Typical behaviour of boundary points of Fatou components of transcendental maps

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Setup

We consider transcendental entire of meromorphic maps

$$f : \mathbb{C} \to \overline{\mathbb{C}}.$$

Basic notation

- **Julia set**
  $J(f) = \{z \in \overline{\mathbb{C}} : \{f^n\}_{n \geq 0} \text{ not normal in any nbhd of } z\}$
- **Fatou set**
  $\mathcal{F}(f) = \overline{\mathbb{C}} \setminus J(f)$
- **Invariant Fatou component**
  U connected component of $\mathcal{F}(f)$ with $f(U) \subset U$.
- **Bounded orbit set**
  $K(f) = \{z \in \mathbb{C} : \{f^n(z)\}_{n=0}^{\infty} \text{ bounded}\}$
- **Escaping set**
  $I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}$
- ∂ = boundary in $\mathbb{C}$, $\dim_H$ = Hausdorff dimension.

Part I: Parabolic and Baker domains

Let U be an invariant Fatou component. We assume that U is simply connected (this always holds if f is entire).

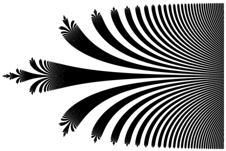
Definition

- U is **parabolic** if there is $p \in \partial U$ with $f(p) = p$, $f'(p) = 1$ and $f^n|_U \to p$ as $n \to \infty$.
- U is **Baker** if $f^n|_U \to \infty$ as $n \to \infty$.

In particular, parabolic domains are in $K(f)$ and Baker domains are in $I(f)$. However, their boundary points can have different dynamical behaviour.

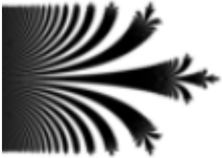
Open question

Is $\partial U \cap I(f)$ non-empty for every invariant Baker domain?



Invariant parabolic basin
for $f(z) = \frac{1}{e}e^z$

[Devaney 1984]



Invariant Baker domain
for $f(z) = z + 1 + e^{-z}$

[Fatou 1926]

Inner function associated to f on U

Let

$$\varphi \colon \mathbb{D} \to U$$

be a **Riemann map**. Then

$$g \colon \mathbb{D} \to \mathbb{D}, \qquad g = \varphi^{-1} \circ f \circ \varphi$$

is the **inner function** associated to f on U.

We have $\lim_{r \to 1^-} g(r\theta) \in \partial\mathbb{D}$ for almost every $\theta \in \partial\mathbb{D}$.

Furthermore,

- if $\deg f|_U < \infty$, then g is a finite Blaschke product and extends to $\overline{\mathbb{C}}$,
- if $\deg f|_U = \infty$, then g has at least one **singular point** in $\partial\mathbb{D}$ (where g does not extend through $\partial\mathbb{D}$).

Since g has no fixed points in $\mathbb{D}$, by the Denjoy–Wolff Theorem, $g^n \to \zeta$ as $n \to \infty$ for a point $\zeta \in \partial\mathbb{D}$ (**Denjoy–Wolff** point of g).

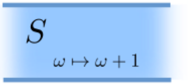
Baker–Pommerenke–Cowen classification

Theorem

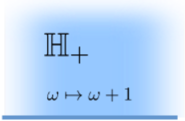
There exists a holomorphic map $\psi\colon \mathbb{D} \to \Omega$ *such that* $\psi \circ g = T \circ \psi$, *where* $T(\omega) = \omega + 1$ *and one of the three following cases holds:*

- $\Omega = S = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < c\},\ c > 0$ **hyperbolic type**
- $\Omega = \mathbb{H}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ **simply parabolic type**
- $\Omega = \mathbb{C}$ **doubly parabolic type**

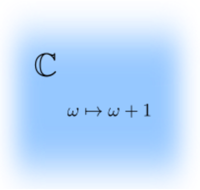
Moreover, ψ *is univalent on an* **absorbing domain** $V \subset \mathbb{D}$ *(where* $g(V) \subset V$ *and* $\bigcup_{n=1}^{\infty} g^{-n}(V) = \mathbb{D}$*).*



hyperbolic



simply parabolic



doubly parabolic

Characterization of the three types

Let ρ_U be the **hyperbolic metric** in U. By the Schwarz–Pick Lemma, the sequence $\rho_U(f^{n+1}(z), f^n(z))$ is decreasing for $z \in U$, so $\lim_{n\to\infty} \rho_U(f^{n+1}(z), f^n(z)) = A(z)$ for some $A(z) \geq 0$.

