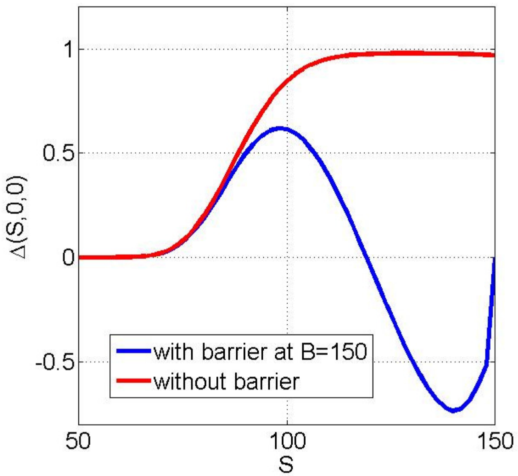
B	T	E	r	σ
150	1	90	0.035	0.2

$[A_{\min} A_{\max}] = [0,5]$

[A.Aimi, C.Guardasoni. Collocation Boundary Element Method for the pricing of Geometric Asian Options, EABE, 2018]

[A.Aimi, L.Diazzi, C.Guardasoni. Numerical Pricing of Geometric Asian Options with Barriers, Mathematical Methods in the Applied Sciences, 2018]





$N_t = N_A$	$S=100$	$S=120$	$S=140$	elapsed time (sec)
10	10.2170	17.3650	8.0877	$3.0 \cdot 10^0$
20	10.1480	17.2561	7.9929	$1.1 \cdot 10^1$
40	10.1419	17.2960	8.1400	$4.3 \cdot 10^1$
80	10.1432	17.3061	8.1507	$1.7 \cdot 10^2$
160	10.1438	17.3086	8.1551	$6.9 \cdot 10^2$
320	10.1439	17.3094	8.1566	$3.0 \cdot 10^3$
640	10.1440	17.3096	8.1570	$1.3 \cdot 10^4$

$V(S,0,0)$ evaluated by SABO at $S = 100, 120, 140$.

• The mathematical model problem 4.: Basket options

[R. Seydel, Tools for Computational Finance 2006]

Two-assets option differential model problem

- $V(S_1, S_2, t)$ option price $\qquad (S_1, S_2) \in \Omega,\ t \in [0, T)$

$$-\frac{\partial V}{\partial t} + \frac{\sigma_1^2 S_1^2}{2}\frac{\partial^2 V}{\partial S_1^2} + rS_1\frac{\partial V}{\partial S_1} + \frac{\sigma_2^2 S_2^2}{2}\frac{\partial^2 V}{\partial S_2^2} + rS_2\frac{\partial V}{\partial S_2} + \rho\sigma_1\sigma_2 S_1 S_2\frac{\partial^2 V}{\partial S_1 \partial S_2} - rV = 0$$

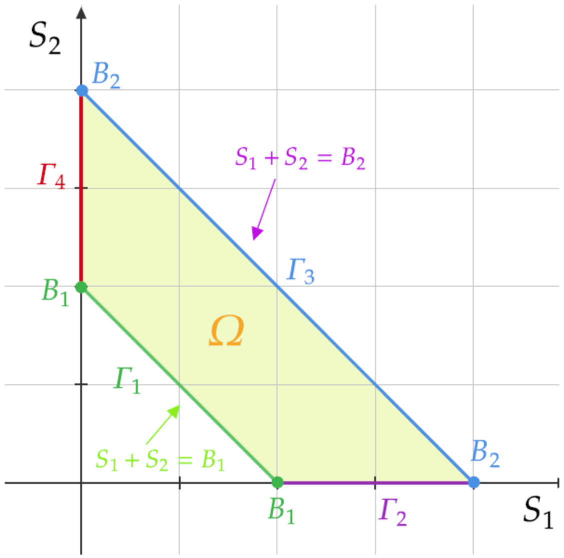
$S_1, S_2 =$ underlying assets value
$r =$ interest rate
$\sigma_1, \sigma_2 =$ volatilities
ρ assets correlation
$T =$ expiry
$K =$ exercise price
$B_1 =$ **DOWN-OUT barrier**
$B_2 =$ **UP-OUT barrier**

- with **Basket call final condition** (payoff) $\qquad (S_1, S_2) \in \Omega,\ t \in [0, T)$

$$V(S_1, S_2, T) = \max(S_1 + S_2 - K, 0)$$

- with **double knock-out boundary conditions**

$$\begin{aligned} V(S_1, S_2, T) &= 0 \qquad S_1 + S_2 \leq B_1 \\ V(S_1, S_2, T) &= 0 \qquad S_1 + S_2 \geq B_2 \end{aligned}$$



