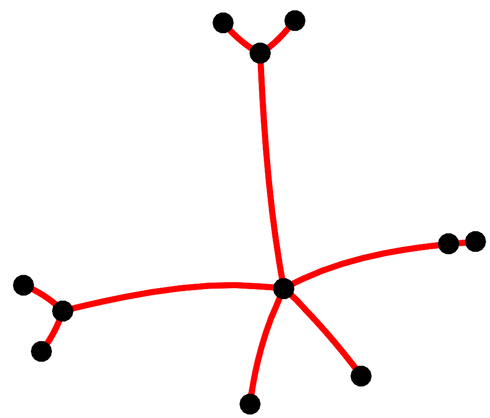
TREES, TRIANGLES AND TRACTS I

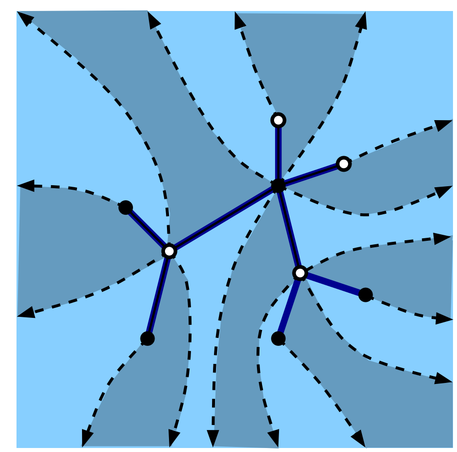
Christopher Bishop, Stony Brook

TOPICS IN COMPLEX DYNAMICS 2021

Transcendental dynamics and beyond, April 19-23, 2021

www.math.sunysb.edu/~bishop/lectures





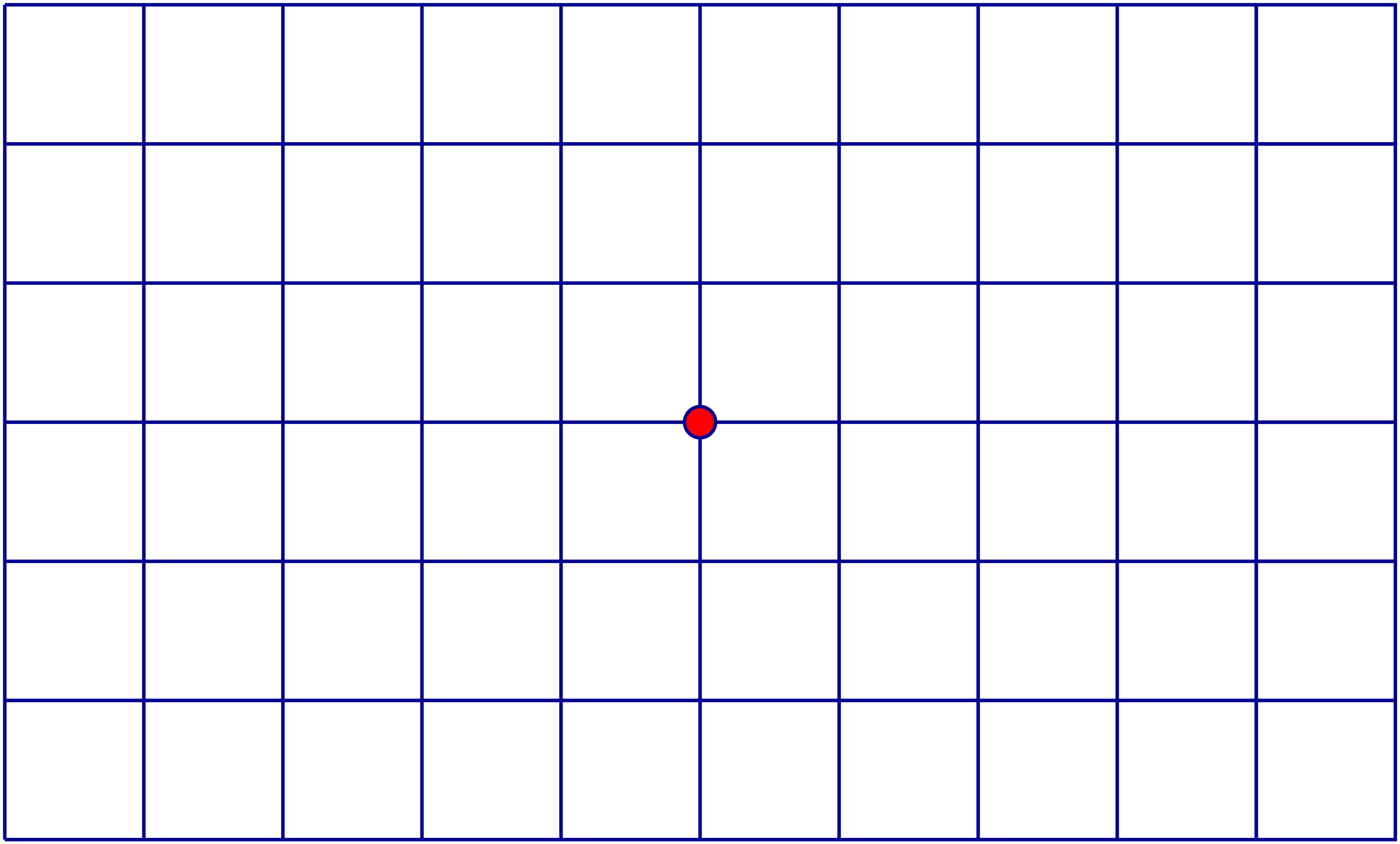
THE PLAN

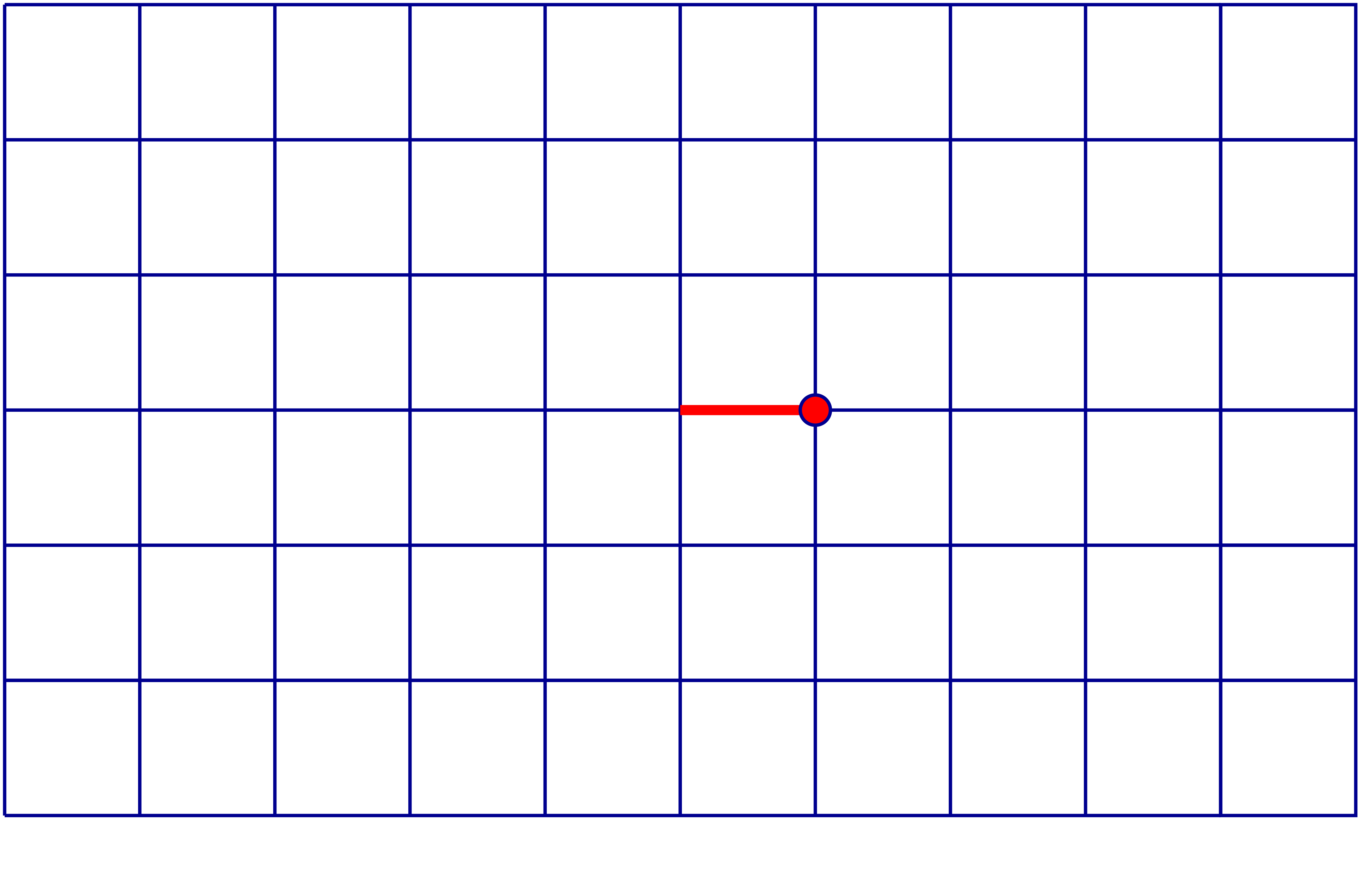
Lecture 1: Finite trees and Riemann surfaces