Theorem

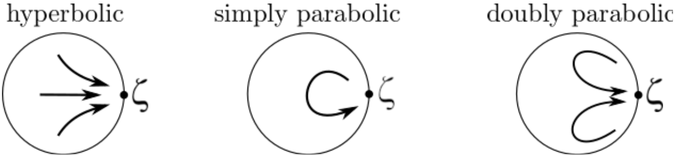
- *$f|_U$ has hyperbolic type* $\iff \inf_{z\in U} A(z) > 0$.
- *$f|_U$ has simply parabolic type* $\iff A(z) > 0$ *and* $\inf_{z\in U} A(z) = 0$.
- *$f|_U$ has doubly parabolic type* $\iff A(z) = 0$.

Furthermore, if the Denjoy–Wolff point ζ is non-singular for g, then:

- *$f|_U$ has hyperbolic type $\iff \zeta$ is an attracting fixed point of g*
- *$f|_U$ has simply parabolic type $\iff \zeta$ is a simply parabolic fixed point of g*
- *$f|_U$ has doubly parabolic type $\iff \zeta$ is a doubly parabolic fixed point of g.*

Definition

- ζ is a **simple parabolic fixed point** of g if $g(z) = z + a(z-\zeta)^2 + \cdots$, $a \neq 0$, z near ζ.
- ζ is a **double parabolic fixed point** of g if $g(z) = z + a(z-\zeta)^3 + \cdots$, $a \neq 0$, z near ζ.



Remark

If U is a parabolic domain, then $f|_U$ has doubly parabolic type (even if p is a simply parabolic fixed point for f). This follows from the existence of Fatou coordinates and the characterization theorem.

Typical behaviour of points in ∂U

Let U be an invariant parabolic or Baker domain and let ω be a **harmonic measure** on ∂U (i.e. $\omega = \varphi_* \lambda$, where λ is the normed Lebesgue measure on $\partial \mathbb{D}$).

Theorem (Fagella–Jarque–Karpińska–B 2019)

Assume that the Denjoy–Wolff point ζ is non-singular for g (this holds e.g. when $\deg f|_U < \infty$).

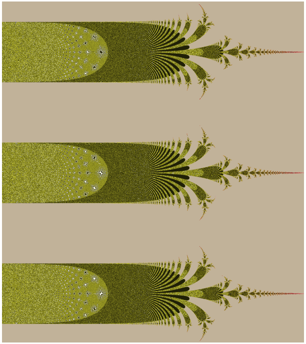
- *If U is a Baker domain and $f|_U$ has hyperbolic or simply parabolic type, then ω-almost every point of ∂U is in $I(f)$.*
- *If U is a parabolic domain, or U is a Baker domain and $f|_U$ has doubly parabolic type, then ω-almost every point of ∂U has dense trajectory in ∂U.*

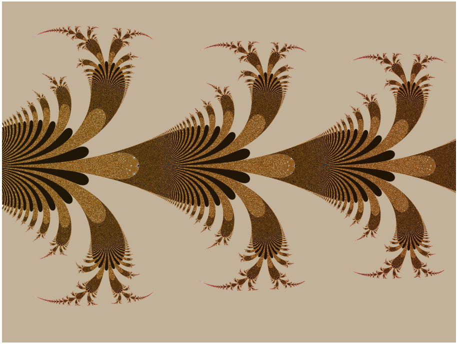
Remark

The case of univalent Baker domains (then $f|_U$ has hyperbolic or simply parabolic type) was proved in [Rippon–Stallard 2018].
The case of parabolic domains was proved in [Doering–Mañé 1991, Aaronson–Denker–Urbański 1993] in the context of rational maps.

