Summer School on
QUALITATIVE THEORY OF
PIECEWISE ORDINARY DIFFERENTIAL EQUATIONS

Mini-course on INTEGRABILITY AND LIMIT CYCLES IN PIECEWISE DIFFERENTIAL SYSTEMS

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Abstract. Nonlinear ordinary differential equations appear in many branches of applied mathematics, physics and, in general, in applied sciences. For a differential system or a vector field defined on the plane the existence of a first integral determines completely its phase portrait; and in higher dimensions allow to reduce the dimension of the space in as many dimensions as independent first integrals we have. Hence to know first integrals it is important, but a natural question arises: *Given a vector field how to recognize if this vector field has a first integral*?

The objective of this mini-course is double. First we shall study how to compute first integrals for polynomial vector fields using the so called Darboux theory of integrability. And second we shall show how to use the existence of first integrals for computing limit cycles in piecewise differential systems.

Part 1: Darboux theory of integrability

For a two dimensional differential system the existence of a first integral completely determines its phase portrait. The simplest planar differential systems having a first integral are the Hamiltonian ones.

A *Hamiltonian differential system* or simple a *Hamiltonian system* is a differential system of the form

$$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial x},$$

where H is a C^2 function.

The integrable planar differential systems which are not Hamiltonian are, in general, very difficult to detect. In this first part we study the existence of first integrals for planar polynomial vector fields through the Darbouxian theory of integrability. This kind of integrability provides a link between the integrability of polynomial vector fields and the number of invariant algebraic curves that they have.

1. Introduction

By definition a two dimensional *planar polynomial differential system* or simply a *polynomial system* is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \tag{1}$$

where the dependent variables x and y, and the independent one (the time) t are real, and P and Q are polynomials in the variables x and y with real coefficients. Throughout these notes $m = \max\{\deg P, \deg Q\}$ denotes the *degree* of the polynomial system, and we always assume that the polynomials P and Q are relatively prime in the ring of real polynomials in the variables x and y.

We want to show the fascinating relationships between integrability (a topological phenomenon) and the existence of exact algebraic solutions for a polynomial differential system.

2. First integrals

The vector field X associated to the differential system (1) is defined by

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}.$$

The polynomial differential system (1) is *integrable* on an open subset U of $\mathbb{R}^2$ if there exists a non–constant analytic function $H : U \to \mathbb{R}$, called a *first integral* of the system on U, which is constant on all solution curves $(x(t), y(t))$ of system (1) contained in U; i.e. $H(x(t), y(t)) =$ constant for all values of t for which the solution $(x(t), y(t))$ is defined and contained in U. Clearly H is a first integral of system (1) on U if and only if

$$\frac{dH}{dt} = H_x\dot{x} + H_y\dot{y} = H_xP + H_yQ = XH = 0 \quad \text{on } U.$$

In U the curves $H(x, y)$ =constant are formed by the orbits or trajectories of the differential system (1).

Example 1. For the Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial y} = P, \qquad \dot{y} = -\frac{\partial H}{\partial x} = Q,$$

the function H called the *Hamiltonian* of the system is a first integral. Indeed

$$XH == P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} = H_yH_x - H_xH_y = 0.$$

3. Integrating factors

Let U be an open subset of $\mathbb{R}^2$ and let $R : U \to \mathbb{R}$ be an analytic function which is not identically zero on U. The function R is an *integrating factor* of the polynomial differential system (1) on U if one of the following three equivalent conditions holds on U:

$$\frac{\partial(RP)}{\partial x} + \frac{\partial(RQ)}{\partial y} = 0, \qquad \operatorname{div}(RP, RQ) = 0, \qquad XR = -R \operatorname{div}(P, Q).$$

As usual the divergence of the vector field $X = (P, Q)$ is defined by

$$\operatorname{div}(X) = \operatorname{div}(P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

Doing the change of time $dt = rds$ the differential system (1) becomes

$$\dot{x} = RP = \frac{\partial H}{\partial y}, \qquad \dot{y} = RQ = -\frac{\partial H}{\partial x}. \tag{2}$$

Note that since system (2) has divergence zero, it is Hamiltonian.

The *first integral H associated to the integrating factor R* is given by

$$H(x, y) = \int R(x, y)P(x, y)\, dy + h(x), \tag{3}$$

where the function h is chosen such that $\dfrac{\partial H}{\partial x} = -RQ$.

Example 2. The quadratic system

$$\dot{x} = -y - b(x^2 + y^2) = P, \qquad \dot{y} = x = Q, \tag{4}$$

has the integrating factor $R = 1/(x^2 + y^2)$. Indeed,

$$\frac{\partial(RP)}{\partial x} + \frac{\partial(RQ)}{\partial y} = \frac{\partial(-y - b(x^2 + y^2))/(x^2 + y^2)}{\partial x} + \frac{\partial x/(x^2 + y^2)}{\partial y} = 0.$$

In order to compute the first integral of system (4) we follow (3), i.e.

$$H = \int RPdy + h(x) = \int \frac{-y - b(x^2 + y^2)}{x^2 + y^2} dy + h(x) = -by - \frac{1}{2}\log(x^2 + y^2) + h(x).$$

Then

$$\frac{\partial H}{\partial x} + RQ = h'(x) = 0.$$

So $h(x)$ is a constant and we can omit it from the first integral H, hence

$$H = -by - \frac{1}{2}\log(x^2 + y^2),$$

or we can take as first integral

$$F = e^H = \frac{e^{-by}}{\sqrt{x^2 + y^2}}.$$

Proposition 1. *If the polynomial differential system* (1) *has two integrating factors* R_1 *and* R_2 *on the open subset* U *of* $\mathbb{F}^2$, *then in the open set* $U\setminus\{R_2=0\}$ *the function* R_1/R_2 *is a first integral, provided* R_1/R_2 *is non–constant.*

Proof. Since R_i is an integrating factor, it satisfies $XR_i = -R_i\mathrm{div}(P,Q)$ for $i=1,2$. Therefore the proposition follows immediately from the computation

$$X\left(\frac{R_1}{R_2}\right) = \frac{(XR_1)R_2 - R_1(XR_2)}{R_2^2} = 0.$$

□

4. Invariant algebraic curves

Let $f\in\mathbb{R}[x,y]$ a non-zero polynoial. The algebraic curve $f(x,y)=0$ is an *invariant algebraic curve* of the polynomial differential system (1) if for some polynomial $K\in\mathbb{R}[x,y]$ we have

$$Xf = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{5}$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f=0$.Since the polynomial differential system has degree m, any cofactor has degree at most $m-1$.

On the points of the algebraic curve $f=0$ the gradient $(\partial f/\partial x, \partial f/\partial y)$ of f is orthogonal to the vector field $X=(P,Q)$ (see (5)). Hence at every point of $f=0$ the vector field X is tangent to the curve $f=0$, so the curve $f=0$ is formed by trajectories of the vector field X. This justifies the name "invariant algebraic curve" since it is invariant under the flow defined by X.

An *irreducible invariant algebraic curve* $f=0$ will be an invariant algebraic curve such that f is an irreducible polynomial in the ring $\mathbb{R}[x,y]$.

Example 3. If $a\neq 0$ the quadratic system

$$\dot{x} = -y(ay+b) - (x^2+y^2-1) = P, \qquad \dot{y} = x(ay+b) = Q, \tag{6}$$

has the following two invariant algebraic solutions $f_1 = ay+b = 0$ with cofactor $K_1 = ax$, and $f_2 = x^2+y^2-1=0$ with cofactor $K_2=-2x$. Indeed,

$$Xf_1 = P\frac{\partial ay+b}{\partial x} + Q\frac{\partial ay+b}{\partial y} = x(ay+b)a = K_1 f_1,$$

and

$$\begin{aligned} Xf_2 = \;& P\frac{\partial f_2}{\partial x} + Q\frac{\partial f_2}{\partial y} = (-y(ay+b) - (x^2+y^2-1))2x + x(ay+b)2y \\ = \;& -(x^2+y^2-1)2x = K_2 f_2. \end{aligned}$$

5. Exponential factors

There is another object, the so–called exponential factor, that plays the same role as the invariant algebraic curves in obtaining a first integral of a polynomial differential system (1).

Let $h, g \in \mathbb{R}[x, y]$ and assume that h and g are relatively prime in the ring $\mathbb{R}[x, y]$ or that $h \equiv 1$. Then the function $\exp(g/h)$ is called an *exponential factor* of the polynomial differential system (1) if for some polynomial $K \in \mathbb{R}[x, y]$ of degree at most $m - 1$ it satisfies

$$X\left(\exp\left(\frac{g}{h}\right)\right) = K \exp\left(\frac{g}{h}\right). \tag{7}$$

As before we say that K is the *cofactor* of the exponential factor $\exp(g/h)$.