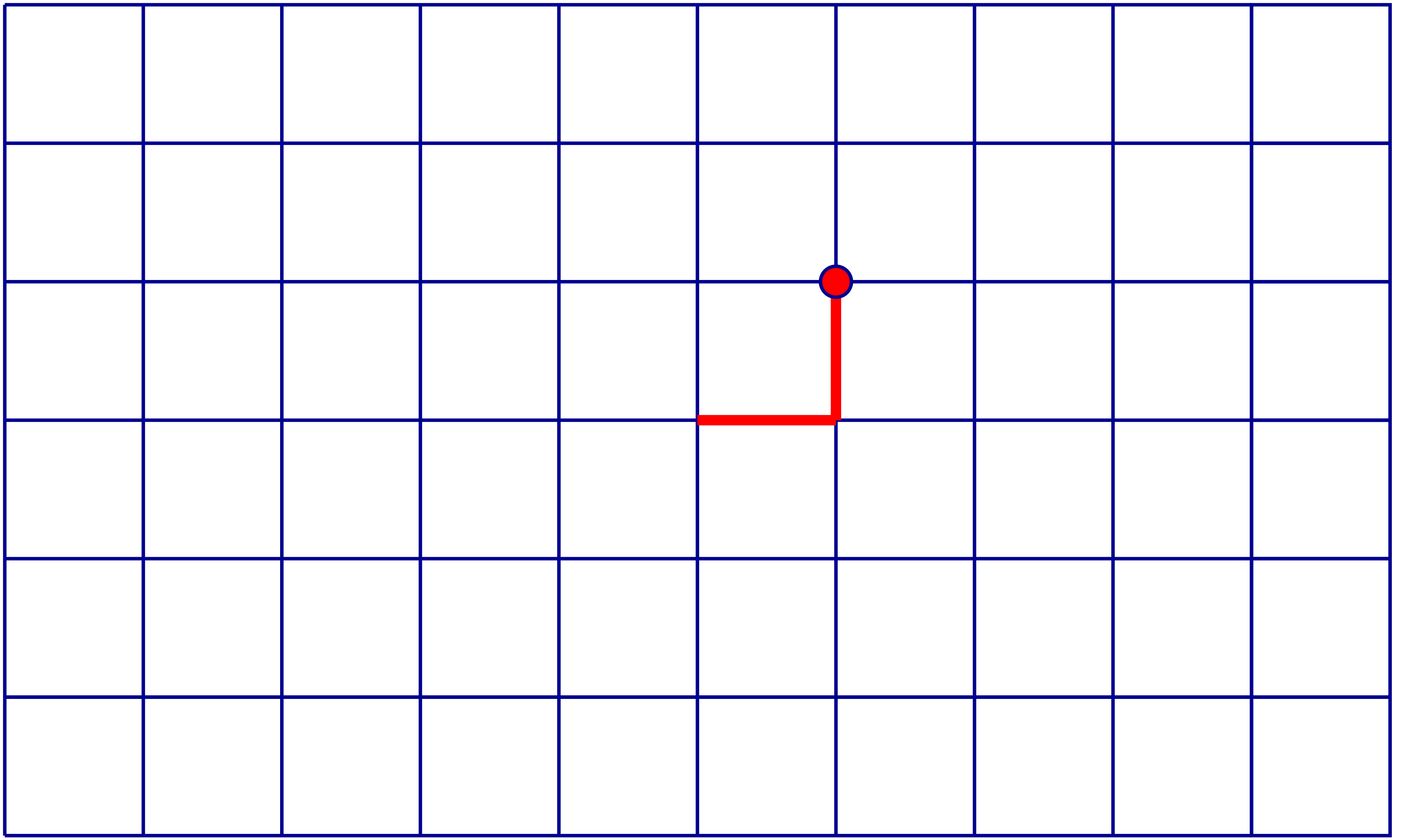
- Harmonic measure
- Finite trees and Shabat polynomials
- True trees exist: 2 proofs
- Possible shapes of a true tree.
- Surfaces with an equilateral triangulation.

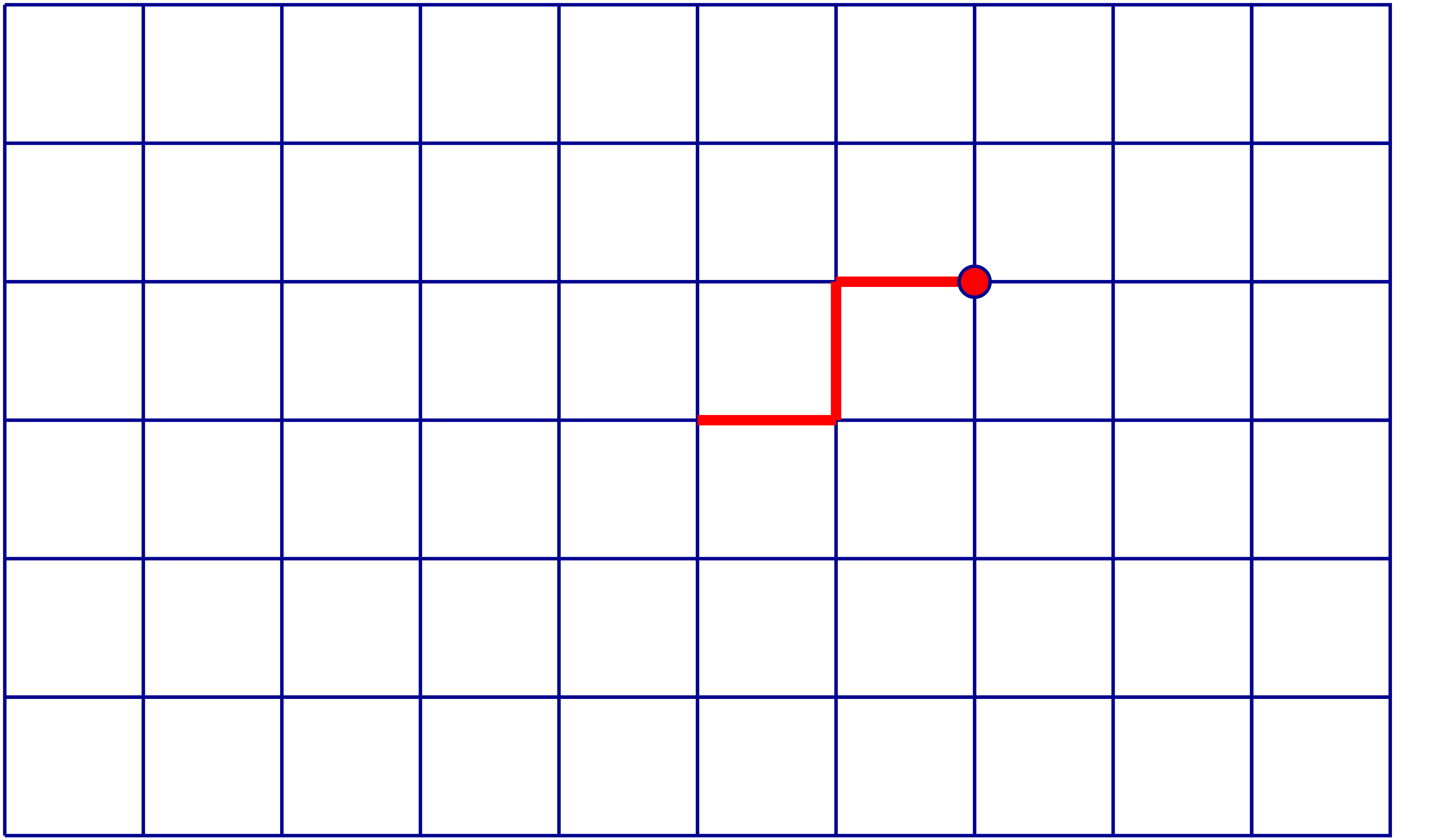
Lecture 2: Infinite trees and transcendental dynamics

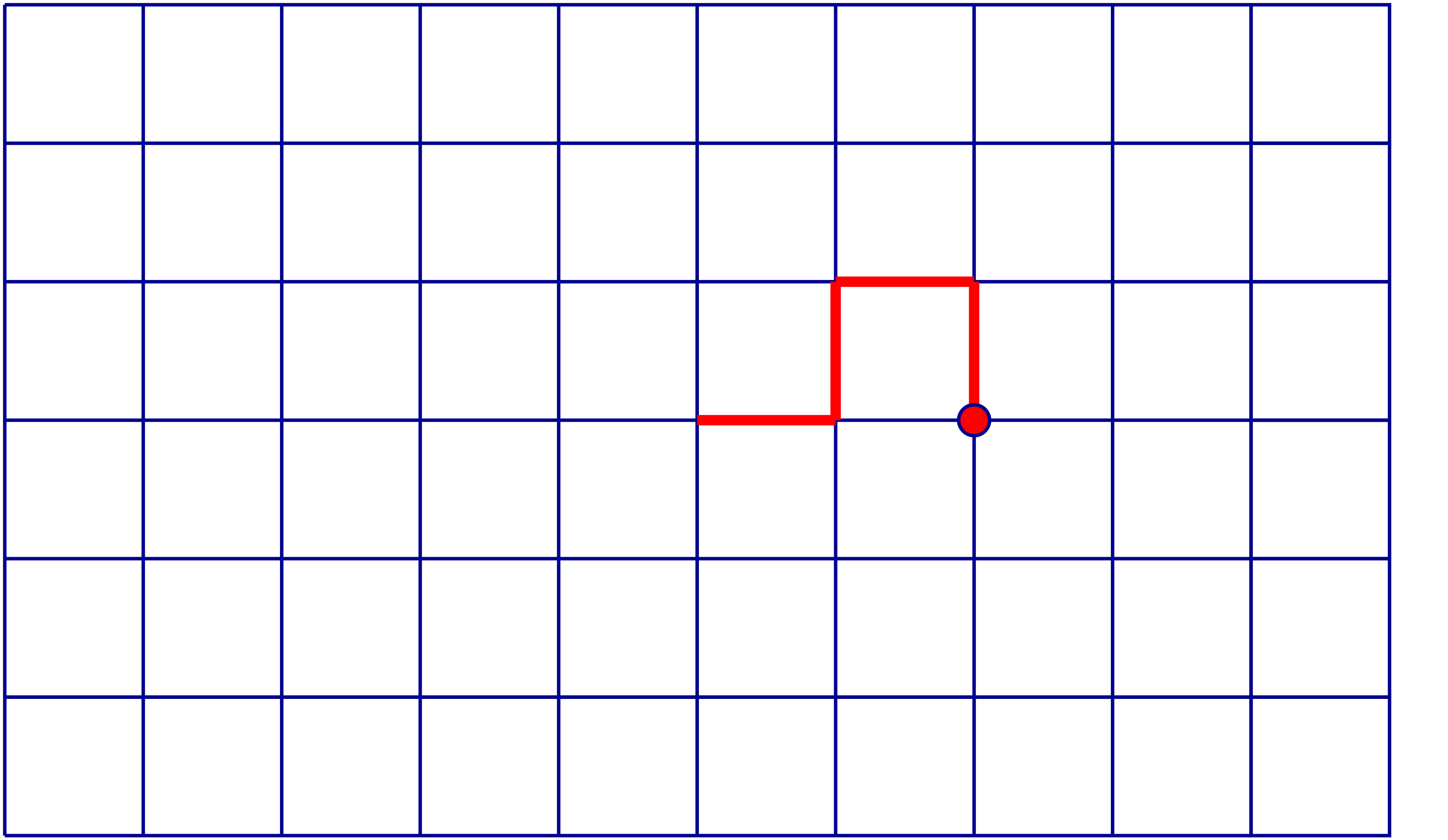
- Infinite trees and entire functions
- The folding theorem
- Application: wandering domains
- Application: Julia sets with small dimension
- Application: prescribing post-singular dynamics