• The mathematical model problem 4.: Basket options

[R. Seydel, Tools for Computational Finance 2006]

Two-assets option differential model problem

- $V(S_1, S_2, t)$ option price $\quad (S_1, S_2) \in \Omega,\ t \in [0, T)$

$$-\frac{\partial V}{\partial t} + \frac{\sigma_1^2 S_1^2}{2}\frac{\partial^2 V}{\partial S_1^2} + rS_1\frac{\partial V}{\partial S_1} + \frac{\sigma_2^2 S_2^2}{2}\frac{\partial^2 V}{\partial S_2^2} + rS_2\frac{\partial V}{\partial S_2} + \rho\sigma_1\sigma_2 S_1 S_2\frac{\partial^2 V}{\partial S_1 \partial S_2} - rV = 0$$

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$B_1=$ **DOWN-OUT barrier**
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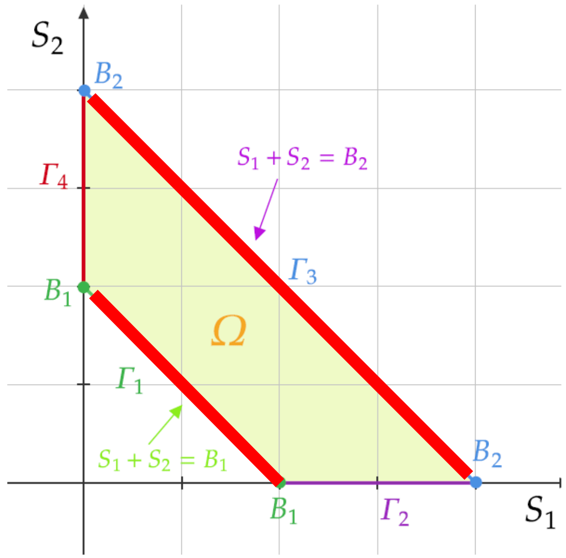


• The mathematical model problem 4.: Basket options

Integral Representation Formula

$V(S_1, S_2, t)$ option price $\qquad (S_1, S_2) \in \Omega,\ t \in [0, T)$

$$\begin{aligned} V(S_1,S_2,t) &= \int_\Omega V(\tilde S_1,\tilde S_2,T)G(S_1,S_2,t;\tilde S_1,\tilde S_2,T)d\tilde S_1 d\tilde S_2 \\ &+ \int_t^T \int_{\Gamma_1\cup\Gamma_3} G(S_1,S_2,t;\tilde S_1,\tilde S_2,\tilde t)\phi(\tilde S_1,\tilde S_2,\tilde t)\ d\tilde t d\tilde S_1 d\tilde S_2 \end{aligned}$$

with

$$\begin{aligned} \phi(\tilde S_1,\tilde S_2,\tilde t) &= \left(\frac{1}{2}\sigma_1^2\tilde S_1^2\frac{\partial V}{\partial \tilde S_1} + \frac{1}{2}\rho\sigma_1\sigma_2\tilde S_1\tilde S_2\frac{\partial V}{\partial \tilde S_2}\right)n_1 \\ &+ \left(\frac{1}{2}\sigma_2^2\tilde S_2^2\frac{\partial V}{\partial \tilde S_2} + \frac{1}{2}\rho\sigma_1\sigma_2\tilde S_1\tilde S_2\frac{\partial V}{\partial \tilde S_1}\right)n_2 \end{aligned}$$