As we will see, from the point of view of the integrability of polynomial differential systems (1) the importance of the exponential factors is twofold. On the one hand, they satisfy equation (7), and on the other hand, their cofactors are polynomials of degree at most $m - 1$. These two facts will imply that they play the same role as the invariant algebraic curves in the integrability of a polynomial differential system (1). We note that the exponential factors do not define invariant curves for the flow of system (1), because they are never zero.

Example 4. Consider the quadratic system

$$\dot{x} = x(y + a) = P, \qquad \dot{y} = y = Q.$$

This system has the exponential factor e^y. Indeed

$$Xe^y = P\frac{\partial e^y}{\partial x} + Q\frac{\partial e^y}{\partial y} = ye^y = Ke^y,$$

with $K = y$.

6. The method of Darboux

The Darboux theory of integrability for polynomial differential systems is summarized in the next theorem.

Theorem 2. *Suppose that a polynomial differential system* (1) *of degree m admits p irreducible invariant algebraic curves $f_i = 0$ with cofactors K_i for $i = 1, \ldots, p$, and q exponential factors* $\exp(g_j/h_j)$ *with cofactors L_j for $j = 1, \ldots, q$.*

(i) *There exist $\lambda_i, \mu_j \in \mathbb{R}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$, if and only if the function*

$$f_1^{\lambda_1} \ldots f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} \ldots \left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q} \tag{8}$$

is a first integral of system (1).

(ii) *If $p + q \geq m(m+1)/2 + 1$, then there exist $\lambda_i, \mu_j \in \mathbb{R}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$.*

(iii) *If $p + q \geq m(m+1)/2 + 2$, then the differential system* (1) *has a rational first integral, and consequently all trajectories of the system are contained in invariant algebraic curves.*

(iv) *There exist $\lambda_i, \mu_j \in \mathbb{R}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -div(P, Q)$, if and only if function* (8) *is an integrating factor of system* (1).

(i) We write $F_j = \exp(g_j/h_j)$. By hypothesis we have p invariant algebraic curves $f_i = 0$ with cofactors K_i, and q exponential factors F_j with cofactors L_j. That is, the f_i's satisfy $Xf_i = K_i f_i$, and the F_j's satisfy $XF_j = L_j F_j$.

Clearly statement (i) follows from the fact that

$$\begin{aligned} X\left(f_1^{\lambda_1} \ldots f_p^{\lambda_p} F_1^{\mu_1} \ldots F_q^{\mu_q}\right) &= \\ \left(f_1^{\lambda_1} \ldots f_p^{\lambda_p} F_1^{\mu_1} \ldots F_q^{\mu_q}\right) \left(\sum_{i=1}^{p} \lambda_i \frac{Xf_i}{f_i} + \sum_{j=1}^{q} \mu_j \frac{XF_j}{F_j}\right) &= \\ \left(f_1^{\lambda_1} \ldots f_p^{\lambda_p} F_1^{\mu_1} \ldots F_q^{\mu_q}\right), \left(\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j\right) &= 0. \end{aligned}$$

Example 5. If $a \neq 0$ the quadratic system

$$(9) \qquad \dot{x} = -y(ay+b) - (x^2+y^2-1), \qquad \dot{y} = x(ay+b),$$

has the algebraic solutions $f_1 = ay+b = 0$ with cofactor $K_1 = ax$, and $f_2 = x^2+y^2-1 = 0$ with cofactor $K_2 = -2x$. Since $2K_1 + aK_2 = 0$, by Theorem 2(i) we have that $H = (ay+b)^2(x^2+y^2-1)^a$ is a first integral of system (9).

(ii) Since the cofactors K_i and L_j are polynomials of degree $m-1$, we have that $K_i, L_j \in \mathbb{R}_{m-1}[x,y]$, the space of all polynomials of $\mathbb{R}[x,y]$ of degree at most $m-1$. We note that the dimension of $\mathbb{C}_{m-1}[x,y]$ as a vector space over $\mathbb{C}$ is $m(m+1)/2$.

Since all the polynomials K_i and L_j belong to the vector space $\mathbb{C}_{m-1}[x,y]$ of dimension $(m+1)/2]$, and we have $p+q$ polynomials K_i and L_j with $p+q > m(m+1)/2$, we obtain that the $p+q$ polynomials must be linearly dependent in $\mathbb{C}_{m-1}[x,y]$. So there are $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$. Hence statement (ii) is proved.

Example 6. If $abc \neq 0$ then the real quadratic system

$$(10) \qquad \dot{x} = x(ax+c), \qquad \dot{y} = y(2ax+by+c),$$

has exactly the following five invariant straight lines (i.e. algebraic solutions of degree 1): $f_1 = x = 0$, $f_2 = ax+c = 0$, $f_3 = y = 0$, $f_4 = ax+by = 0$, $f_5 = ax+by+c = 0$. Then by Theorem 2(ii) we know that system (10) must have a first integral of the form $H = f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3} f_4^{\lambda_4} f_5^{\lambda_5}$ with $\lambda_i \in \mathbb{R}$ satisfying $\sum_{i=1}^{5} \lambda_i K_i = 0$, where K_i is the cofactor of f_i. An easy computation shows that $K_1 = ax+c$, $K_2 = ax$, $K_3 = 2ax+by+c$, $K_4 = ax+by+c$ and $K_5 = ax+by$. Then a solution of $\sum_{i=1}^{5} \lambda_i K_i = 0$ is $\lambda_1 = \lambda_5 = -1$, $\lambda_2 = \lambda_4 = 1$ and $\lambda_3 = 0$. Therefore a first integral of system (10) is

$$H = \frac{(ax+c)(ax+by)}{x(ax+by+c)}.$$

(iii) Under the assumptions of statement(iii) we apply statement (ii) to two subsets of $p+q-1>0$ functions defining invariant algebraic curves or exponential factors. Thus we get two linear dependencies between the corresponding cofactors, which after some linear algebra and relabeling, we can write in the following form

$$M_1+\alpha_3 M_3+\ldots+\alpha_{p+q-1}M_{p+q}=0,\ M_2+\beta_3 M_3+\ldots+\beta_{p+q-1}M_{p+q}=0,$$

where M_l are the cofactors K_i and L_j, and the α_l and β_l are real numbers. Then by statement (i), it follows that the two functions

$$G_1 G_3^{\alpha_3}\ldots G_{p+q-1}^{\alpha_{p+q}},\qquad G_2 G_3^{\beta_3}\ldots G_{p+q-1}^{\beta_{p+q}},$$

are first integrals of the differential system (1), where G_l is the polynomial defining an invariant algebraic curve or the exponential factor having cofactor M_l for $l=1,\ldots,p+q$. Then taking logarithms of the two first integrals above, we obtain that

$$\begin{array}{rcl} H_1 & = & \log(G_1)+\alpha_3\log(G_3)+\ldots+\alpha_{p+q}\log(G_{p+q}), \\ H_2 & = & \log(G_2)+\beta_3\log(G_3)+\ldots+\beta_{p+q}\log(G_{p+q}), \end{array}$$

are first integrals of system (1) on their domain of definition. Each provides an integrating factor R_i such that

$$P=R_i\frac{\partial H_i}{\partial y},\qquad Q=-R_i\frac{\partial H_i}{\partial x}.$$

Therefore we obtain that

$$\frac{R_1}{R_2}=\frac{\partial H_2}{\partial x}/\frac{\partial H_1}{\partial x}.$$

Since the functions G_l are polynomials or exponentials of a quotient of polynomials, it follows that the functions $\partial H_i/\partial x$ are rational for $i=1,2$. So from the last equality, we get that the quotient of the two integrating factors R_1/R_2 is a rational function. Proposition 1 implies statement (iii).

In the example 5 we note that since this system has five invariant algebraic curves, by Theorem 2(iii) it must have a rational first integral, as we have found.

(iv) Since the equality $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -\mathrm{div}(P,Q)$ is equivalent to the equality

$$
\begin{aligned}
X\left(f_1^{\lambda_1}\dots f_p^{\lambda_p} F_1^{\mu_1}\dots F_q^{\mu_q}\right) &= \\
\left(f_1^{\lambda_1}\dots f_p^{\lambda_p} F_1^{\mu_1}\dots F_q^{\mu_q}\right)\left(\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j\right) &= \\
-\left(f_1^{\lambda_1}\dots f_p^{\lambda_p} F_1^{\mu_1}\dots F_q^{\mu_q}\right)\mathrm{div}(P,Q)\,,
\end{aligned}
$$

statement (iv) follows.