Example

The map $f(z) = z + e^{-z}$, studied in [Baker–Domínguez 1999] has infinitely many invariant Baker domains of doubly parabolic type.





The set of escaping points in the boundaries of these domains has harmonic measure zero.

Theorem (Fagella–Jarque–Karpińska–B 2019)

Let U be an invariant parabolic or Baker domain, such that

$$\rho_U(f^{n+1}(z), f^n(z)) \leq \frac{1}{n} + O\left(\frac{1}{n^r}\right), \quad r > 1$$

for some $z \in U$. Then ω-almost every point of ∂U has dense trajectory in ∂U.

Example (Aaronson 1981)

Let

$$f\colon \mathbb{C} \to \overline{\mathbb{C}}, \qquad f(z) = z - \sum_{n=0}^{\infty} \frac{2z}{z^2 - n^\delta,} \qquad \delta \in (1, 2)$$

Then $U = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ is an invariant Baker domain,

$$\frac{c_1}{n} \leq \rho_U(f^{n+1}(z), f^n(z)) \leq \frac{c_2}{n} \quad \text{for } z \in U$$

and ω-almost every point of ∂U is in $I(f)$.

A family of examples

Proposition

Let f be a meromorphic map of the form

$$f(z) = z + a + h(z),$$

where $a \in \mathbb{C} \setminus \{0\}$ *and*

$$|h(z)| < \frac{c}{(\operatorname{Re}(z/a))^r} \quad \text{for } \operatorname{Re}\left(\frac{z}{a}\right) > c_1, \quad r > 1.$$

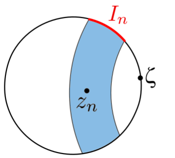
Then f has an invariant Baker domain U containing a half-plane. If U is simply connected (e.g. if f is entire), then f on U satisfies the assumptions of the previous theorem and hence ω*-almost every point of* ∂U *has dense trajectory in* ∂U*.*

Fatou's example

The map $f(z) = z + 1 + e^{-z}$ satisfies the above assumptions.

Idea of proof — hyperbolic and simply parabolic case

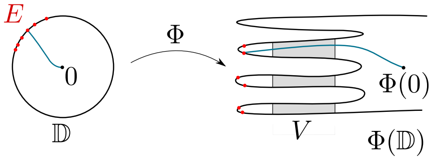
The map g is defined almost everywhere on $\partial\mathbb{D}$ in the sense of radial limits. Let $\zeta \in \partial\mathbb{D}$ be the Denjoy-Wolff point and let $I \subset \partial\mathbb{D}$ be an arc. Set $I_n = g^n(I)$.



Let $z_n = g^n(z_0)$ for a suitable $z_0 \in \mathbb{D}$. Then $z_n \to \zeta$ and $f^n(\varphi(z_0)) = \varphi(z_n) \to \infty$. We show that the set E of points $w \in I_n$ such that $f^n(\varphi(w)) = \varphi(g^n(w))$ is far from $\varphi(z_n)$ has small measure (this gives $f^n(\varphi(w)) \to \infty$ for almost every $w \in I$).

To do so, we use the following **Pflüger-type estimate**:

If for a Riemann map $\Phi\colon \mathbb{D} \to \Phi(\mathbb{D})$, the image of every curve joining 0 to a set $E \subset \partial\mathbb{D}$ has to pass through long 'corridors' in $V \subset U$ of small area, then the measure of E is small.



Idea of proof — doubly parabolic case

Fact

The map g on $\partial\mathbb{D}$ preserves the infinite measure

$$\mu(E) = \int_E \frac{d\lambda(w)}{|w-\zeta|^2}.$$

We use results of **ergodic theory of inner functions** established in [Aaronson 1978–1981, Doering–Mañé 1991].

- If $\sum_{n=1}^{\infty}(1-|g^n(z)|) = \infty$ for some $z \in \mathbb{D}$, then g on $\partial\mathbb{D}$ is **conservative** with respect to the Lebesgue measure.
- The map g on $\partial\mathbb{D}$ is **exact** with respect to the Lebesgue measure $\iff$ g has doubly parabolic type.
- If g on $\partial\mathbb{D}$ is **non-singular**, conservative and **ergodic**, then for every $E \subset \partial\mathbb{D}$ of positive Lebesgue measure, the trajectory of almost every point of $\partial\mathbb{D}$ visits E infinitely many times.