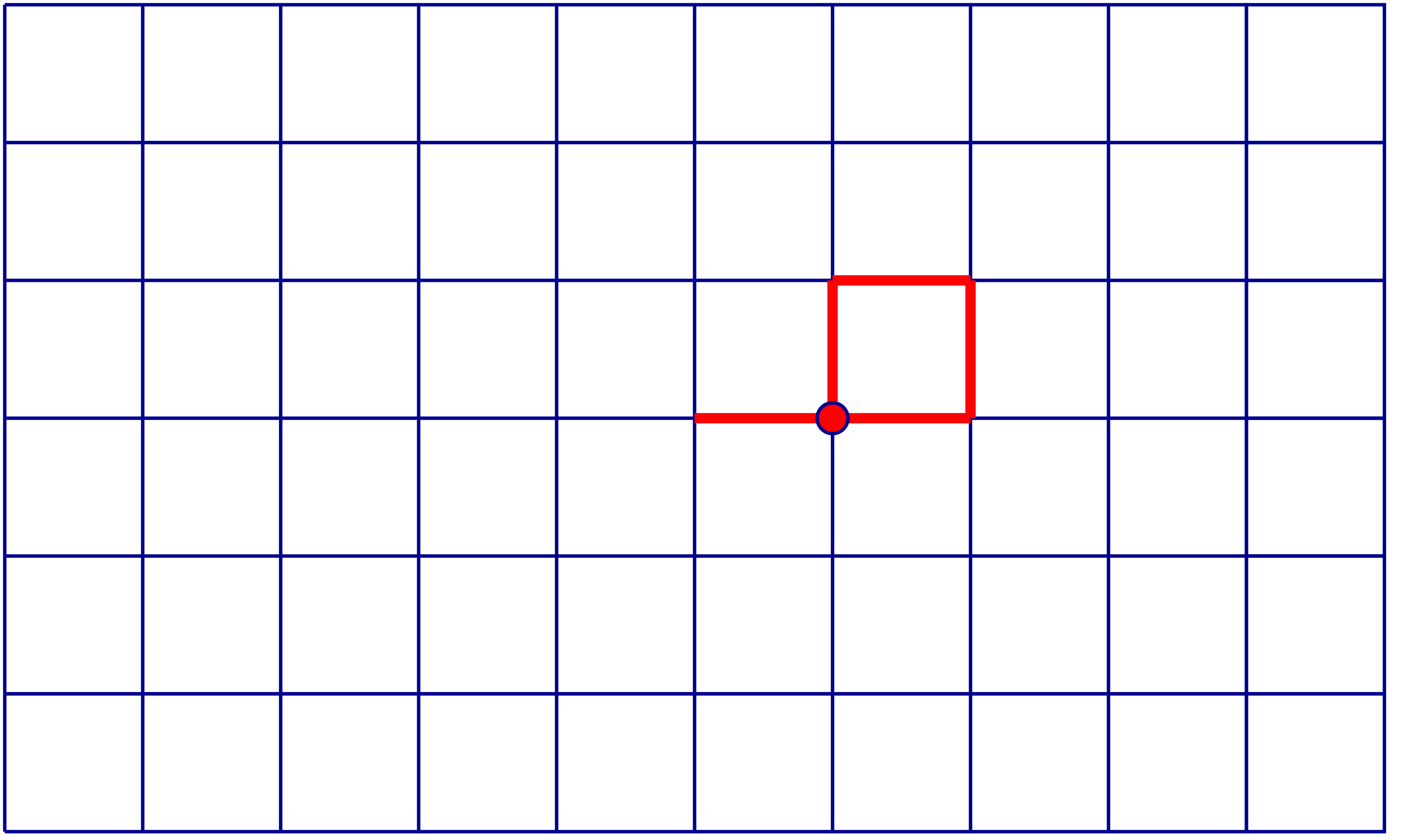


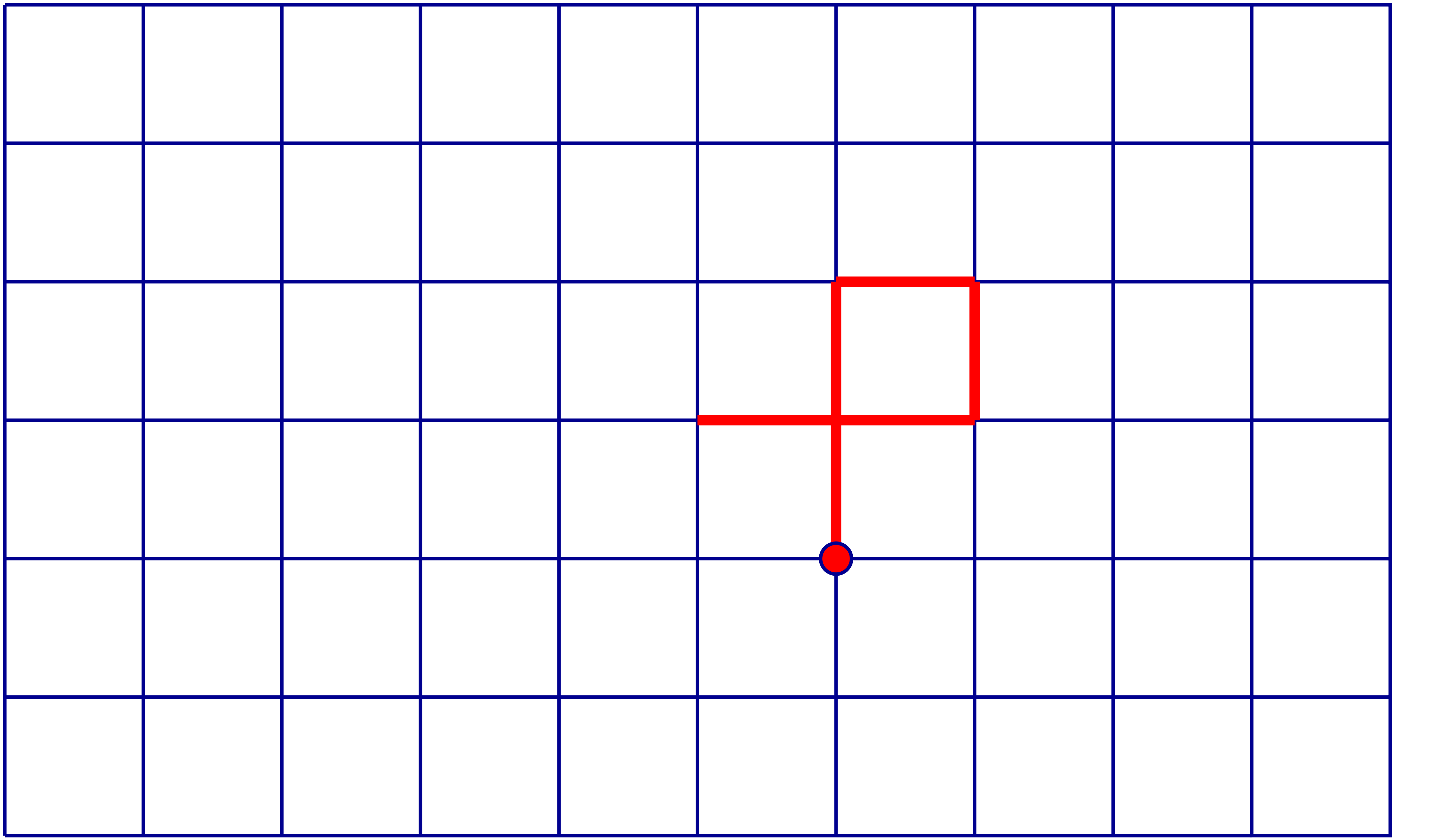


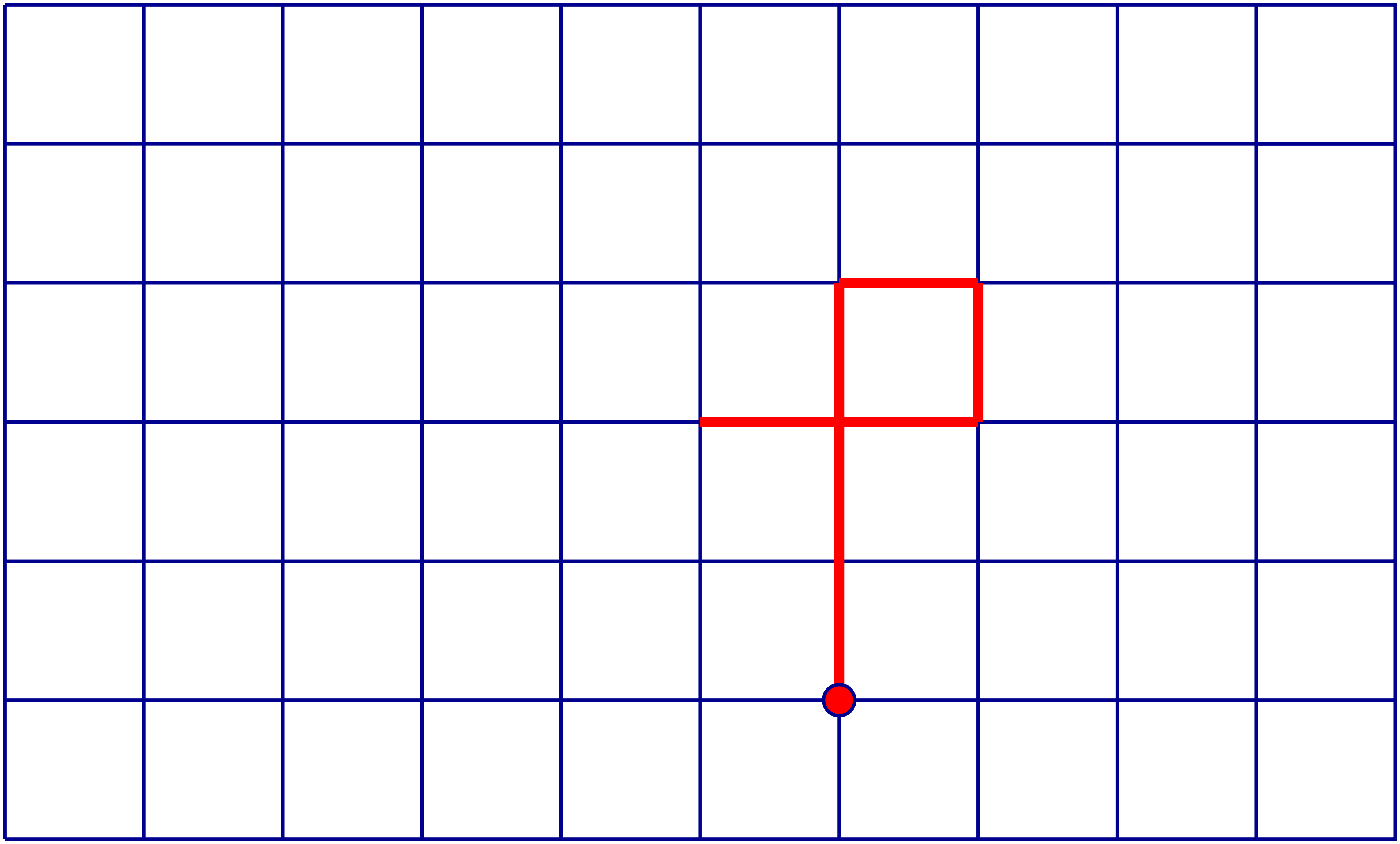