Example 7. The quadratic system

$$\dot{x} = -y - b(x^2+y^2) = P, \qquad \dot{y} = x = Q, \tag{11}$$

has the invariant algebraic curve $f_1 = x^2 + y^2$ with cofactor $K_1 = -2bx$. Since $K_1 = \mathrm{div}(P,Q)$, from Theorem 2(iv), it follows that f_1^{-1} is an integrating factor. Then an easy computation shows that $H = \exp(-2by)(x^2+y^2)$ is a first integral of system (11).

The Darboux theory of integrability here presented for polynomial differential systems in $\mathbb{R}^2$ has been extended to polynomial differential differential systems in $\mathbb{R}^n$ and $\mathbb{C}^n$. For more information on the Darboux theory of integrability see Chapter 8 of [3].

Part 2: LIMIT CYCLES IN PIECEWISE DIFFERENTIAL SYSTEMS VIA FIRST INTEGRALS

The study of the limit cycles is one of the most important objectives in the qualitative theory of the planar ordinary differential equations. We remark that to provide an upper bound for the maximum number of limit cycles for a given differential system in the plane $\mathbb{R}^2$, in general, is a very difficult problem.

The study of the discontinuous piecewise differential systems, more recently also called Filippov systems, has attracted the attention of the mathematicians during these past decades due to their applications. These piecewise differential systems in the plane are formed by different differential systems defined in distinct regions separated by a curve. A pioneering work on this subject was due to Andronov, Vitt and Khaikin in 1920's, and later on Filippov in 1988 provided the theoretical bases for this kind of differential systems. Nowadays a vast literature on these differential systems is available. As for the smooth differential systems the study of the existence and location of limit cycles in the piecewise differential systems is also of great importance.

The main tools for computing analytically limit cycles of differential systems are based on the averaging theory, the Melnikov integral, the Poincaré map, and the Poincaré map together with the Newton-Kantorovich Theorem or the Poincaré-Miranda theorem. To these tools, in the particular case of the piecewise differential systems, we must add the use of the first integrals of the differential systems forming the piecewise differential systems for computing their limit cycles.

To show how to use the first integrals for computing the limit cycles and the periodic orbits of the piecewise differential systems is the objective of this second part of this mini-curs. Of course, this tool is restricted to the piecewise differential systems such that all their differential systems be integrable, in the sense that we know for each of them a first integral.

7. DISCONTINUOUS PIECEWISE DIFFERENTIAL SYSTEMS

A *discontinuous piecewise differential system* on $\mathbb{R}^2$ is a pair of C^r (with $r \geq 1$) differential systems in $\mathbb{R}^2$ separated by a smooth codimension one manifold Σ. The *line of discontinuity* Σ of the discontinuous piecewise differential system is defined by $\Sigma = h^{-1}(0)$, where $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a differentiable function having 0 as a regular value. Note that Σ is the separating boundary of the regions $\Sigma^+ = \{(x, y) \in \mathbb{R}^2 \mid h(x, y) > 0\}$ and $\Sigma^- = \{(x, y) \in \mathbb{R}^2 \mid h(x, y) < 0\}$. So the piecewise C^r vector field associated to a piecewise differential system with line of discontinuity Σ is

$$Z(x,y) = \begin{cases} X(x,y), & \text{if } h(x,y) \geq 0, \\ Y(x,y), & \text{if } h(x,y) \leq 0. \end{cases} \tag{12}$$

As usual, system (12) is denoted by $Z = (X, Y, \Sigma)$ or simply by $Z = (X, Y)$, when the separation line Σ is well understood. In order to establish a definition for the trajectories of Z and investigate its behavior, we need a criterion for the transition of the orbits between Σ^+ and Σ^- across Σ. The contact between the vector field X (or Y) and the line of discontinuity Σ is characterized by the derivative of h in the direction of the vector field X i.e.

$$Xh(p) = \langle \nabla h(p), X(p) \rangle ,$$

where $\langle .,. \rangle$ is the usual inner product in $\mathbb{R}^2$. The basic results of the discontinuous piecewise differential systems in this context were stated by Filippov [5]. We can divide the line of discontinuity Σ in the following sets:

(a) *Crossing set*: $\Sigma^c : \{p \in \Sigma : Xh(\mathbf{x}) \cdot Yh(\mathbf{x}) > 0\}$.
(b) *Escaping set*: $\Sigma^e : \{p \in \Sigma : Xh(\mathbf{x}) > 0 \text{ and } Yh(\mathbf{x}) < 0\}$.
(c) *Sliding set*: $\Sigma^s : \{p \in \Sigma : Xh(\mathbf{x}) < 0 \text{ and } Yh(\mathbf{x}) > 0\}$.

The *escaping* Σ^e or *sliding* Σ^s regions are respectively defined on points of Σ where both vector fields X and Y simultaneously point outwards or inwards from Σ while the interior of its complement in Σ defines the *crossing region* Σ^c (see Figure 1). The complementary of the union of these regions is the set formed by the *tangency* points between X or Y with Σ.

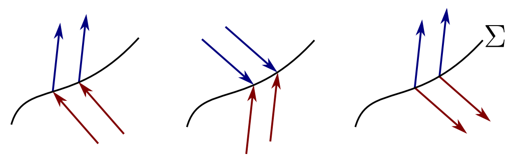

FIGURE 1. Crossing, sliding and escaping regions, respectively.

8. Limit cycles of a piecewise differential system formed by a linear differential system and a quadratic polynomial differential system separated by the straight line $x = 0$

In what follows we want to study the limit cycles of the following discontinuous piecewise differential system separated by the straight line $x = 0$, in the half-plane $x \geq 0$ there is the linear differential system

$$\dot{x} = 2 + 2x - 2y, \qquad \dot{y} = 6 - 2y, \tag{13}$$

and in the half-plane $x \leq 0$ there is the quadratic polynomial differential system

$$\begin{aligned} \dot{x} &= -9x + 15y + 4x^2 + 8xy - 28y^2, \\ \dot{y} &= -6x + 9y - 4x^2 + 32xy - 44y^2. \end{aligned} \tag{14}$$

Theorem 3. *The discontinuous piecewise differential system* (13)-(14) *has a unique limit cycle shwon in Figure 1.*

Proof. Since we want to compute the limit cycles of this piecewise differential system using their first integrals, we must find such first integrals.

It is known that all the linear differential systems in $\mathbb{R}^n$ for $n \geq 2$ are are Darboux integrable (see [6]), so in particular the differential system (13) must have a first integral. Clearly the differential system (13) is Hamiltonian with Hamiltonian

$$H_1(x, y) = 6x - 2y - 2xy + y^2.$$

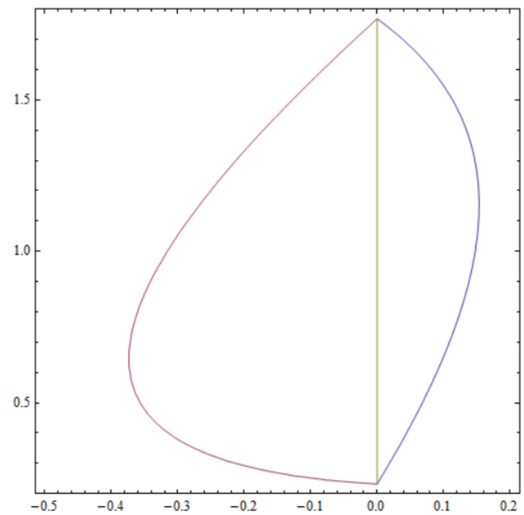

FIGURE 2. The unique limit cycle that exists for the piecewise differential system (13)–(14). The limit cycle is travelled in counterclocwise sense.

So we have a first integral of system (13).

System (14) has a center at the origin of coordinates, because the eigenvalues of the linear part of the system at the origin are $\pm 3i$, so the origen is either a weak focus or a center, but computing their Lyapunov constants we see that all of them are zero, so it is a center, see for more details Chapter 5 of [3]. Moreover, it is well known that all quadratic polynomial differential systems having a center are Darboux integrable, see for more details the proof of Theorem 8.15 of [3] or the paper [9].

Now in order to find a first integral for the differential system (14) we shall apply the Darboux theory of integrability. We start looking for their invariant algebraic curves of degree 1 and 2.