Part II: Attracting basins for exponential maps

Let

$$E_\lambda(z) = \lambda \exp(z)$$

for $z \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \{0\}$.

The point 0 is the unique singular (asymptotic) value of E_λ.

Assumption

We assume that E_λ is **hyperbolic**. Then there exists an **attracting periodic point** of (minimal) period $p \geq 1$, i.e. a point $z_0 \in \mathbb{C}$ with

$$E_\lambda^p(z_0) = z_0, \quad |(E_\lambda^p)'(z_0)| < 1.$$

Let

$$B = \{z \in \mathbb{C} : E_\lambda^{pn}(z) \xrightarrow[n \to \infty]{} E_\lambda^j(z_0) \text{ for some } j \in \{0, \ldots, p-1\}\}$$

be the **entire basin of attraction** of this cycle. Then

$$J(E_\lambda) = \mathbb{C} \setminus B = \partial B, \qquad 0 \in B.$$

Case $p = 1$

- B consists of a unique simply connected component U.
- [McMullen 1987]

$$\dim_H(\partial U) = \dim_H(\partial U \cap I(E_\lambda)) = \dim_H(I(E_\lambda)) = 2.$$

- [Karpińska 1999, Urbański–Zdunik 2003]

$$1 < \dim_H(\partial U \cap K(E_\lambda)) \leq \dim_H(\partial U \setminus I(E_\lambda)) < 2.$$

Corollary

In case $p = 1$ a (dimensionally) typical point of ∂U is escaping.

Case $p > 1$

- B consists of infinitely many simply connected components.
- [McMullen 1987] $\dim_H(\partial B) = \dim_H(\partial B \cap I(E_\lambda)) = 2$.

Theorem (Karpińska–Zdunik–B 2009)

If $p > 1$, then for every connected component U of B,

- $1 < \dim_H(\partial U) < 2$.
- $1 < \dim_H(\partial U \cap K(E_\lambda)) \le \dim_H(\partial U \setminus I(E_\lambda)) < 2$,
- $\dim_H(\partial U \cap I(E_\lambda)) = 1$.

Corollary

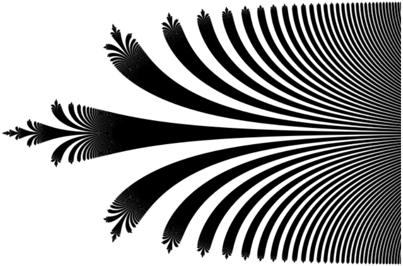
The set of **buried points** *(i.e. points in $J(E_\lambda)$ outside the boundary of any Fatou component) has Hausdorff dimension 2, while the set of non-buried points has dimension smaller than 2. Hence, in case $p > 1$ a (dimensionally) typical point of $J(E_\lambda)$ is buried.*

Corollary

In case $p > 1$ a (dimensionally) typical point of ∂U is non-escaping.

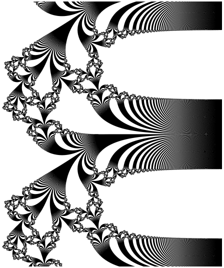
Topological properties of the Julia set — case $p = 1$

- $J(E_\lambda)$ consists of disjoint **hairs** homeomorphic to $[0, \infty)$, tending to ∞, composed of points with given symbolic itineraries [Devaney et al. 1984–1999].
- All points of hairs except their **endpoints** are in $I(E_\lambda)$ and form a set of Hausdorff dimension 1 [Karpińska 1999].
- The Julia set is homeomorphic to a **straight brush** [Aarts–Oversteegen 1993].