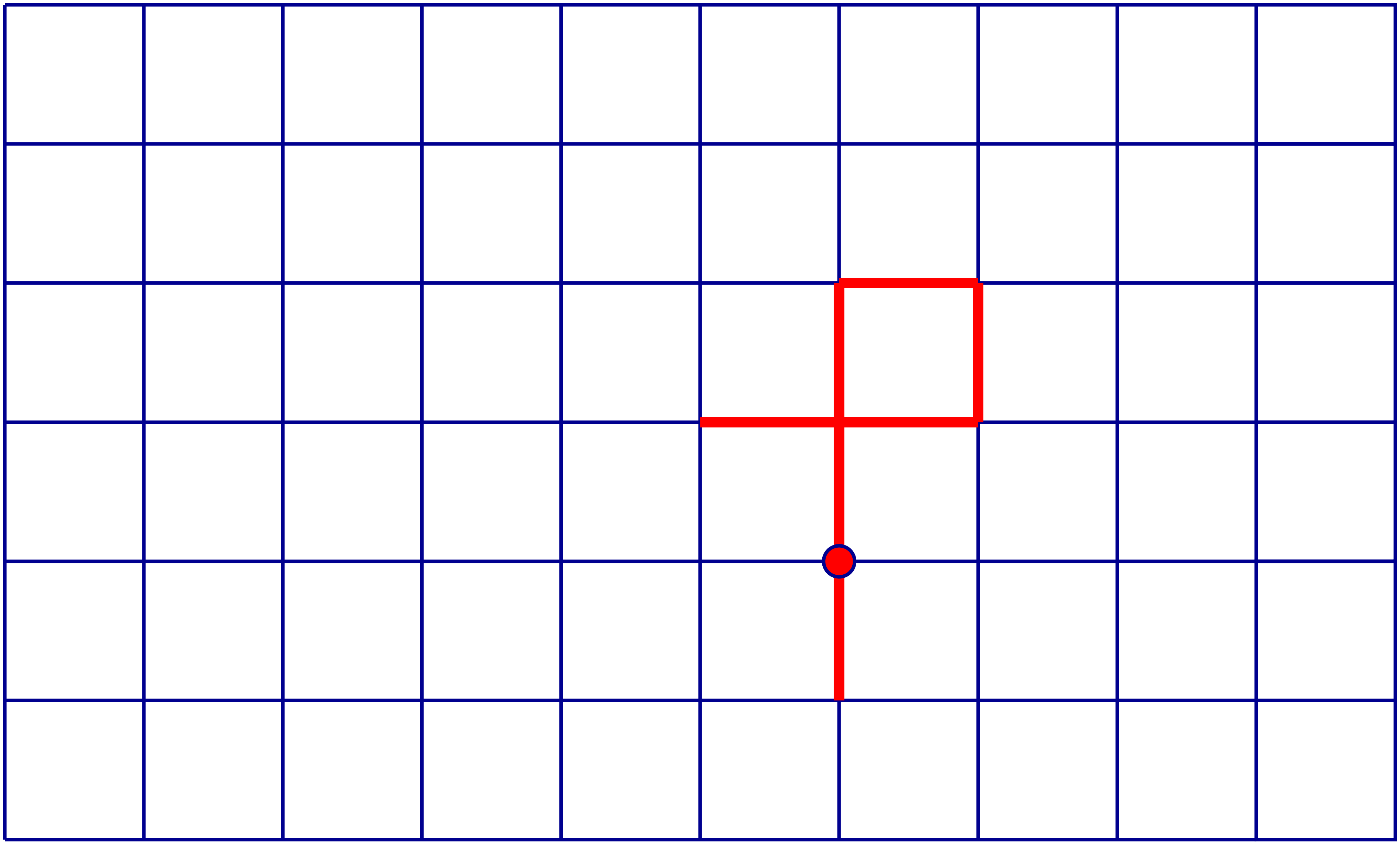


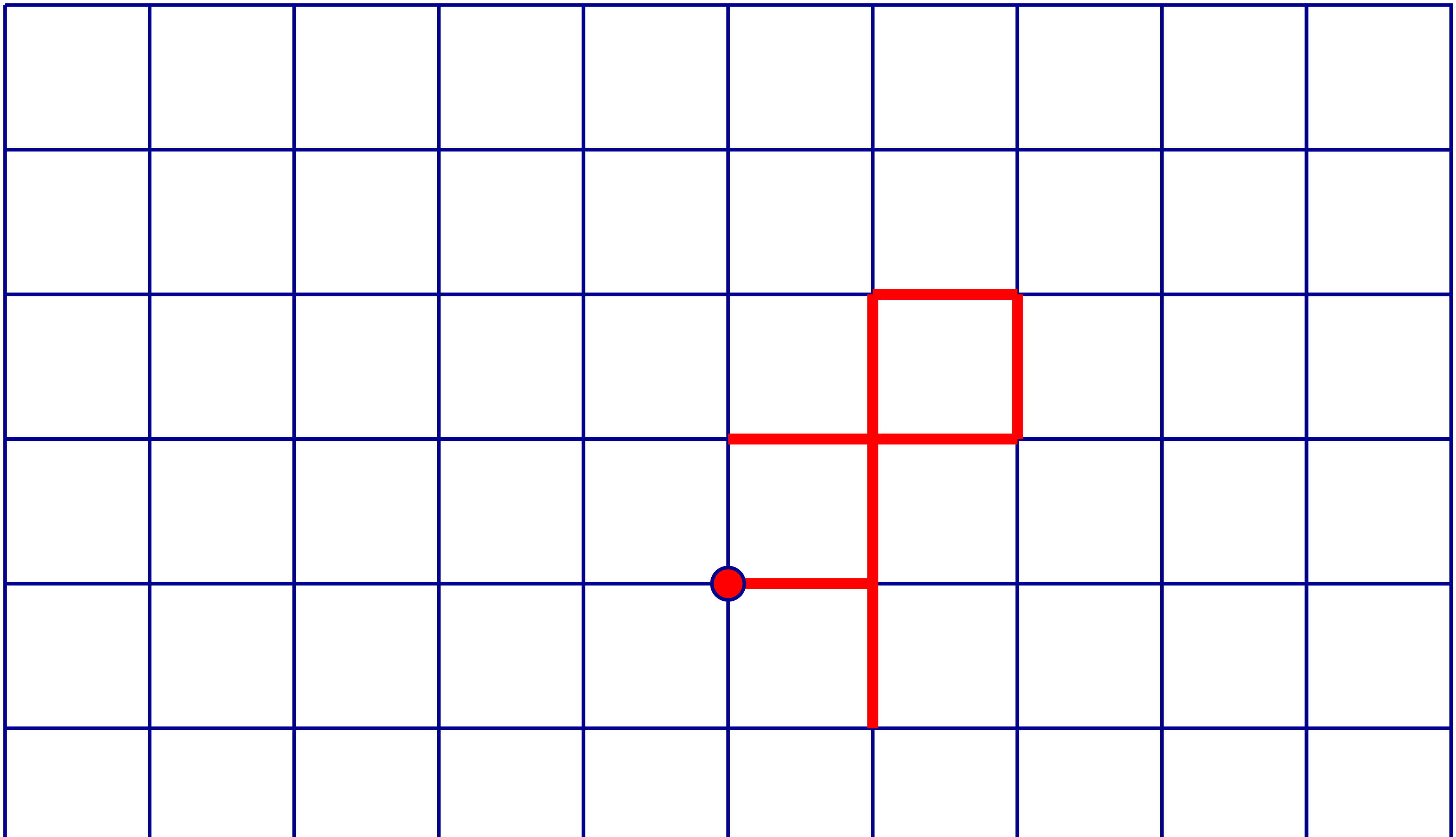


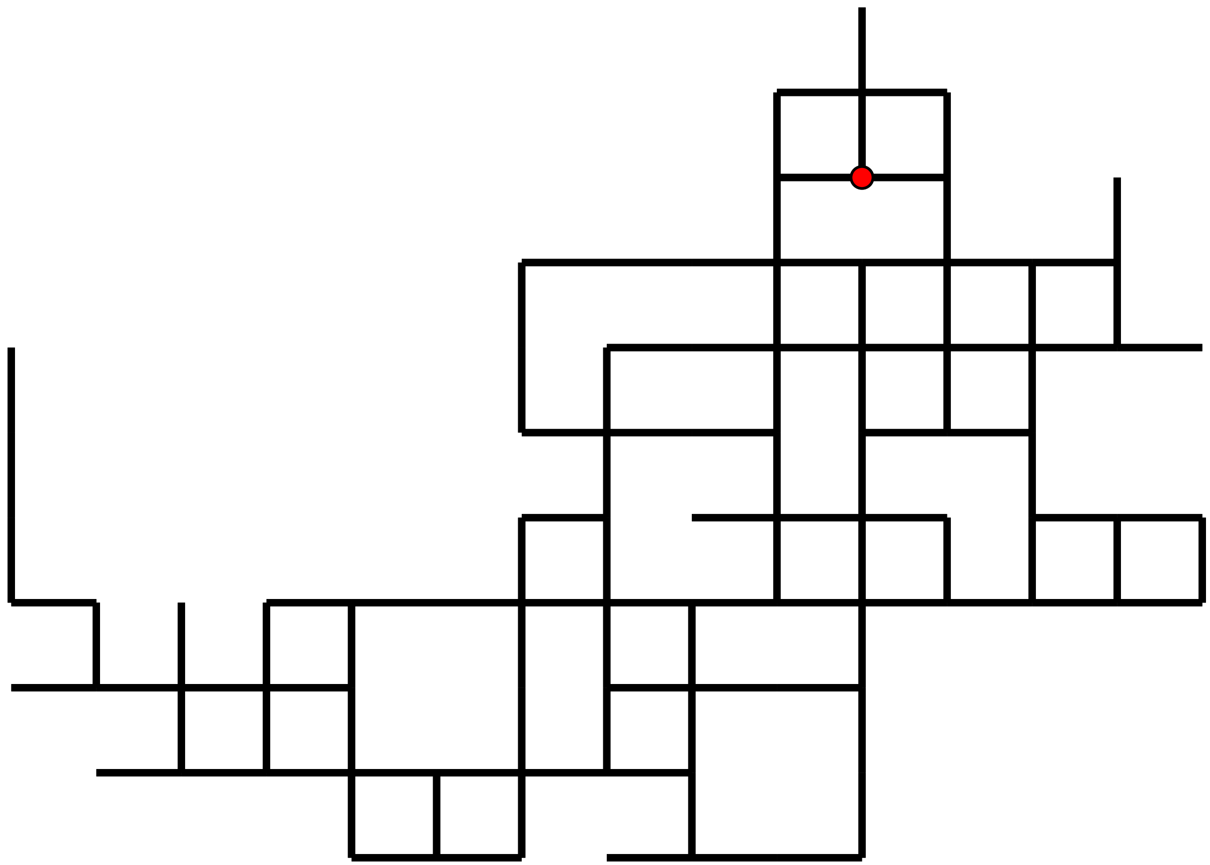




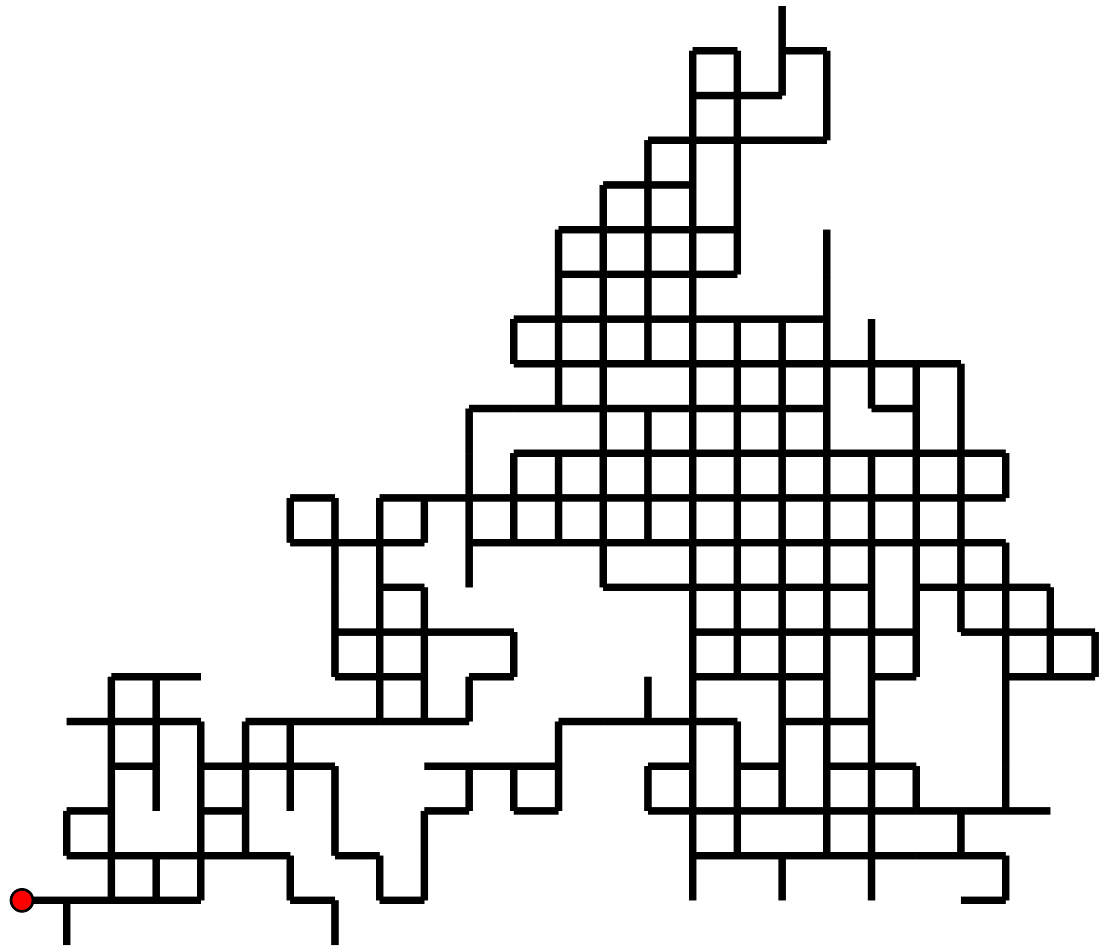