and

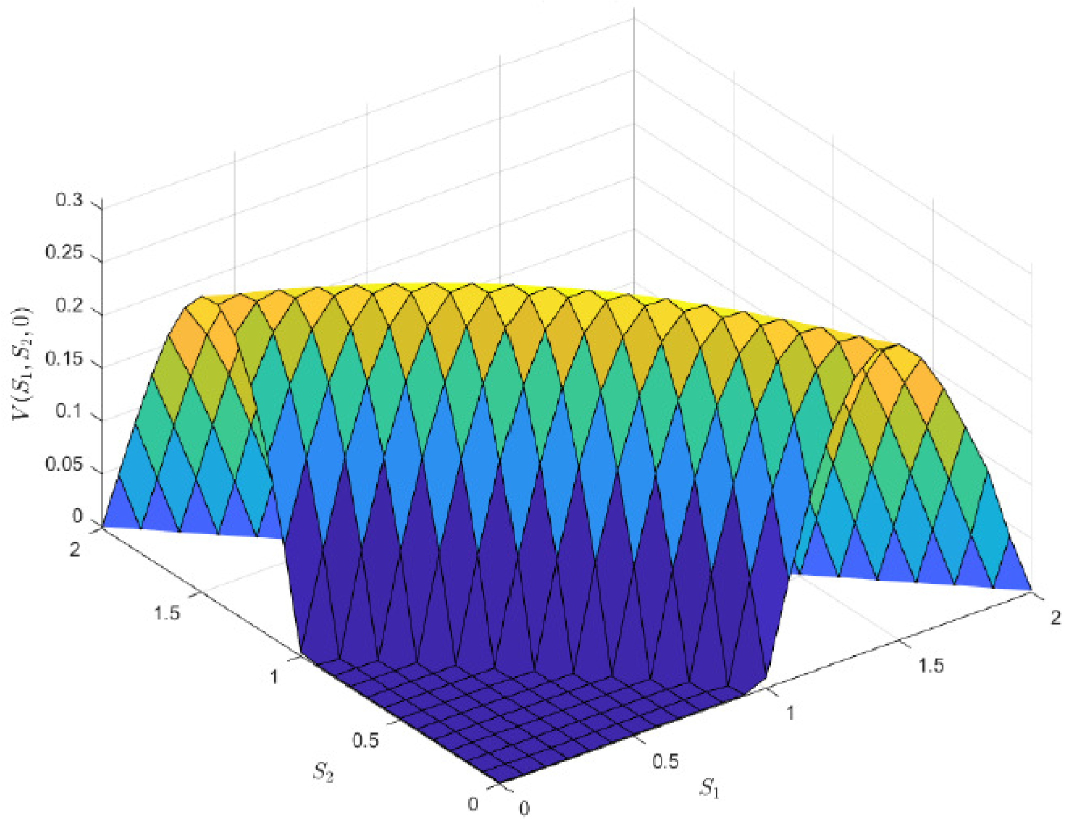
$$G(S_1,S_2,t;\tilde S_1,\tilde S_2,\tilde t) = \frac{e^{-r(\tilde t - t)}}{2\pi(\tilde t - t)}\frac{\exp\left(-\frac{\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}}{2}\right)}{\sigma_1\sigma_2\tilde S_1\tilde S_2\sqrt{\det\boldsymbol{\Sigma}}}$$

$$\alpha_i = \frac{\log\frac{S_i}{\tilde S_i} + \left(r - \frac{\sigma_i^2}{2}\right)(\tilde t - t)}{\sigma_i\sqrt{\tilde t - t}}$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{pmatrix}$$

Numerical example

K	r	σ_1	σ_2	ρ	T	B_1	B_2
1	0.05	0.25	0.25	0.7	1	1	2



Thanks to S. Dallospedale, G. Geroldi, M.A. Leoni, master students in Mathematics

Observations

Advantages of SABO:

- implicit satisfaction of asset infinity boundary conditions
- reduction of the discretization domain of one dimension
- high precision and stability
- direct evaluation of derivated functions (greeks)

Costs of SABO are due to:

- discretization in time and of domain boundary
- numerical quadrature for linear system entries

Needs of SABO:

- **Green fundamental solution in a closed or approximated form**

Thank you for the attention

Thank you for CRM invitation

A. Aimi, G. Di Credico, **C. Guardasoni**, S. Sanfelici

University of Parma, Italy

Numerical Example

[C. Guardasoni, S. Sanfelici. Fast numerical pricing of barrier options under stochastic volatility and jumps, SIAM Journal on Applied Mathematics, 2016]

Option value by SABO with $N_{\Delta t}$ and $N_{\Delta v}$ intervals

S =115	$N_{\Delta v}$ =3	$N_{\Delta v}$ =6	$N_{\Delta v}$ =9	$N_{\Delta v}$ =12	$N_{\Delta v}$ =15
$N_{\Delta t}$ =3	8.3170E+00	8.3167E+00	8.3127E+00	8.3113E+00	8.3107E+00
$N_{\Delta t}$ =6	8.3356E+00	8.3244E+00	8.3212E+00	8.3200E+00	8.3195E+00
$N_{\Delta t}$ =9	8.3354E+00	8.3259E+00	8.3227E+00	8.3215E+00	8.3210E+00
$N_{\Delta t}$ =12	8.3353E+00	8.3265E+00	8.3232E+00	8.3220E+00	8.3215E+00
$N_{\Delta t}$ =15	8.3356E+00	8.3267E+00	8.3235E+00	8.3223E+00	8.3218E+00

$N_{\Delta t} = N_{\Delta v}$	times
3	1.5E+02 s.
6	7.5E+02 s.
9	3.4E+03 s.
12	3.7E+03 s.
15	6.2E+03 s.

