Picard Groups of the Stable Module Category for Quaternion Groups

Richard Wong

University of Texas, United States of America.
E-mail address: richard.wong@math.utexas.edu
URL: http://ma.utexas.edu/users/richard.wong

One problem of interest in modular representation theory of finite groups is in computing the group of endo-trivial modules. In homotopy theory, this group is known as the Picard group of the stable module category. This group was originally computed by Carlson–Thévenaz using the theory of support varieties. However, jointly with Jeroen van der Meer, we provide new, homotopical proofs of their results for the quaternion group of order 8, generalized quaternion groups, and cyclic p-groups using descent methods.

Let G denote a finite group, and let k be a field of characteristic p, where p divides the order of G (i.e. the characteristic is modular). In this setting, one can study the representation theory of G over k. As p divides $|G|$, Maschke's theorem fails, which infamously implies that the structural phenomena of representation theory over modular characteristics are wildly different than the usual theory over other characteristics. Central to modular representation theory, then, is the study of the structural property of the category of kG-modules.

One particular instance of this is the problem of computing the group of **endotrivial modules**

$$T(G) := \left\{ M \in \mathsf{Mod}(kG) \mid \mathsf{End}_k(M) \cong k \oplus P \right\}.$$

That is, the $k[G]$-modules M such that the endomorphism module decomposes as the direct sum of k, the trivial kG-module, and a projective kG-module P. This forms a group under tensor product. The group of endotrivial modules was first studied by Dade for elementary abelian groups $G \cong (C_p)^n$, who regarded endotrivial modules as a stepping stone towards the study of the more general endopermutation modules. Endotrivial modules over p-groups were later classified by work of Carlson and Thévenaz using purely representation-theoretic techniques, such the theory of support varieties, cf. for instance [1]. The classification for arbitrary finite groups is an active problem that has been studied by numerous people.

The group of endotrivial modules can be approached through homotopy theory. This is the approach taken in [2], and it's the approach we too will make profound use of throughout this paper. In Section 2, we realise the group of endotrivial modules as the Picard group of the stable module ∞-category. The latter admits a decomposition, which implies a corresponding decomposition of the Picard space; the resulting decomposition is then shown to be amenable to spectral sequence techniques.

In certain cases, the decomposition of the stable module ∞-category can be viewed through the lens of Galois theory. We take up this topic in Section 3 and we use a result of Rognes to give new proofs of the decomposition for cyclic p-groups and quaternion groups.

Finally, in Section 4 we evaluate the limit spectral sequences associated to the decomposition of the stable module category to explicitly compute the group of endotrivial modules for cyclic p-groups and generalized quaternion groups. Although these groups have already been computed, the method given here is entirely new. In particular, our approach allows for a new interpretation of the fact that the group of endotrivial modules over Q_8 depends on the arithmetic structure of the base field; we shall see that it arises naturally from a certain nonlinear differential in the limit spectral sequence.

References

[1] Doyle, J. and J. Carlson, (2000), Power Laws, Highly Optimized Systems, and Generalized Source Coding, Physical Review Letters, 84, 24, 5656-ŋ5659.

[2] Grodal, J., (2018), "Endotrivial modules for finite groups via homotopy theory". `arXiv:1608.00499 [math.GR]`