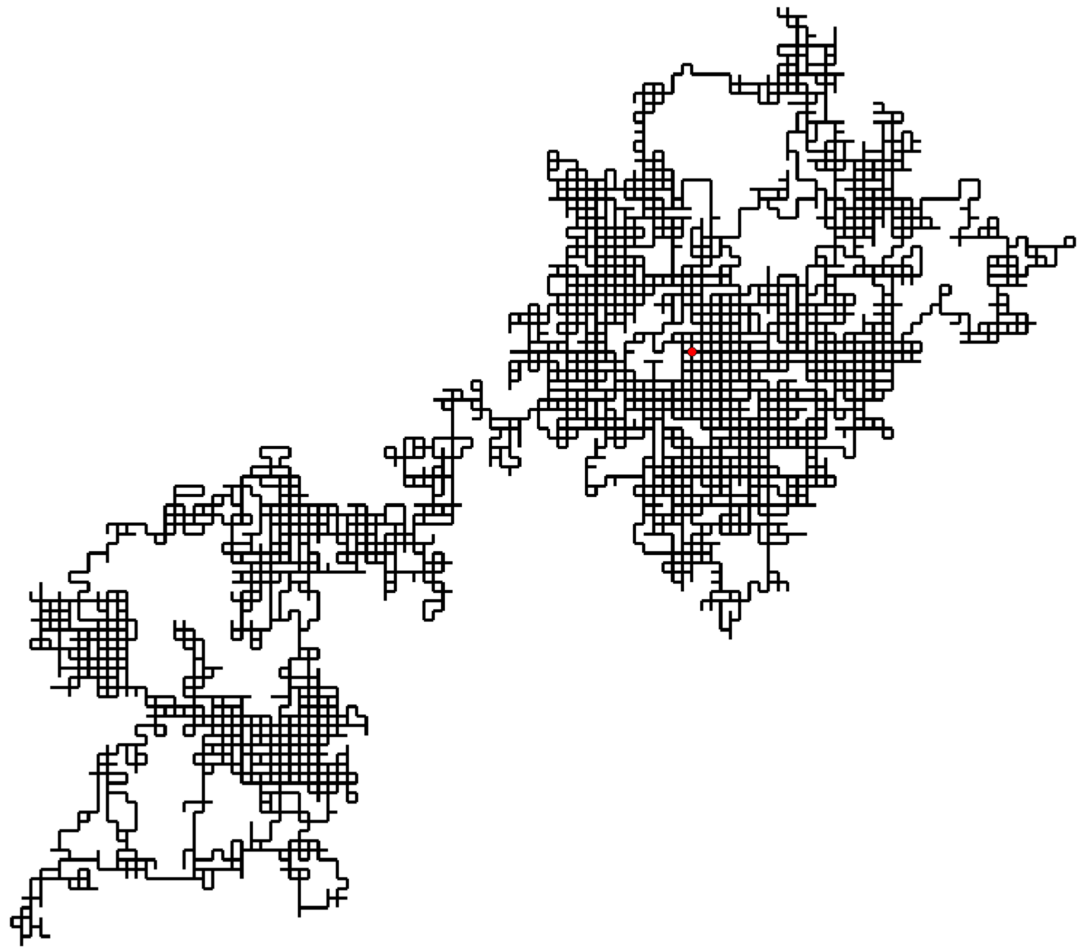




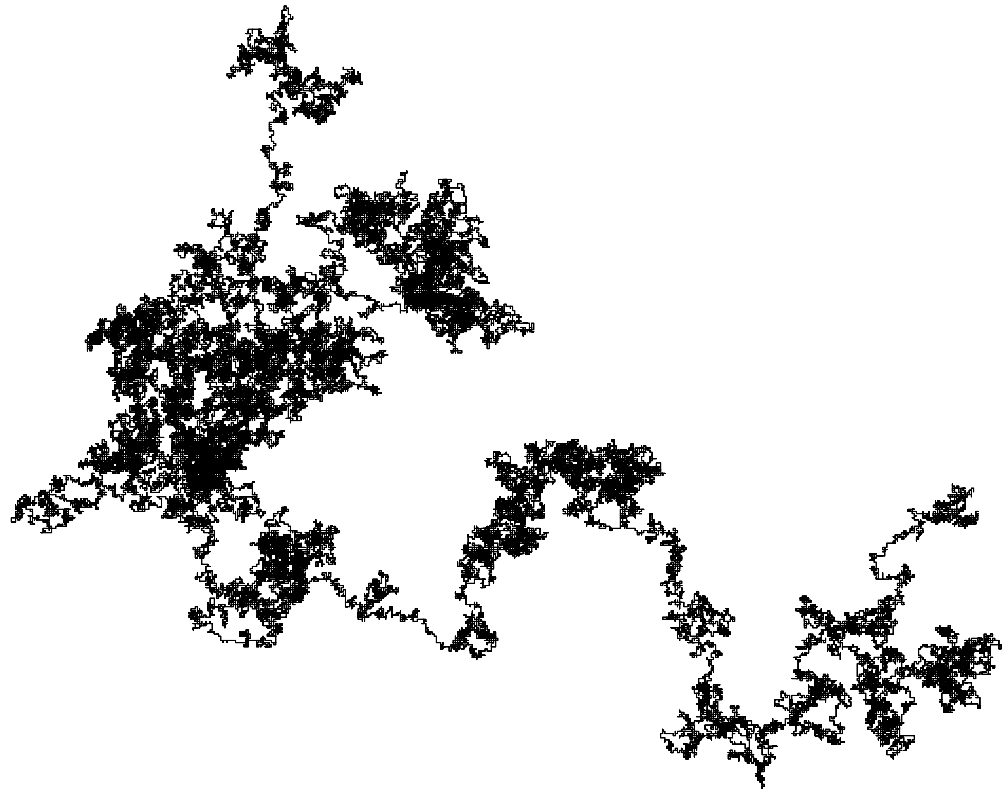
200 step random walk.



1000 step random walk.