Computation times

$$V(S = 115, v = 0.01, t = 0)$$

Option value by Monte Carlo method with N samples and M time steps

M	$N = 10^4$	95% conf. int.	$N = 10^6$	95% conf. int.	$N = 10^8$	95% conf. int.
100	8.3410E+00	[8.01,8.67]	8.3198E+00	[8.29,8.35]	8.3353E+00	[8.33,8.34]
200	8.1367E+00	[7.81,8.47]	8.3356E+00	[8.30,8.37]	8.3291E+00	[8.33,8.33]
400	8.2543E+00	[7.92,8.59]	8.3295E+00	[8.30,8.36]	8.3254E+00	[8.32,8.33]
800	8.3002E+00	[7.96,8.64]	8.3256E+00	[8.29,8.36]	8.3261E+00	[8.32,8.33]
1600	8.1777E+00	[7.85,8.51]	8.3229E+00	[8.29,8.36]	8.3231E+00	[8.32,8.33]

M	$N = 10^4$	$N = 10^6$	$N = 10^8$
100	4.8E-01 s.	4.4E+01 s.	4.5E+03 s.
200	7.6E-01 s.	6.5E+01 s.	5.1E+03 s.
400	1.2E+00 s.	1.1E+02 s.	1.7E+04 s.
800	2.0E+00 s.	1.9E+02 s.	2.1E+04 s.
1600	3.6E+00 s.	3.5E+02 s.	4.2E+04 s.

Computation times

The mathematical model problem 2.: Heston model

European CALL option differential model problem

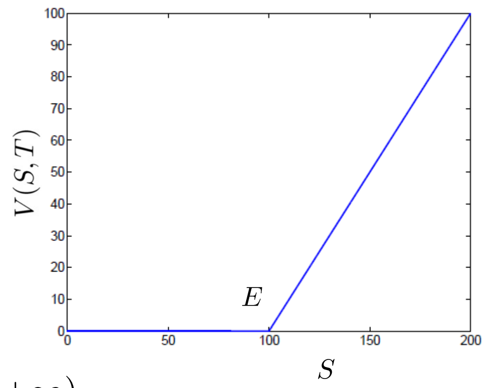
$V(x,v,t)$ option price: V depends also on v (the square of volatility)

$x \in \Omega_x = (-\infty,+\infty)$, $v \in \Omega_v = (0,+\infty)$, $t \in [0,T)$

- $$\frac{\partial V}{\partial t} + \frac{1}{2} v \frac{\partial^2 V}{\partial x^2} + \rho \eta v \frac{\partial^2 V}{\partial x \partial v} + \frac{1}{2}\eta^2 v \frac{\partial^2 V}{\partial v^2} + \left(r - d - \frac{1}{2} v\right)\frac{\partial V}{\partial x} - (\lambda(v - \bar{v}) - \theta v)\frac{\partial V}{\partial v} - rV = 0$$

- with **final condition** (payoff)

$$V(x,v,T) = \max(e^x - E, 0) \qquad x \in (-\infty,+\infty) \quad v \in (0,+\infty)$$



- with **boundary conditions** **[E. Miglio-C. Sgarra (2011)]**

on the asset

$$\lim_{x \to -\infty} V(x,v,t) = 0 \qquad \lim_{x \to +\infty} V(x,v,t) \simeq e^{x-dt} \qquad t \in [0,T) \quad v \in (0,+\infty)$$

on the variance

$$\lim_{v \to +\infty} S(x,v,t) = e^x \qquad S(x,0,t) = \sum_{n=0}^{+\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} S_{BS}(t, e^x, B, \bar{\sigma}_n, \bar{r}_n) \qquad x \in (-\infty,+\infty) \quad t \in [0,T)$$

$S_{BS}(t, e^x, B, \bar{\sigma}_n, \bar{r}_n)$ Black-Scholes value with

variance $\bar{\sigma}_n^2 = \frac{n\sigma^2}{t}$ and rate $\bar{r}_n = r - \delta + \lambda(1 - e^{\mu+\sigma^2/2}) + n\frac{\mu + \sigma^2/2}{t}$

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