First we look for invariant straight lines $f = f(x, y) = ax + by + c = 0$, and since the polynomial differential system has degree 2 their cofactors must be polynomials of degree at most 1, so they must be of the form $K = k_0 + k_1 x + k_2 y$, and f and K must satisfy equation (5). Passing the right hand side of this equation to the left we obtain the polynomial

$$-ck_0 + (-9a - 6b - ak_0 - ck_1)x + (15a + 9b - bk_0 - ck_2)y + (4a - 4b - ak_1)x^2 + \\ (8a + 32b - bk_1 - ak_2)xy + (-28a - 44b - bk_2)y^2 = 0.$$

So we must solve the system

$$\begin{aligned} &ck_0 = 0, \\ &9a + 6b + ak_0 + ck_1 = 0, \\ &15a + 9b - bk_0 - ck_2 = 0, \\ &4a - 4b - ak_1 = 0, \\ &8a + 32b - bk_1 - ak_2 = 0, \\ &28a + 44b + bk_2 = 0, \end{aligned}$$

in the unknowns a, b, c, k_0, k_1, k_2 for obtaining the possible invariant straight lines of the differential system (14). This system has a unique solution with $(a, b, c) = (0, 0, 0)$. Namely $b = -a$, $c = -3a/8$, $k_0 = 0$, $k_1 = 8$ and $k_2 = -16$. So we have

$$f = \frac{1}{8}a(8x - 8y - 3) \quad \text{and} \quad K = 8(x - 2y).$$

So, without loss of generality we can assume that the invariant straight line is $f_1 = 8x - 8y - 3 = 0$ with the cofactor $K_1 = 8(x - 2y)$.

Now we look for possible invariant algebraic curves of degree two. So we must solve the equation (5) with $f = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2$ and $K = k_0 + k_1x + k_2y$. Solving it, in a similar way for obtaining the invariant algebraic curves of degree 1, we obtain only two invariant algebraic curves of degree 2 for f, namely

$$-\frac{1}{128}a_4(8x - 8y - 3)^2 = 0, \text{ and } -\frac{1}{768}a_4(-9 + 96x - 96y + 128x^2 - 768xy + 896y^2).$$

The first solution really it is the invariant straight line. So essentially there is a unique invariant algebraic curve of degree two, that we can take $f_2 = -9 + 96x - 96y + 128x^2 - 768xy + 896y^2 = 0$ with cofactor $K_2 = 32(x - 2y)$.

We have that the equation $\lambda_1 K_1 + \lambda_2 K_2 = 0$ is satisfied with $\lambda_1 = -4$ and $\lambda_2 = 1$, therofore by Theorem 2(ii) we obtain the first integral

$$H_2(x, y) = \frac{-9 + 96x - 96y + 128x^2 - 768xy + 896y^2}{(8x - 8y - 3)^4}.$$

Clearly in the half-planes $x \geq 0$ and $x \leq 0$ the differential systems (13) and (14) has no limit cycles, because the polynomial H_1 and rational first integral H_2 prevent their existence, respectively. So the piecewise differential system formed with the systems (13) and (14) separated by the straight line $x = 0$ has limit cycles, these must cross the line $x = 0$ in exactly two points, denoted by $(0, y)$ and $0, Y)$ with $y < Y$. These two points must be crossing points and satisfy the system

$$e_1 = H_1[0, y] - H_1[0, Y] = 0, \qquad e_2 = H_2[0, y] - H_2[0, Y] = 0.$$

The unique solution of this system satisfying $y < Y$ is

$$(15) \quad (y, Y) = \frac{1}{1616}\left(1616 - \sqrt{2222(1013 - 9\sqrt{1257})}, 1616 + \sqrt{2222(1013 - 9\sqrt{1257})}\right).$$

This solution provides the limit cycle of Figure 6. □

9. Limit cycles of piecewise differential systems formed by three linear centers

These last years the interest for studying the piecewise linear differential systems has increased strongly, mainly due to their applications to many physical phenomena. In the study of these differential systems the limit cycles play a main role. Up to now the major part of papers which study the limit cycles of the piecewise linear differential systems consider only two pieces. Here we consider piecewise linear differential systems with three pieces.

In the paper [8] we studied the limit cycles of the discontinuous piecewise linear differential systems in the plane $\mathbb{R}^2$ formed by three arbitrary linear centers separated by the set

$$\Sigma = \{(x, y) \in \mathbb{R}^2 : y = 0 \text{ or } x = 0 \text{ and } y \geq 0\}.$$

There it is proved that such discontinuous piecewise linear differential systems can have 1, 2 or 3 limit cycles, with 3 the maximum number of limit cycles that such systems can exhibit. In particular it is proved that there are such piecewise linear differential having 3 limit cycles intersecting in a unique point each of the three branches of $\Sigma \setminus \{(0, 0)\}$.

The three components of $\mathbb{R}^2 \setminus \Sigma$ are the positive or first quadrant $Q_1 = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 0\}$, the second quadrant $Q_2 = \{(x, y) \in \mathbb{R}^2 : x < 0 \text{ and } y > 0\}$, and the half-plane $H = \{(x, y) \in \mathbb{R}^2 : y < 0\}$.

The objective of this section is to study the limit cycles of the discontinuous piecewise linear differential system with three pieces separated by the set Σ defined by

$$\begin{aligned}
&\dot{x} = \frac{2}{1565}y + \frac{379}{1565}, && \dot{y} = -2x + \frac{237}{313}, && \text{in } Q_1,\\
&\dot{x} = \frac{4}{1565}y + \frac{11566}{10955}, && \dot{y} = -8x - \frac{4\sqrt{4430533}}{2191}, && \text{in } Q_2,\\
&\dot{x} = 2y, && \dot{y} = -8x - \frac{2\left(\sqrt{4430533} - 1299\right)}{2191}, && \text{in } H.
\end{aligned} \tag{16}$$

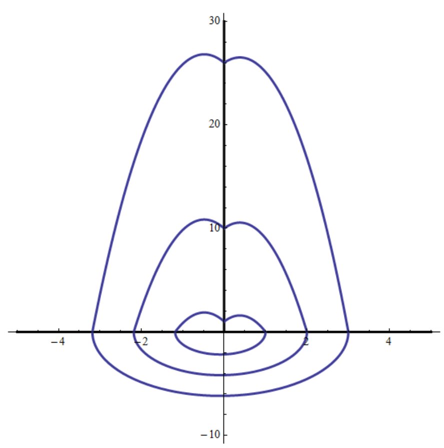

FIGURE 3. The three limit cycles of the discontinuous piecewise linear differential system (22). These limit cycles are travelled in counterclockwise sense.

Theorem 4. *The discontinuous piecewise differential system* (22) *has exactly three limit cycles intersecting the set* Σ *exactly in three points, and they are traveled in clockwise sense, see Figure 2.*

Proof. Of course we want to prove this objective using the first integrals of the three linear differential centers. It is well known that all the linear differential centers can be obtained doing an affine transformation of the linear differential center $\dot{x} = -y$, $\dot{y} = x$, which has the first integral $H = x^2 + y^2$. So all the linear differential centers are

Hamiltonian systems, and then computing the Hamiltonians of the linear centers in Q_1, Q_2 and H we obtain the first integrals

$$\begin{aligned} H_1(x,y) &= 1565x^2 + y^2 - 1185x + 379y, \\ H_2(x,y) &= 21910x^2 + 7y^2 + 10\sqrt{4430533}x + 5783y, \\ H_3(x,y) &= 8764x^2 + 2191y^2 + (2\sqrt{4430533} - 2598)x, \end{aligned}$$

for each one of these three systems, respectively.

These limit cycles must intersect each branch of $\Sigma \setminus \{(0,0)\}$ in one point, namely $(x_+,0)$ with $x_+ > 0$, $(0,y_+)$ with $y_+ > 0$, and $(x_-,0)$ with $x_- < 0$. Of course these three points must be crossing points. Then the first integrals H_1, H_2 and H_3 must satisfy the following three equations

$$\begin{aligned} H_1(x_+,0) - H_1(0,y_+) &= 0, \\ H_2(0,y_+) - H_2(x_-,0) &= 0, \\ H_3(x_-,0) - H_3(x_+,0) &= 0, \end{aligned} \tag{17}$$

or equivalently

$$\begin{aligned} &1565x_+^2 - y_+^2 - 1185x_+ - 379y_+ = 0, \\ &21910x_-^2 - 7y_+^2 + 10\sqrt{4430533}x_- - 5783y_+ = 0, \\ &(x_+ - x_-)\left(4382x_+ + 4382x_- + \sqrt{4430533} - 1299\right) = 0. \end{aligned} \tag{18}$$

Taking into account that we are only interested in the solutions (x_+, y_+, x_-) satisfying $x_+ > 0$, $x_- < 0$ and $y_+ > 0$, the unique three solutions of the previous system are

$$\begin{aligned} (x_+^1, y_+^1, x_-^1) &= \left(1, \frac{-3083 - \sqrt{4430533}}{4382}, 1\right), \\ (x_+^2, y_+^2, x_-^2) &= \left(2, \frac{-7465 - \sqrt{4430533}}{4382}, 10\right), \\ (x_+^3, y_+^3, x_-^3) &= \left(3, \frac{-11847 - \sqrt{4430533}}{4382}, 26\right). \end{aligned}$$

The first linear differential system of (22) has the solution

$$
\begin{aligned}
x_1(t) = \; & \frac{1}{21910}\sin\left(2\sqrt{\frac{2}{1565}}t\right)\left(\sqrt{3130}(14u+5783)\cos\left(2\sqrt{\frac{2}{1565}}t\right)\right.\\
& \left.-10\sqrt{4430533}\sin\left(2\sqrt{\frac{2}{1565}}t\right)\right),\\
y_1(t) = \; & \left(u+\frac{5783}{14}\right)\cos\left(4\sqrt{\frac{2}{1565}}t\right)-\frac{1}{7}\sqrt{\frac{22152665}{626}}\sin\left(4\sqrt{\frac{2}{1565}}t\right)\\
& -\frac{5783}{14},
\end{aligned}
$$

satisfying the initial conditions $x_1(0) = u$ and $y_1(0) = 0$.