Topological properties of the Julia set — case $p > 1$

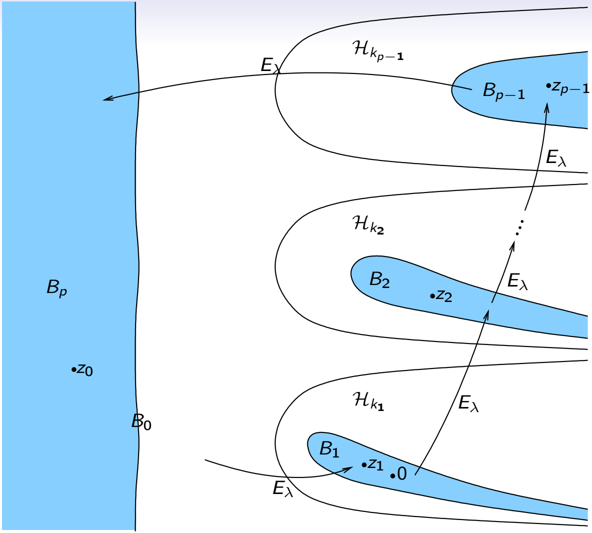
- $J(E_\lambda)$ consists of hairs with the same properties, but some hairs have common endpoints.
- The Julia set is homeomorphic to a **modified straight brush** [Bhattacharjee, Devaney et al. 2001].



Combinatorial description (Bhattacharjee–Devaney)

Structure of the attracting basin

- $z_0, \ldots, z_p = z_0$ attracting cycle
- $U_0, \ldots, U_p = U_0$ components of B with $z_j \in U_j$, $0 \in U_1$.
- $B_{p+1} \subset U_1$ Jordan domain with $0, z_1 \in B_{p+1}$, $\overline{E_\lambda^p(B_{p+1})} \subset B_{p+1}$
- $B_p = E_\lambda^{-1}(B_{p+1})$
- B_j component of $E_\lambda^{-1}(B_{j+1})$ with $z_j \in B_j$, $j = p-1, \ldots, 0$



$$B_1 \xrightarrow[1-1]{E_\lambda} B_2 \xrightarrow[1-1]{E_\lambda} \cdots \xrightarrow[1-1]{E_\lambda} B_{p-1} \xrightarrow[1-1]{E_\lambda} B_p \Subset B_0 \xrightarrow[\text{univ. cover}]{E_\lambda} B_1 \setminus \{0\}$$

Symbolic coding

$$\mathbb{C}\backslash B_0 = \bigcup_{s\in\mathbb{Z}} \mathcal{H}_s, \qquad \mathcal{H}_s = \mathcal{H}_0 + 2\pi i s \quad \text{disjoint}, \qquad J(E_\lambda) \subset \bigcup_{s\in\mathbb{Z}} \mathcal{H}_s$$

$$\mathcal{L} = \mathbb{C} \setminus \bigcup_{j=1}^{p} B_j \quad \text{simply connected, away from } 0$$

$$g_s : \mathcal{L} \to \mathcal{H}_s \quad \text{branches of } E_\lambda^{-1}, \qquad g_s(\mathcal{L}) \subset \mathcal{L}$$

Corollary

$$J(E_\lambda) = \bigcap_{n=0}^{\infty} \bigcup_{s_0,\ldots,s_n\in\mathbb{Z}} g_{s_n} \circ \cdots \circ g_{s_0}(\mathcal{L}).$$

Each $z \in J(E_\lambda)$ has **itinerary** $(s_0, s_1, \ldots) \in \mathbb{Z}^{\mathbb{N}}$*, defined by* $E_\lambda^n(z) \in \mathcal{H}_{s_n}$ *for every* $n \geq 0$.

Symbolic description of ∂U_0

Definition

- $\underline{k} = (k_1, \ldots, k_{p-1})$, where $k_j \in \mathbb{Z}$ such that $z_j \in U_j \subset \mathcal{H}_{k_j}$
- $\Sigma_{\underline{k}} = \{(s_0, \underline{k}, s_1, \underline{k}, \ldots) : s_0, s_1, \ldots \in \mathbb{Z}\}$
- $\Sigma'_{\underline{k}} = \{(s_0, \underline{k}, s_1, \underline{k}, \ldots) : s_0, s_1, \ldots \in \mathbb{Z}$ and $s_n \notin \{k_1, \ldots, k_{p-1}\}$ for infinitely many $n\}$

Expanding property

For every $z \in \mathcal{L}$, $n \geq 0$, $|(g_{s_0} \circ \cdots \circ g_{s_n})'(z)| < cq^n$, where $q < 1$.