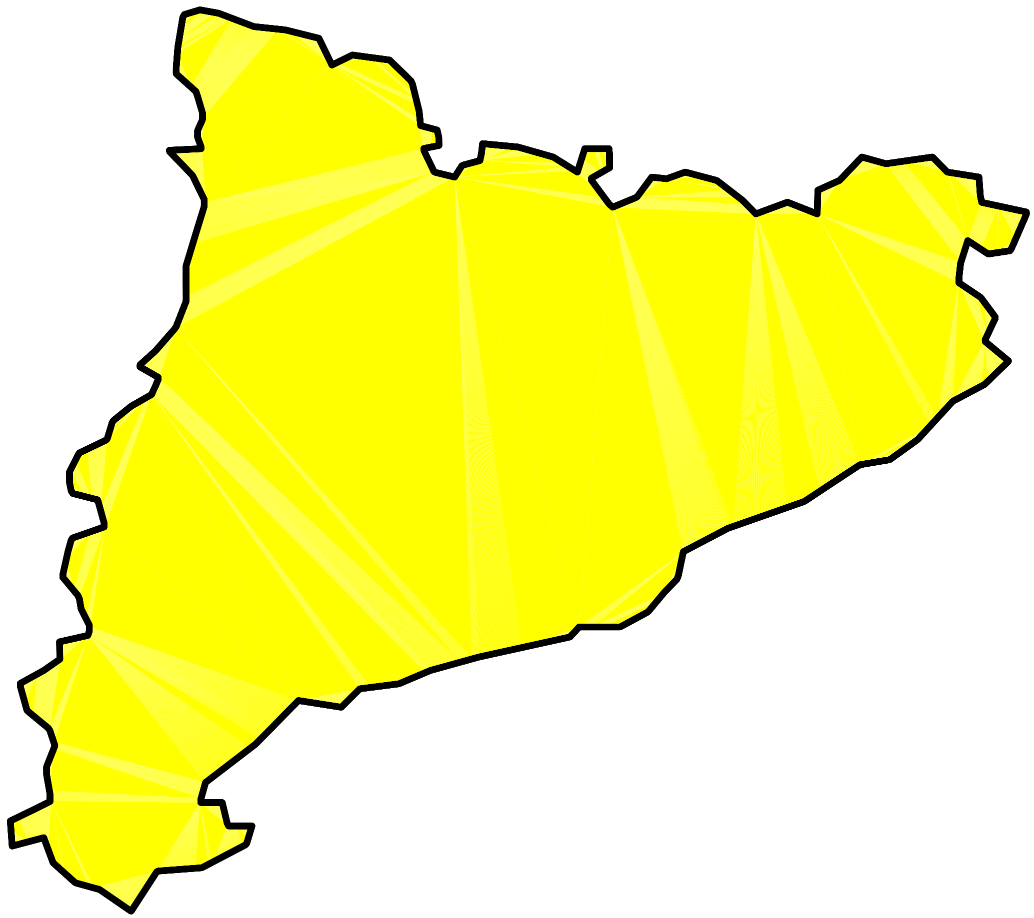


10,000 step random walk.



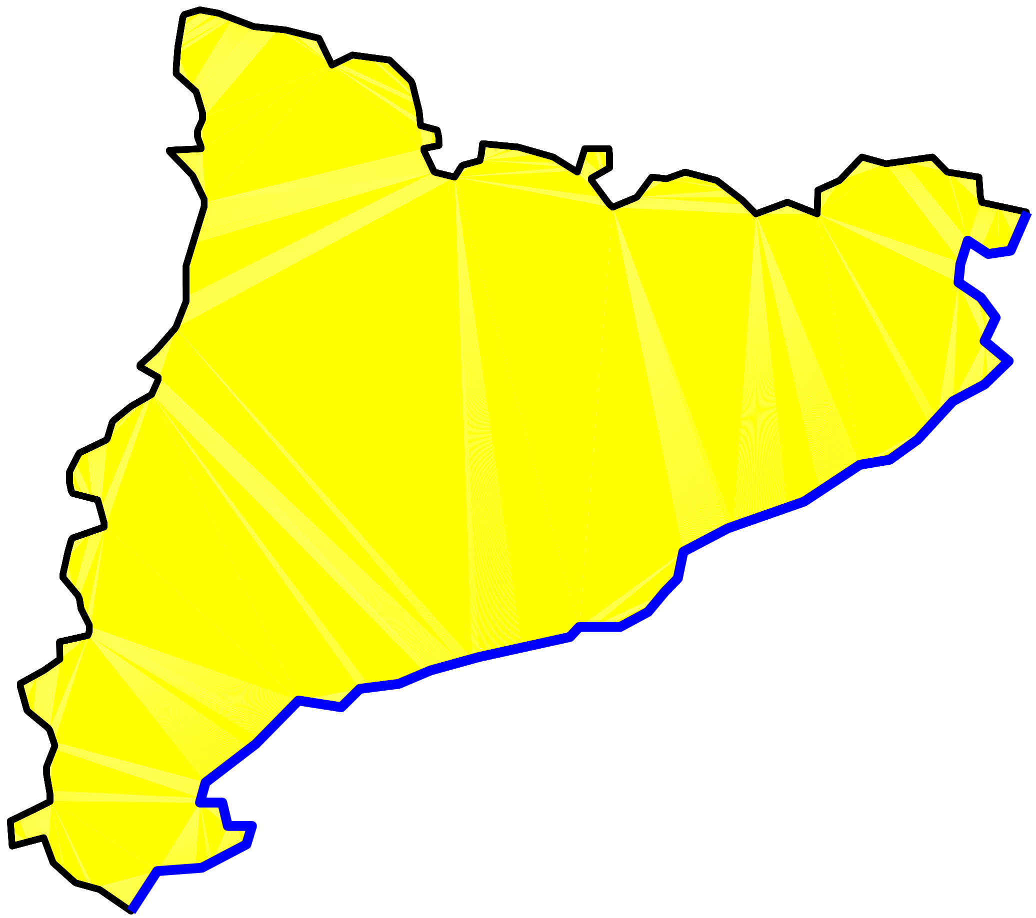
100,000 step random walk.