The second linear differential system of (22) has the solution

$$
\begin{aligned}
x_2(t) &= \left(v-\frac{237}{626}\right)\cos\left(\frac{2t}{\sqrt{1565}}\right)+\frac{379}{2\sqrt{1565}}\sin\left(\frac{2t}{\sqrt{1565}}\right)+\frac{237}{626},\\
y_2(t) &= \frac{1}{2}\sqrt{\frac{5}{313}}(237-626v)\sin\left(\frac{2t}{\sqrt{1565}}\right)+\frac{379}{2}\cos\left(\frac{2t}{\sqrt{1565}}\right)-\frac{379}{2},
\end{aligned}
$$

satisfying the initial conditions $x_3(0) = v$ and $y_3(0) = 0$.

The third linear differential system of (22) has the solution

$$
\begin{aligned}
x_3(t) &= \frac{1}{8764}\left(8764w\cos(4t)+\left(\sqrt{4430533}-1299\right)\left(\cos(4t)-1\right)\right),\\
y_3(t) &= \frac{\left(1299-8764w-\sqrt{4430533}\right)\sin(4t)}{4382},
\end{aligned}
$$

satisfying the initial conditions $x_3(0) = w$ and $y_3(0) = 0$.

Now we consider the solution $(x_k^1(t), y_k^1(t))$ for $k = 1, 2, 3$ of the discontinuous piecewise linear differential system (22) corresponding to the solution (x_+^1, y_+^1, x_-^1) of system (18). Then the time that the solution $(x_1^1(t), y_1^1(t))$ contained in Q_1 needs to reach the point $(0, v)$ is $t_1 = 0.785398163397448..$ The time that the solution $(x_2^1(t), y_2^1(t))$ contained in Q_2 needs to reach the point $(w, 0)$ is $t_2 = 4.10363864680248..$ Finally $t_3 = 1.11762450719575..$ is the time that the solution $(x_3^1(t), y_3^1(t))$ contained in H needs to reach the point $(u, 0)$.

Let $(x_k^2(t), y_k^2(t))$ for $k = 1, 2, 3$ be the solution of the discontinuous piecewise linear differential system (22) corresponding to the solution (x_+^2, y_+^2, x_-^2) of system (18). Then the time that the solution $(x_1^2(t), y_1^2(t))$ contained in Q_1 needs to reach the point $(0, v)$ is $r_1 = 0.785398163397448..$ The time that the solution $(x_2^2(t), y_2^2(t))$ contained in Q_2 needs to reach the point $(w, 0)$ is $r_2 = 7.93799227264621..$ The time that the solution $(x_3^2(t), y_3^2(t))$ contained in H needs to reach the point $(u, 0)$ is $r_3 = 2.02943545009903..$

Let $(x_k^3(t), y_k^3(t))$ for $k = 1, 2, 3$ be the solution of the discontinuous piecewise linear differential system (22) corresponding to the solution $(x_+^3, y_+^3, x_-^3) =$ of system (18).

Then the time that the solution $(x_1^3(t), y_1^3(t))$ contained in Q_1 needs to reach the point $(0, v)$ is $s_1 = 0.785398163397448..$ The time that the solution $(x_2^3(t), y_2^3(t))$ contained in Q_2 needs to reach the point $(w, 0)$ is $s_2 = 11.27688306691738..$ The time that the solution $(x_3^3(t), y_3^3(t))$ contained in H needs to reach the point $(u, 0)$ is $s_3 = 2.88219547492608..$

Drawing the three orbits $(x_k^j(t), y_k^j(t))$ for $j = 1, 2, 3$ and for the times $t \in [0, t_k]$, $t \in [0, r_k]$ and $t \in [0, s_k]$ for $k = 1, 2, 3$, respectively, we obtain the 3 limit cycles of Figure 3 which are traveled in clockwise sense. $\square$

10. Periodic orbits of a relay system in $\mathbb{R}^3$

The goal of this section is to study analytically the periodic orbits of the following discontinuous piecewise linear differential system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -\text{sign}(x) y, \end{aligned} \tag{19}$$

using their first integrals. As usually the sign function is defined as follows

$$\text{sign}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

So $x = 0$ is the plane of discontinuity of system (19). System (19) has been studied in [7].

Control theory is a natural source of mathematical models for the discontinuous piecewise differential system (19). In fact system (19) is a particular relay system of the ones studied in [1].

To study analytically the periodic solutions of a differential system, in general, is a very difficult task, usually impossible of doing. Our goal here is to show that the periodic orbits of piecewise differential systems, continuous or discontinuous, which are completely integrable in each piece, "sometimes" can be studied analytically using the first integrals of these systems. This "sometimes" means that for such differential systems in theory always can be studied their periodic orbits using the first integrals, but in the practice the computations that must be done using the first integrals may be difficult and cannot allow to do this study.

The objective of this section is to prove the next result.

Theorem 5. *The following statements hold.*

(a) *Let γ be a periodic orbit of the discontinuous piecewise linear differential system* (19) *which intersects the discontinuous plane $x = 0$ in two points. Then γ intersects the plane $x = 0$ in the points $(0, y, z)$ and $(0, -y, z)$ satisfying $z > 0$ and $y^2 - z^2 < 0$. See Figure 6.*

(b) *For every pair of points $(0, y, z)$ and $(0, -y, z)$ satisfying $z > 0$ and $y^2 - z^2 < 0$, there exists a periodic orbit γ of he discontinuous piecewise linear differential system* (19) *which intersects the discontinuous plane $x = 0$ in these two points.*

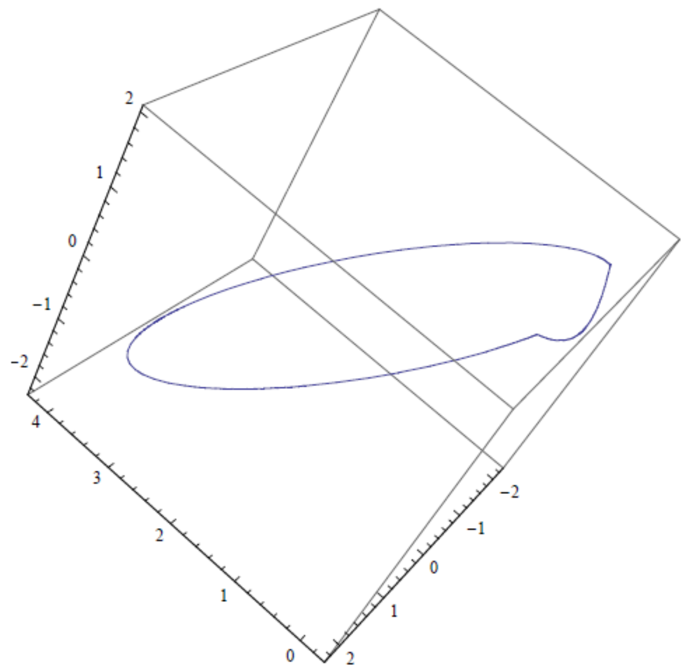

FIGURE 4. The periodic orbit of Theorem 5 which intersects the discontinuous plane $x = 0$ in the points $(0, -1, 2)$ and $(0, 1, 2)$.

Proof. The discontinuous piecewise linear differential system (19) in $\mathbb{R}^3$ is formed by the following two linear differential systems

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -y, \end{aligned} \tag{20}$$

in the half–space $x > 0$, and

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= y, \end{aligned} \tag{21}$$

in the half–space $x < 0$. Note that both systems have the points of the x–axis contained in the half–spaces where they are defined as equilibrium points.