Lemma

$$\{z \textit{ has itinerary in } \Sigma'_{\underline{k}}\} \subset \partial U_0 \subset \{z \textit{ has itinerary in } \Sigma_{\underline{k}}\}.$$

Proof.

By the combinatorics of U_j and expanding property. □

Plan of the proof

(1) $\dim_H(\{z \text{ has itinerary in } \Sigma_{\underline{k}}\} \cap I(E_\lambda)) = 1$.
(2) $\dim_H(\{z \text{ has itinerary in } \Sigma'_{\underline{k}}\} \cap K(E_\lambda)) > 1$.
(3) $\dim_H(J(E_\lambda) \setminus I(E_\lambda)) < 2$ [Urbański–Zdunik 2003].

Proof of (1).
Instead of $\Sigma_{\underline{k}}$, we can take

$$\tilde{\Sigma}_{\underline{k}} = \{(\underline{k}, s_1, \underline{k}, s_2, \ldots) : s_1, s_2, \ldots \in \mathbb{Z}\}.$$

We have $\dim_H \geq 1$, since the set contains curves (hairs).
To get $\dim_H \leq 1$, it is sufficient to prove:

Proposition

Fix $\delta > 0$ and let

$$A_M = \{z \textit{ has itinerary in } \tilde{\Sigma}_{\underline{k}} \textit{ and } \operatorname{Re}(E_\lambda^n(z)) > M \textit{ for every } n \geq 0\}.$$

Then $\dim_H(A_M) < 1 + \delta$ for sufficiently large M.

Proof of Proposition

Inductive construction of covers of A_M

$$\mathcal{K}_0 = \{\text{comp. of } \mathcal{L} \cap \mathcal{H}_{k_1} \cap \{r \leq \mathrm{Re}(z) < r+1\}, r \in \mathbb{N}, r \geq M\}$$
$$\mathcal{K}_{n+1} = \{G_{\underline{k},s}(K) : K \in \mathcal{K}_n, s \in \mathbb{Z} \text{ and } G_{\underline{k},s}(K) \cap \{\mathrm{Re}(z) \geq M\} \neq \emptyset\}$$
$$\text{for } G_{\underline{k},s} = g_{k_1} \circ \cdots \circ g_{k_{p-1}} \circ g_s.$$

Fact

$\mathcal{K}_n$ is a sequence of covers of A_M with diameters shrinking to 0.

Estimate of the $(1+\delta)$-Hausdorff measure of A_M

Lemma

For $z \in \bigcup \mathcal{K}_n$,

$$|G'_{\underline{k},s}(z)| < \frac{c}{\mathrm{Re}(z)|s|} \leq \frac{c}{M|s|}.$$

Corollary

$$\sum_{K \in \mathcal{K}_n} (\mathrm{diam}\, K)^{1+\delta} = \sum_{K_0 \in \mathcal{K}_0} \sum_{s_1,\dots,s_n \in \mathbb{Z}} (\mathrm{diam}\, G_{\underline{k},s_n} \circ \cdots \circ G_{\underline{k},s_1}(K_0))^{1+\delta}$$

$$\leq c' \sum_{r=M}^{\infty} \sum_{s_1,\dots,s_n \in \mathbb{Z}} \left(\frac{c}{r|s_n|} \prod_{j=1}^{n-1} \frac{c}{M|s_j|} \right)^{1+\delta}$$

$$\leq c' \sum_{r=M}^{\infty} \frac{1}{r^{1+\delta}} \left(\frac{\sum_{s \in \mathbb{Z}} \left(\frac{c}{|s|}\right)^{1+\delta}}{M^{1+\delta}} \right)^n \leq c' \sum_{r=M}^{\infty} \frac{1}{r^{1+\delta}} \leq \frac{c''}{M^{\delta}} < 1.$$

Proof of (2)

Idea

Construct a finite conformal IFS with a compact E_λ^{2p}-invariant repeller X, check that the topological pressure at 1 is positive and use Bowen's formula.