Harmonic measure = hitting distribution of Brownian motion



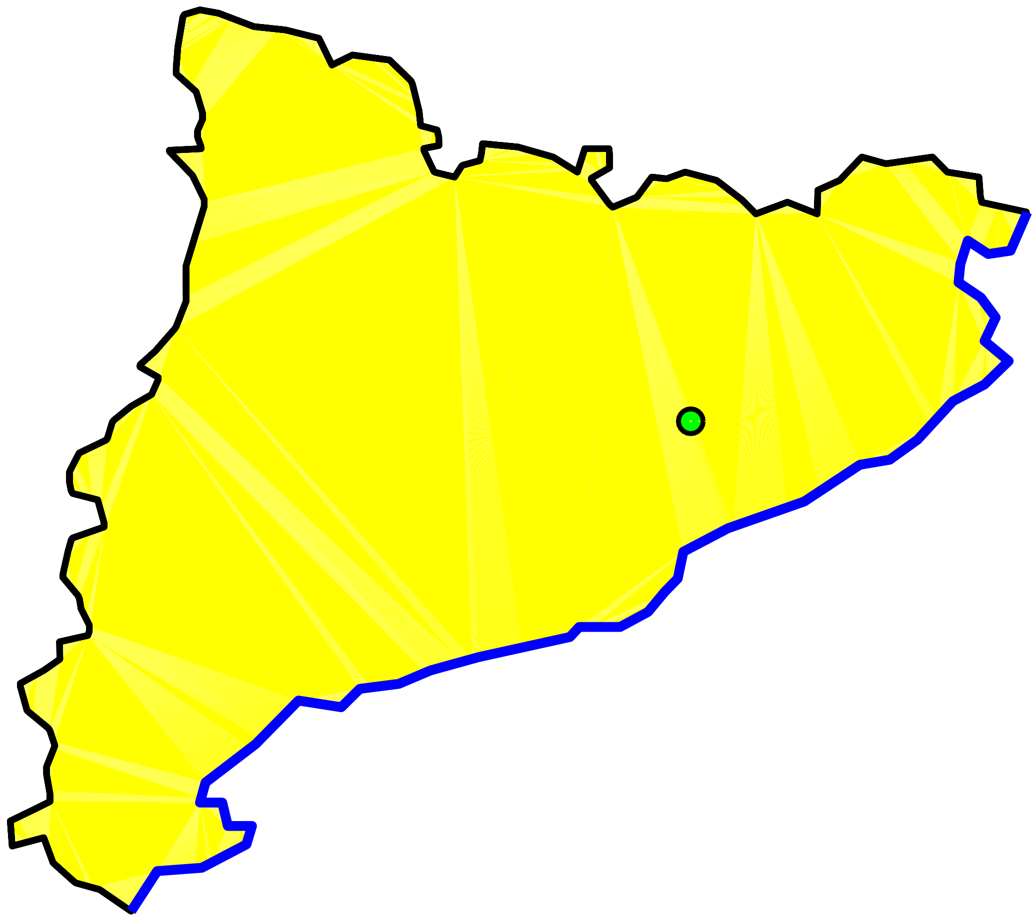
Suppose Ω is a planar Jordan domain.

Harmonic measure = hitting distribution of Brownian motion



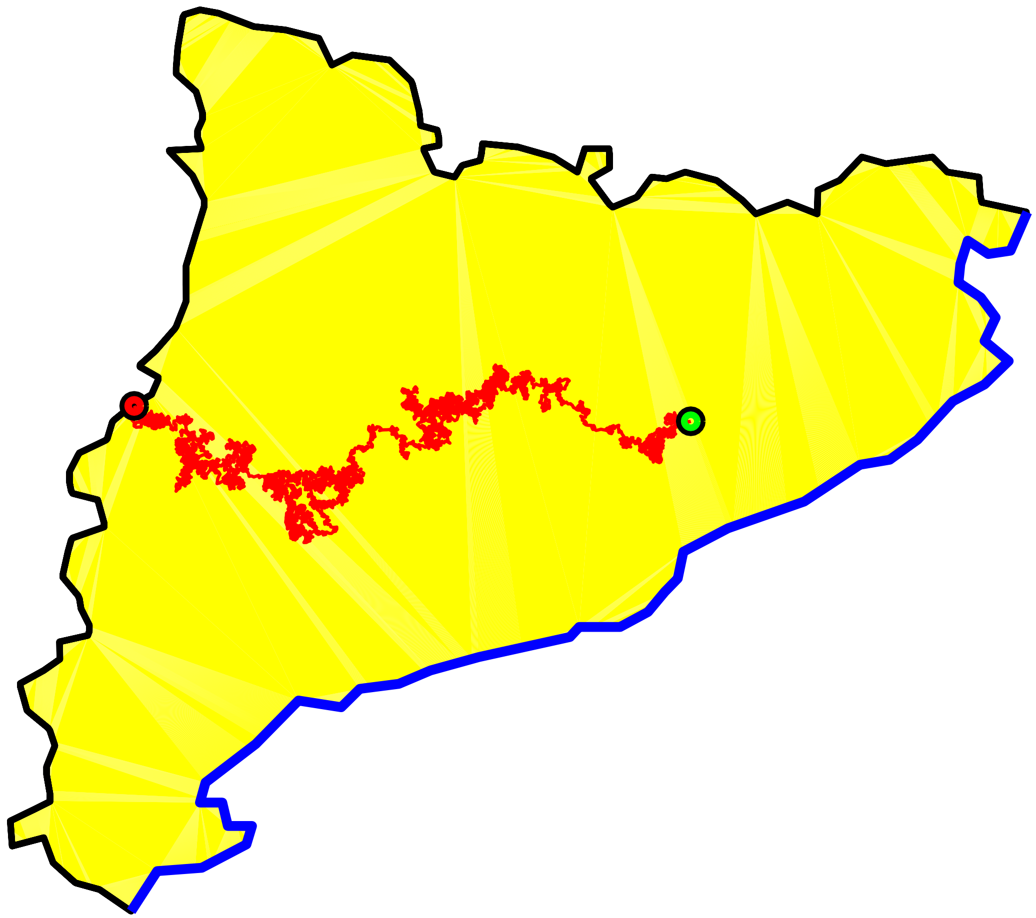
Let E be a subset of the boundary, $\partial\Omega$.

Harmonic measure = hitting distribution of Brownian motion



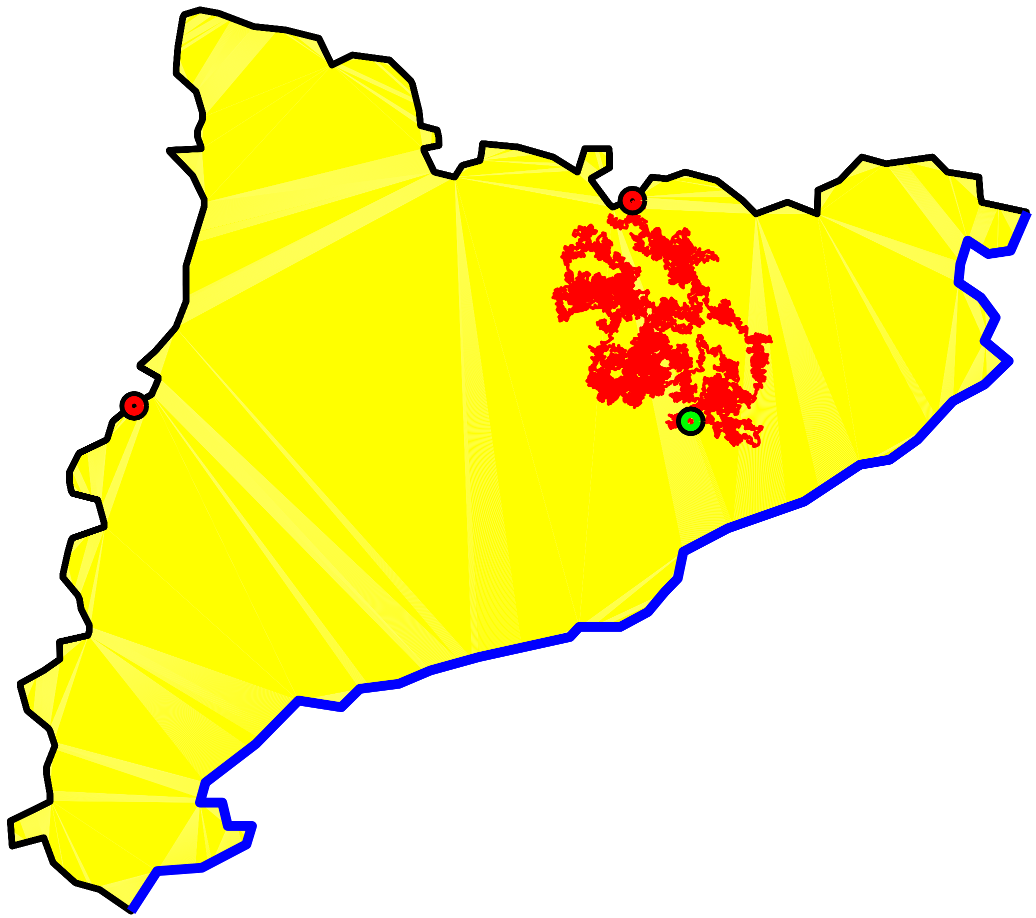
Choose an interior point $z \in \Omega$.

Harmonic measure = hitting distribution of Brownian motion



$\omega(z, E, \Omega)$ = probability a particle started at z first hits $\partial\Omega$ in E.

Harmonic measure = hitting distribution of Brownian motion



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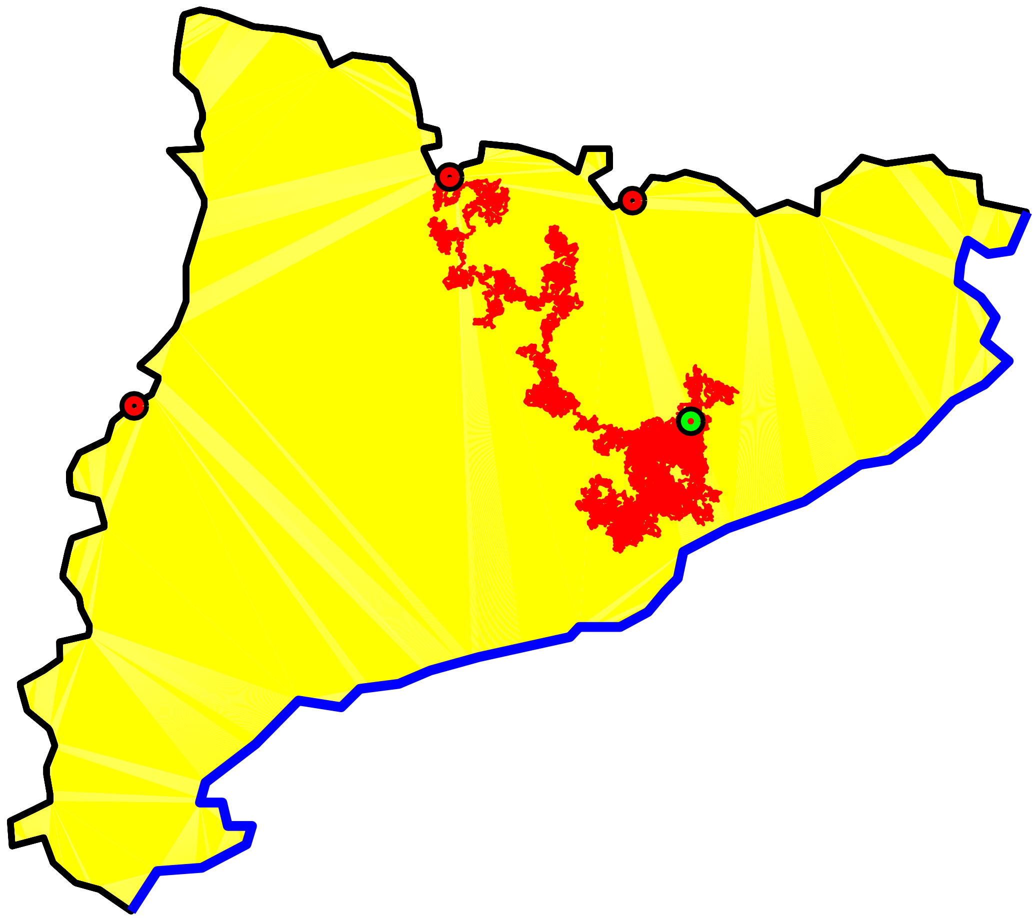