Since both differential systems are linear we always can compute two independent first integrals for each system. In fact linear differential systems always are completely Darboux integrable, see for more details [6]. Thus two independent first integrals for system (20) are

$$H_1 = x + z, \quad \text{and} \quad H_2 = y^2 + z^2.$$

So the orbits of system (20) are contained in the sets

$$\gamma_{h_1 h_2} = \{H_1 = h_1\} \cap \{H_2 = h_2\} \cap \{x > 0\},$$

for all $(h_1, h_2) \in \mathbb{R}^2$ when these sets are non–empty.

In a similar way two independent first integrals for system (21) are

$$F_1 = x - z, \quad \text{and} \quad F_2 = y^2 - z^2.$$

So the orbits of system (21) are contained in the sets

$$\gamma_{f_1 f_2} = \{F_1 = f_1\} \cap \{F_2 = f_2\} \cap \{x < 0\},$$

for all $(f_1, f_2) \in \mathbb{R}^2$ when these sets are non–empty.

We note that the set $\gamma_{h_1h_2}$, when it is non–empty and $h_2 > 0$, is formed by the piece of the connected curve obtained from the intersection of the plane $H_1 = h_1$ with the cylinder $H_2 = h_2$ and with the half–space $x > 0$. So under these assumptions the set $\gamma_{h_1h_2}$ is a connected arc contained in $x > 0$ which does not contain equilibria, so it is an orbit of system (20). If $\gamma_{h_1h_2}$ is non–empty and $h_2 = 0$, then $\gamma_{h_1h_2}$ is an equilibrium point. In short, always that the set $\gamma_{h_1h_2}$ is non–empty it is formed by a unique orbit of system (20).

It is easy to check that the set $\gamma_{f_1f_2}$ is always non–empty. If $f_2 \neq 0$, it is formed by one or two arcs of the curve obtained from the intersection of the plane $F_1 = f_1$ with the hyperboloid cylinder $F_2 = f_2$ contained in the half–space $x < 0$. Moreover these one or two arcs do not contain equilibria (because the equilibria need that $f_2 = 0$), so $\gamma_{f_1f_2}$ is formed by one or two orbits of system (21). If $f_2 = 0$, then $\gamma_{f_1f_2}$ is the intersection of the plane $F_1 = f_1$ with the two planes $F_2 = 0$ and with the half–space $x < 0$. The intersection $\{F_1 = f_1\} \cap \{F_2 = 0\}$ is formed by two straight lines intersecting at the equilibrium point $(f_1, 0, 0)$. The intersection of these two straight lines with the half–space $x < 0$, can contain either 5 orbits (one of them is the equilibrium point $(f_1, 0, 0)$), or 2 orbits.

We want to study when an orbit of $\gamma_{h_1h_2}$ and an orbit of $\gamma_{f_1f_2}$ can connect forming a periodic orbit of the discontinuous piecewise linear differential system (19), i.e. when the two orbits reach the plane $x = 0$ in the same two points. In such a case they form a periodic solution because from system (19) these two points are crossing points. More precisely, let X_+ (resp. X_-) be the vector field associated to the linear differential system (20) (resp. (21)) in the half–space $x \geq 0$ (resp. $x \leq 0$). Let p be a point of the discontinuity plane $x = 0$, when the segment connecting the endpoints of the vectors $X_+(p)$ and $X_-(p)$ does not intersect the plane $x = 0$, then p is a *crossing point*. For more details on crossing points see section 7.

From the previous study done on the orbits of systems (20) and (21) it follows that an orbit of $\gamma_{h_1h_2}$ and an orbit of $\gamma_{f_1f_2}$ can connect forming a periodic orbit only if $h_2 > 0$ and $f_2 \neq 0$.

We take an arbitrary point of the plane of discontinuity, for instance the point $(0, y_0, z_0)$. We evaluate the four first integrals H_1, H_2, F_1 and F_2 in this point and we get the four values $h_1 = z_0$, $h_2 = y_0^2 + z_0^2$, $f_1 = -z_0$ and $f_2 = y_0^2 - z_0^2$, respectively. Now we study how many points of the orbit $\{H_1 = h_1\} \cap \{H_2 = h_2\} \cap \{x \geq 0\}$ are in the plane $x = 0$, solving the system

$$H_1 = h_1, \quad H_2 = h_2, \quad x = 0,$$

we get two points $(0, \pm y_0, z_0)$. We also analyze how many points of the orbits $\{F_1 = f_1\} \cap \{F_2 = f_2\} \cap \{x \leq 0\}$ are in the plane $x = 0$, solving the system

$$F_1 = f_1, \quad F_2 = f_2, \quad x = 0,$$

again we get the two points $(0, \pm y_0, z_0)$. If these two points belong to the same orbit of the set $\{F_1 = f_1\} \cap \{F_2 = f_2\} \cap \{x \leq 0\}$, then we have a periodic orbit of the discontinuous piecewise linear differential system (19).

If we parameterize the orbit $\{H_1 = h_1\} \cap \{H_2 = h_2\}$ in the half–space $x \geq 0$ using the variable x we obtain the arc

$$\{(x, \pm\sqrt{y_0^2 + z_0^2 - (z_0 - x)^2}, z_0 - x) : 0 \leq x \leq z_0 + \sqrt{y_0^2 + z_0^2}\}. \tag{22}$$

Note that this orbit is symmetric with respect to the y–axis, it is contained in $x \geq 0$ and its endpoints are the two points $(0, \pm y_0, z_0)$ on the plane $x = 0$.

Again we parameterize the curve $\{F_1 = f_1\} \cap \{F_2 = f_2\}$ in the half–space $x \leq 0$ using the variable x. This curve is formed by

- the two orbits $\{(x, \sqrt{x^2 + y_0^2}, x) : x \leq 0\}$ and $\{(x, -\sqrt{x^2 + y_0^2}, x) : x \leq 0\}$ if $z_0 = 0$, each orbit has only one endpoint in the plane $x = 0$;
- the two orbits $\{x, \sqrt{(x + z_0)^2 + y_0^2 - z_0^2}, z_0 + x)\} : x \leq 0\}$ and $\{x, -\sqrt{(x + z_0)^2 + y_0^2 - z_0^2}, z_0 + x)\} : x \leq 0\}$ if either $z_0 < 0$ or $z_0 > 0$ and $y_0^2 - z_0^2 > 0$, each orbit has only one endpoint in the plane $x = 0$;
- the two orbits
$$\{x, \pm\sqrt{(x + z_0)^2 + y_0^2 - z_0^2}, z_0 + x)\} : -z_0 + \sqrt{z_0^2 - y_0^2} \leq x \leq 0\} \tag{23}$$
and
$$\{x, \pm\sqrt{(x + z_0)^2 + y_0^2 - z_0^2}, z_0 + x)\} : x \leq -z_0 - \sqrt{z_0^2 - y_0^2}\},$$
if $z_0 > 0$ and $y_0^2 - z_0^2 < 0$, the first orbit has its two endpoints at the points $(0, \pm y_0, z_0)$ of the plane $x = 0$, and the second orbit has its endpoints at infinity;

We note that $y_0^2 - z_0^2 \neq 0$ otherwise $f_2 = 0$.

In short, if $z_0 > 0$ and $y_0^2 - z_0^2 < 0$ then the orbit (22) of the linear differential system (20) together with the orbit (23) of the linear differential system (21) form a periodic orbit of the discontinuous piecewise linear differential system (19), and this periodic orbit intersects the plane of discontinuity $x = 0$ at the two points $(0, \pm y_0, z_0)$. This completes the proof of Theorem 5. $\square$

11. Limit cycles of a class of piecewise differential systems separated by a parabola

.

The objective of this section is to study the limit cycles of discontinuous piecewise differential systems separated by the parabola $y = x^2$ and formed by two linear Hamiltonian systems without equilibrium points.

Easy computations show that a linear Hamiltonian systems without equilibrium points must be of the form

$$X_i(x, y) = (-\lambda_i b_i x + b_i y + \gamma_i, -\lambda_i^2 b_i x + \lambda_i b_i y + \delta_i),$$

$\delta_i \neq \lambda_i \gamma_i$ and $b_i \neq 0$, with $i = 1 \dots 4$, and its corresponding Hamiltonian function is

$$H_i(x, y) = (-\lambda_i^2 b_i/2)x^2 + \lambda_i b_i xy - (b_i/2)y^2 + \delta_i x - \gamma_i y.$$

We want to prove the following result, which comes from [2].