Construction

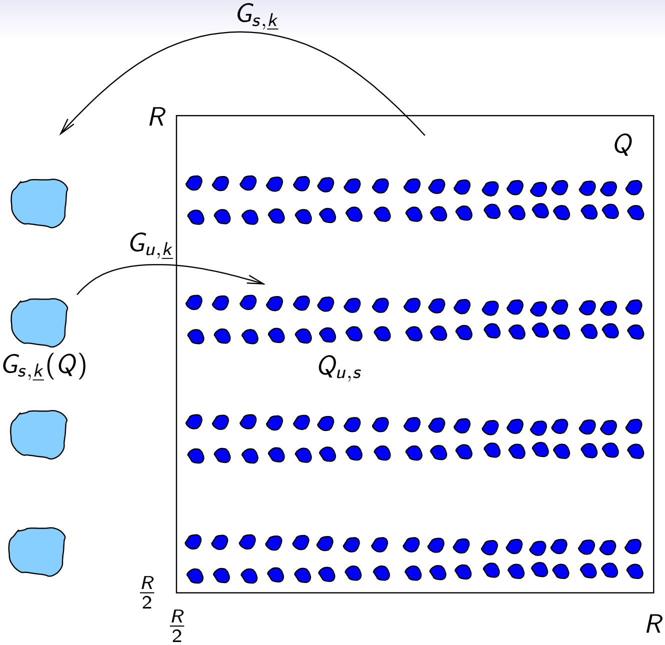
$$Q = \left[\frac{R}{2}, R\right] \times \left[\frac{R}{2}, R\right] \qquad R \text{ large}$$

$$Q_{u,s} = G_{u,\underline{k}} \circ G_{s,\underline{k}}(Q), \qquad s, u \in \mathbb{Z}$$

$$\mathcal{G} = \{(u,s) : \ Q_{u,s} \subset Q\}$$

$$X = \bigcap_{n=1}^{\infty} \bigcup_{s_1,u_1,\ldots,s_n,u_n} (G_{u_n,\underline{k}} \circ G_{s_n,\underline{k}}) \circ \cdots \circ (G_{u_1,\underline{k}} \circ G_{s_1,\underline{k}})(Q),$$

where $(u_1, s_1), \ldots, (u_n, s_n) \in \mathcal{G}$.



$G_{s,\underline{k}}$
R
Q
$G_{u,\underline{k}}$
$G_{s,\underline{k}}(Q)$
$Q_{u,s}$
$\frac{R}{2}$
$\frac{R}{2}$
R

Estimate of the topological pressure

- $|G'_{s,\underline{k}}| > \dfrac{\text{const}}{R(\ln R)^{p-1}}$
- If $\mathcal{U} = \{u : 2\pi u \in [R/2 + c, R - c]\}$, then $\#\mathcal{U} \asymp R$
- For every $u \in \mathcal{U}$, $\displaystyle\sum_{s:(u,s)\in\mathcal{G}} |G'_{u,\underline{k}}| > \text{const } R$

Corollary

$$\sum_{(u,s)\in\mathcal{G}} |(G_{u,\underline{k}} \circ G_{s,\underline{k}})'| > const \, \frac{R^2}{R(\ln R)^{p-1}} \xrightarrow[R\to\infty]{} \infty$$

Conclusion

$$P(1) \geq \lim_{n\to\infty} \frac{1}{n} \ln \left(\inf_{z\in Q} \sum_{(u,s)\in\mathcal{G}} |(G_{u,\underline{k}} \circ G_{s,\underline{k}})'(z)| \right)^n > 0.$$

Work in progress
(Karpińska–Martí-Pete–Pardo Simón–Zdunik–B)

Prove $\dim_H(\partial U) > 1$ for simply connected invariant attracting basins U for more general classes of transcendental maps f.

Motivation:

Theorem (Przytycki–Urbański–Zdunik∼1990)

For a simply connected invariant attracting basin U *of a rational map, either* $\dim_H(\partial U) > 1$ *or* ∂U *is an analytic curve.*

Thank you for attention!