Equivariant Euler characteristics

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Let L be something with an Euler characteristic (finite CW-complex, compact manifold, partially ordered set, ...) and G a finite group acting on L. Atiyah and Segal defined the rth (reduced) *equivariant Euler characteristic* of (L, G) as the normalized sum

$$\widetilde{\chi}_r(L, G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbf{Z}^r, G)} \widetilde{\chi}(C_L(X))$$

of the reduced Euler characteristics $\widetilde{\chi}(C_L(X))$ of the subobject of L fixed by X where X runs through all group homomorphisms of $\mathbf{Z}^r$ to G. Special cases are:

- If $G = 1$ is the trivial group, then $\widetilde{\chi}_r(L, 1) = \widetilde{\chi}(L)$ for all r
- If $L = 1$ is the one-point set, then $\chi_r(1, G) = |\mathrm{Hom}(\mathbf{Z}^r, G)|/|G|$
- If $r = 1$, then $\chi_1(G, L) = \frac{1}{|G|} \sum_{X \in G} \chi(C_L(X)) = \chi(L/G)$ by the Lefschetz formula

We shall consider two situations where equivariant Euler characteristics arise:

- Finite groups of Lie type acting on their buildings, for example $\mathrm{GL}_n^+(\mathbf{F}_q)$ acting on the poset $\mathrm{L}_n^*(\mathbf{F}_q)$ of nontrivial and proper subspaces of $\mathbf{F}_q^n$
- Let C be a finite EI-category. To every object a of C, we may associate the equivariant Euler characteristics $\chi_r([a/C], C(a))$ for the action of the automorphism group $C(a)$ of a on the poset $[a/C]$ of isomorphism classes of objects under a.

A sample result of the first item is

$$\widetilde{\chi}_{r+1}(\mathrm{L}_n^*(\mathbf{F}_q), \mathrm{GL}_n^+(\mathbf{F}_q)) = \frac{(-1)^n}{|W(A_n)|} \sum_{w \in W(A_n)} \det(w) \det(q - w)^r$$

where $W(A_n)$ is the permutation representation of the symmetric group Σ_n on $\mathbf{R}^n$.

A sample result of the second item is

$$\chi_r(\mathrm{D}_n^-(G)^*, G \wr \Sigma_n) = \binom{\chi_r(1,G)}{n}$$

where $\mathrm{D}_n^-(G)^*$ is the Dowling $G \wr \Sigma_n$-poset for the group G.

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References

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