Theorem 6. *Generically the maximum number of limit cycles of the piecewise differential systems separated by the parabola $y = x^2$ and formed by two linear Hamiltonian systems without equilibrium points is two, and this maximum is reached, see Figure 5.*

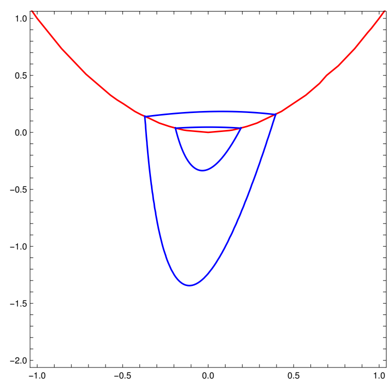

FIGURE 5. Two limit cycles of a piecewise differential system separated by the parabola $y = x^2$ and formed by two linear Hamiltonian systems without equilibrium points. Both limit cycles are travelled in counterclockwise sense.

Proof. In the region $R_1 = \{(x, y) : y - x^2 \geq 0\}$ we consider the linear Hamiltonian system without equilibrium points

$$\dot{x} = -\lambda_1 b_1 x + b_1 y + \gamma_1, \quad \dot{y} = -\lambda_1^2 b_1 x + \lambda_1 b_1 y + \delta_1, \tag{24}$$

with $b_1 \neq 0$ and $\delta_1 \neq \lambda_1 \gamma_1$. Its corresponding Hamiltonian function is

$$H_1(x, y) = -(\lambda_1^2 b_1/2)x^2 + \lambda_1 b_1 xy - (b_1/2)y^2 + \delta_1 x - \gamma_1 y. \tag{25}$$

In the region $R_2 = \{(x, y) : y - x^2 \leq 0\}$ we consider another linear Hamiltonian system without equilibrium points

$$\dot{x} = -\lambda_2 b_2 x + b_2 y + \gamma_2, \quad \dot{y} = -\lambda_2^2 b_2 x + \lambda_2 b_2 y + \delta_2, \tag{26}$$

with $b_2 \neq 0$ and $\delta_2 \neq \lambda_2 \gamma_2$. Its corresponding Hamiltonian function is

$$H_2(x, y) = -(\lambda_2^2 b_2/2)x^2 + \lambda_2 b_2 xy - (b_2/2)y^2 + \delta_2 x - \gamma_2 y. \tag{27}$$

In order to have a crossing limit cycle which intersects the parabola $y - x^2 = 0$ in the points (x_i, x_i^2) and (x_k, x_k^2), these points must satisfy the following system

$$\begin{aligned} H_1(x_i, x_i^2) - H_1(x_k, x_k^2) &= 0, \\ H_2(x_i, x_i^2) - H_2(x_k, x_k^2) &= 0, \end{aligned} \tag{28}$$

We suppose that the two systems (24) and (26) have three crossing limit cycles, and we shall arrive to a contradiction. Then system (28) must have three pairs of points as solutions, namely $p_i = (r_i, r_i^2)$ and $q_i = (s_i, s_i^2)$, with $i = 1, 2, 3$.

Since the points $p_1 = (r_1, r_1^2)$ and $q_1 = (s_1, s_1^2)$ satisfy system (28), we obtain that the parameters γ_1 and γ_2 must be

$$\begin{aligned}\gamma_1 = \ & \frac{1}{2(r_1+s_1)}(-r_1 r_1^3 - b_1 r_1^2 s_1 - b_1 r_1 s_1^2 - b_1 s_1^3 + 2\delta_1 + 2b_1 r_1^2 \lambda_1 + 2b_1 r_1 s_1 \lambda_1 \\ & +2b_1 s_1^2 \lambda_1 - b_1 r_1 \lambda_1^2 - b_1 s_1 \lambda_1^2),\end{aligned}$$

and γ_2 has the same expression that γ_1 changing $(b_1, \lambda_1, \delta_1)$ by $(b_2, \lambda_2, \delta_2)$.

If the second points $p_2 = (r_2, r_2^2)$ and $q_2 = (s_2, s_2^2)$ satisfy system (28), then the parameters δ_1 and δ_2 must be

$$\begin{aligned}\delta_1 = \ & \frac{b_1}{2(r_1 - r_2 + s_1 - s_2)}(-r_1^3 r_2 - r_1 r_2^3 + r_1^2 r_2 s_1 - r_2^3 s_1 + r_1 r_2 s_1^2 + r_2 s_1^3 + r_1^3 s_2 \\ & -r_1 r_2^2 s_2 + r_1^2 s_1 s_2 - r_2^2 s_1 s_2 + r_1 s_1^2 s_2 + s_1^3 s_2 - r_1 r_2 s_2^2 - r_2 s_1 s_2^2 - r_1 s_2^3 - s_1 s_2^3 \\ & -2r_1^2 r_2 \lambda_1 + 2r_1 r_2^2 \lambda_1 - 2r_1 r_2 s_1 \lambda_1 + 2r_2^2 s_1 \lambda_1 - 2r_2 s_1^2 \lambda_1 - 2r_1^2 s_2 \lambda_1 + 2r_1 r_2 s_2 \lambda_1 \\ & -2r_1 s_1 s_2 \lambda_1 + 2r_2 s_1 s_2 \lambda_1 - 2s_1^2 s_2 \lambda_1 + 2r_1 s_2^2 \lambda_1 + 2s_1 s_2^2 \lambda_1),\end{aligned}$$

and δ_2 has the same expression that δ_1 changing (b_1, λ_1) by (b_2, λ_2).

Finally, we suppose that the points $p_3 = (r_3, r_3^2)$ and $q_3 = (s_3, s_3^2)$ satisfy system (28), then the parameters λ_1 and λ_2 must be $\lambda_1 = A/B$ where

$$\begin{aligned}A = \ & r_1^3(r_2 - r_3 + s_2 - s_3) + r_1^2 s_1 (r_2 - r_3 + s_2 - s_3) + r_2^3(r_3 - s_1 + s_3) + r_2^2 s_2 (r_3 - s_1 \\ & +s_3) + r_1(-r_2^3 + r_3^3 - r_3 s_1^2 - r_2^2 s_2 + s_1^2 s_2 - s_2^3 + r_2(s_1^2 - s_2^2) + r_3^2 s_3 - s_1^2 s_3 + r_3 s_3^2 \\ & +s_3^3) + (s_1 - s_2)(r_3^3 + r_3^2 s_3 + (s_1 - s_3)(s_2 - s_3)(s_1 + s_2 + s_3) - r_3(s_1^2 + s_1 s_2 + s_2^2 \\ & -s_3^2)) - r_2(r_3^3 - s_1^3 + s_1 s_2^2 + r_3^2 s_3 - s_2^2 s_3 + s_3^3 + r_3(-s_2^2 + s_3^2)),\\ B = \ & 2((s_1 - s_2)(r_3^2 + (s_1 - s_3)(s_2 - s_3) - r_3(s_1 + s_2 - s_3)) + r_1^2(r_2 - r_3 + s_2 - s_3) \\ & +r_2^2(r_3 - s_1 + s_3) + r_1(-r_2^2 + r_3^2 - r_3 s_1 + r_2(s_1 - s_2) + s_1 s_2 - s_2^2 + r_3 s_3 - s_1 s_3 \\ & +s_3^2) - r_2(r_3^2 + r_3(-s_2 + s_3) - (s_1 - s_3)(s_1 - s_2 + s_3)).\end{aligned}$$

And λ_2 has the same expression that λ_1 changing b_1 by b_2.

We replace γ_1, λ_1 and δ_1 in the expression of $H_1(x, y)$, and γ_2, λ_2 and δ_2 in the expression of $H_2(x, y)$ and we obtain $H_1(x, y) = H_2(x, y)$. So the piecewise linear differential system becomes a linear differential system, which does not have limit cycles. So the maximum number of limit cycles is two.

To complete the proof of the theorem we present a discontinuous piecewise differential system satisfying the assumptions of the theorem and having two limit cycles.

Consider the planar discontinuous piecewise linear Hamiltonian system without equilibrium points separated by the parabola $y = x^2$:

$$\dot{x} = 5.5x - 0.5y + 3, \quad \dot{y} = 60.5x - 5.5y + 0.2, \tag{29}$$

in the region R_1, its corresponding Hamiltonian function is

$$H_1(x, y) = 30.25x^2 - 5.5xy + 0.2x + 0.25y^2 - 3y.$$

The second system is

(30) $$\dot{x} = 0.2x - 0.1y - 0.778814, \quad \dot{y} = 0.4x - 0.2y + 0.00727332,$$

in the region R_2, its corresponding Hamiltonian function is

$$H_2(x, y) = 0.2x^2 - 0.2xy + 0.00727332x + 0.05y^2 + 0.778814y.$$

This piecewise differential system has the limit cycles shown in Figure 5. This completes the proof of Theorem 6. □

12. Piecewise differential system with a non-regular discontinuity line

The *extended* 16*th Hilbert problem*, that is, to find an upper bound for the maximum number of limit cycles that a given class of differential systems can exhibit, is in general an unsolved problem.

Only for very few classes of differential systems this problem has been solved.

Now we shall study the maximum number of limit cycles of piecewise differential systems separated by the the non-regular line $\mathcal{R}$ formed by the two positive half-axes x and y, and formed by two linear centers.

We denote by $\mathcal{R}_1$ the open positive quadrant of $\mathbb{R}^2$, and by $\mathcal{R}_2$ the interior of $\mathbb{R}^2 \setminus \mathcal{R}_1$.

It is known that an arbitrary linear center can be written as

(31) $$\begin{aligned} \dot{x} &= -Ax - (A^2 + \Omega^2)y + B, \\ \dot{y} &= x + Ay + C, \end{aligned} \qquad \text{for } (x, y) \in \mathcal{R}_1,$$

and

(32) $$\begin{aligned} \dot{x} &= -ax - (a^2 + \omega^2)y + b, \\ \dot{y} &= x + ay + c, \end{aligned} \qquad \text{for } (x, y) \in \mathcal{R}_2,$$

with $\Omega, \omega > 0,\ A, B, C, a, b, c \in \mathbb{R},\ A, a \neq 0$.

Each system (31) and (32) have, respectively, the first integrals

(33) $$\begin{aligned} H_1(x, y) &= (x + Ay)^2 + 2(Cx - By) + y^2\Omega^2, \\ H_2(x, y) &= (x + ay)^2 + 2(cx - by) + y^2\omega^2. \end{aligned}$$

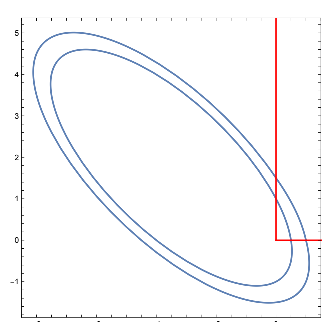

Figure 6. Both limit cycles are travelled in counter-clockwise sense.

The next result appears in [4].

Theorem 7. *Consider the discontinuous piecewise differential systems separated by the non-regular line $\mathcal{R}$ and formed by two arbitrary linear differential centers* (31)-(32). *The maximum number of limit cycles of these discontinuous piecewise linear differential systems intersecting $\mathcal{R}$ in two points is two. Moreover there exists systems with exactly two limit cycles of this type, see Figure* 6.

Proof. We consider the discontinuous planar linear differential system (31)-(32). If there exists a crossing limit cycle intersecting the non-regular separation curve $\mathcal{R}$ in two points of the form $(x,0)$ and $(0,y)$, both different from the origin. Since the functions H_1 and H_2 defined in (33) are first integrals of the systems (31) and (32) respectively, these points must satisfy the equations

$$\begin{aligned} e_1 &:= H_1(x,0) - H_1(0,y) = 2Cx + x^2 + 2By - A^2y^2 - y^2\Omega^2 = 0, \\ e_2 &:= H_2(x,0) - H_2(0,y) = 2cx + x^2 + 2by - a^2y^2 - y^2\omega^2 = 0. \end{aligned} \tag{34}$$

By Bézout theorem this system can have at most four isolated solutions, one of them being the origin.

In order to obtain crossing limit cycles equations $e_1 = 0$ and $e_2 = 0$ must have isolated solutions (x_i, y_i) with $x_i, y_i > 0$. So, there are at most three crossing limit cycles of system $e_1 = e_2 = 0$. In order that these three solutions (x_i, y_i) can produce limit cycles it is necessary that

$$0 < x_1 < x_2 < x_3 \quad \text{and} \quad 0 < y_1 < y_2 < y_3.$$

We claim that there are at most two solutions (x_1, y_1) and (x_2, y_2) providing limit cycles, that is, satisfying

$$0 < x_1 < x_2 \quad \text{and} \quad 0 < y_1 < y_2. \tag{35}$$

Now we prove the claim.

If $a^2 + \omega^2 - A^2 - \Omega^2 = 0$ then the piecewise system has at most one limit cycle. Indeed the resultant of the polynomial e_1 and e_2 with respect to the variable y is

$$\begin{aligned} 4x\,(a^2+\omega^2)\,(x\,(&a^2c^2 - 2a^2cC + a^2C^2 - b^2 + 2bB - B^2 + c^2\omega^2 - 2cC\omega^2 + C^2\omega^2) \\ &-2(b-B)(bC - Bc)). \end{aligned}$$

So at most one positive solution of x, consequently at most one limit cycle.

Assume now that $a^2 + \omega^2 - A^2 - \Omega^2 \neq 0$.

Equations $e_1 = e_2 = 0$ are equivalent to equations $E_1 = e_1 - e_2 = 0$ and $E_2 = e_1(a^2+\omega^2) - e_2(A^2+\Omega^2) = 0$, i.e.

$$
\begin{aligned}
E_1 =& 2(C-c)x + 2(B-b)y - (A^2+\Omega^2-a^2-\omega^2))y^2 = 0,\\
E_2 =& 2\big((a^2+\omega^2)C - (A^2+\Omega^2)c\big)x + 2\big((a^2+\omega^2)B - (A^2+\Omega^2)b\big)y\\
&+ (a^2+\omega^2-A^2-\Omega^2)x^2 = 0.
\end{aligned} \tag{36}
$$

If $C = c$, then $E_1 = 0$ reduces to either one horizontal straight line, or two horizontal parallel straight lines passing one of these two straight lines through the origin. The equation $E_2 = 0$ is either a parabola symmetric with respect to some vertical straight line, or one vertical straight line, or two vertical parallel straight lines passing one of these two straight lines through the origin. Since $E_1 = E_2 = 0$ pass through the origin, there are at most two intersection points satisfying (35) and so at most two limit cycles.

Assume now that $C \neq c$. In this case $E_1 = 0$ is a parabola symmetric with respect to some horizontal straight line and $E_2 = 0$ is a parabola symmetric with respect to some vertical straight line. Since both parabolas intersect at the origin, there are at most two intersection points satisfying (35) and so at most two limit cycles. This proves the claim and consequently the theorem once we provide an example with two limit cycles.

Now we give a discontinuous piecewise linear differential system (31)-(32) having exactly two limit cycles intersecting in two points the discontinuity line $\mathcal{R}$. In region $\mathcal{R}_1$ we consider the linear differential center

$$\dot{x} = -2x - 8y - \frac{3}{2}, \qquad \dot{y} = x + 2y + \frac{43}{4}, \tag{37}$$

with the first integral

$$H_1(x,y) = 4y^2 + 2\left(\frac{43}{4}x + \frac{3}{2}y\right) + (x+2y)^2;$$

and in region $\mathcal{R}_2$ we consider the linear differential center

$$\dot{x} = -x - 2y, \qquad \dot{y} = x + y + \frac{7}{4}, \tag{38}$$

with the first integral

$$H_2(x,y) = y^2 + \frac{7}{2}x + (x+y)^2.$$

In this case, the two solutions of equations (36) are

$$(x_1,y_1) = \left(\frac{1}{2}, 1\right), \qquad (x_2,y_2) = \left(1, \frac{3}{2}\right),$$

and the corresponding limit cycles are shown in Figure 6. □

References

[1] D.V. Anosov, *Stability of the equilibrium positions in relay systems*, Automation and remote control **XX**, 2, (1959); auk SSSR **14** (1937), 247–250.

[2] R. Benterki and J. Llibre, *Crossing Limit cycles of planar piecewise linear Hamiltonian systems without equilibrium points*, Mathematics **2020**(8) (2020), 755–pp. 14.

[3] F. Dumortier, J. Llibre and J.C. Artés, *Qualitative theory of planar differential systems*, UniversiText, Springer–Verlag, New York, 2006.

[4] M. Esteban, J. Llibre and C. Valls, *The extended* 16*-th Hilbert problem for discontinuous piecewise linear centers separated by a non-regular line*, to appear in 2021.

[5] A. F. Filippov, *Differential Equations with Discontinuous Right-Hand Sides*, Kluwer Academic, Dordrecht, 1988.

[6] M. Falconi and J. Llibre, $n-1$ *independent first integrals for Linear differential systems in* $\mathbb{R}^n$ *and* $\mathbb{C}^n$, Qualitative Theory and Dynamical Systems **4** (2004), 233–254.

[7] J. Llibre and M.A. Teixeira, *Periodic orbits of continuous and discontinuous piecewise linear differential systems via first integrals*, Sao Paulo J. of Math. Sci. **12** (2018), 121–135.

[8] J. Llibre and X. Zhang, *Limit cycles created by piecewise linear centers*, Chaos **29** (2019), 053116, pp. 14.

[9] D. Schlomiuk, *Algebraic particular integrals, integrability and the problem of the center*, Trans. Amer. Math. Soc. **338** (1993), 799–841.