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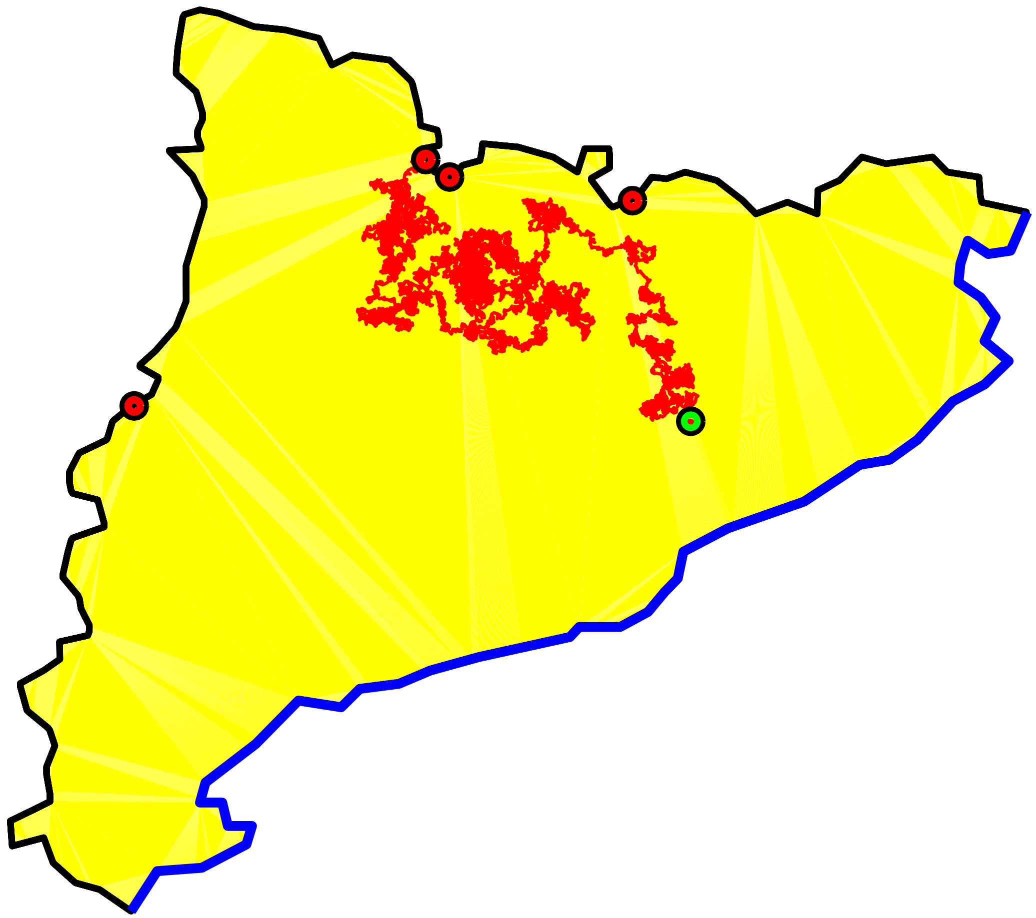

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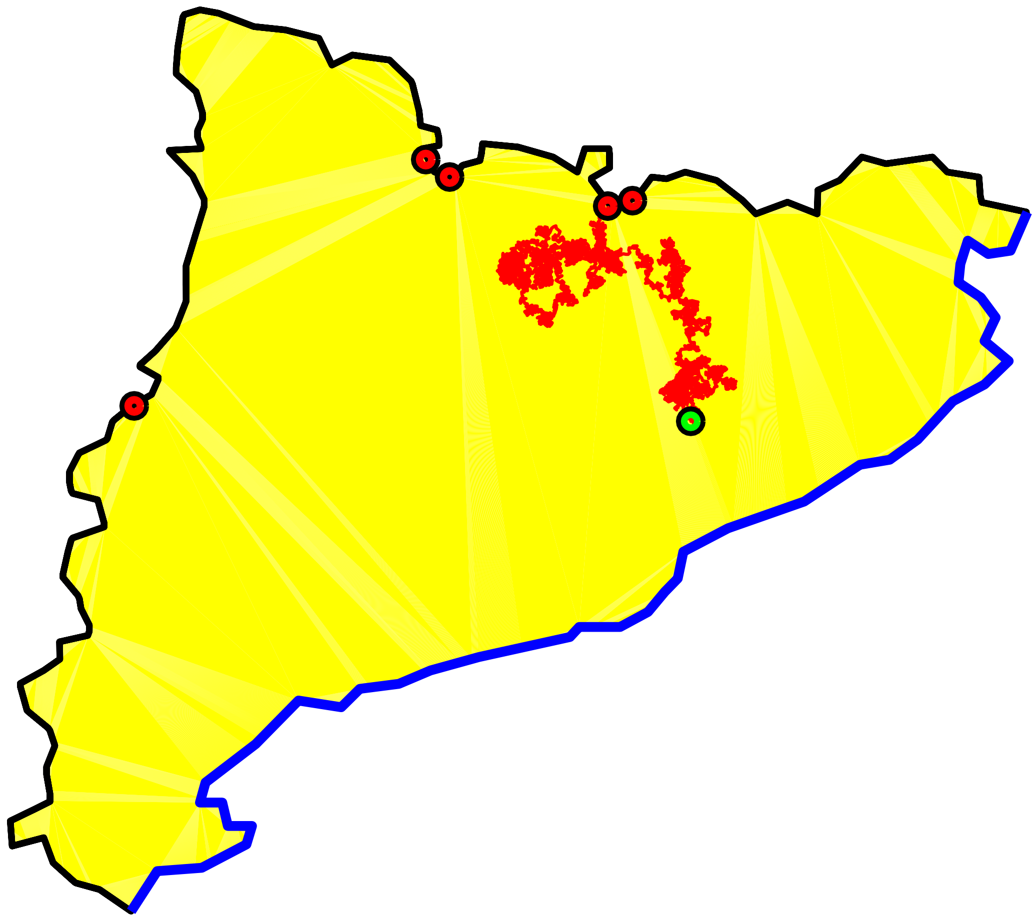

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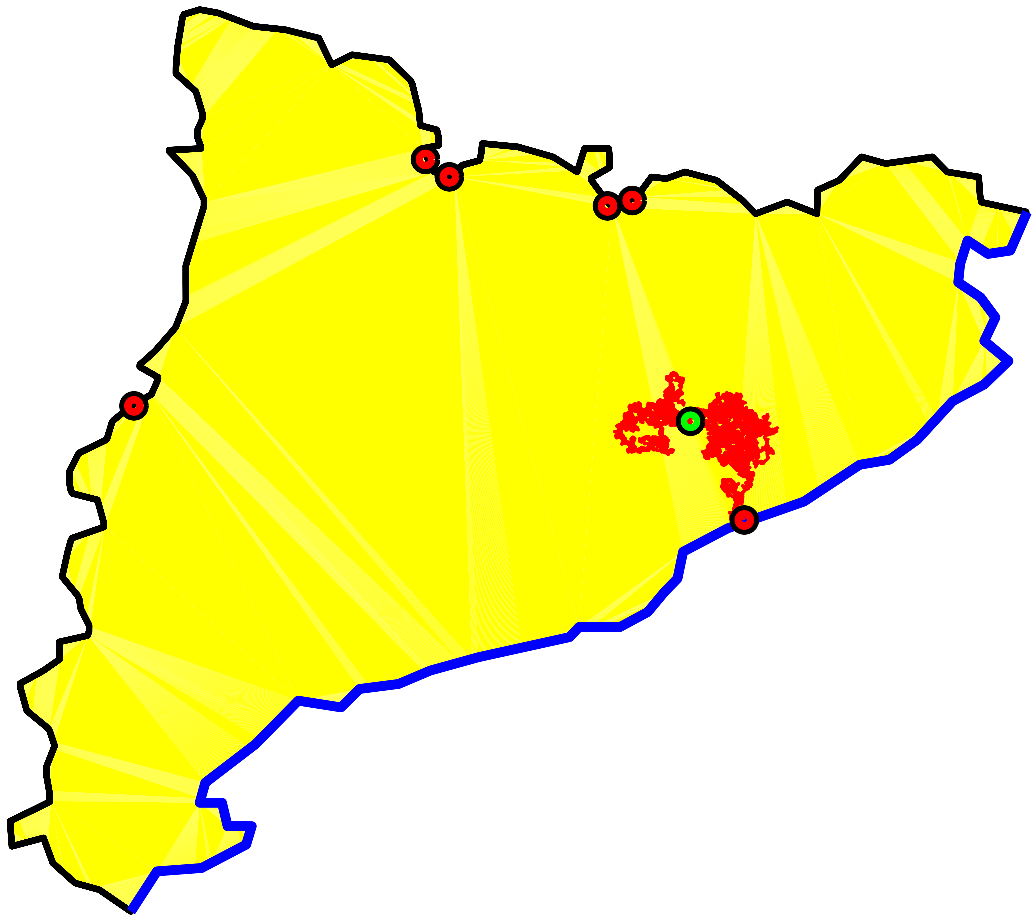

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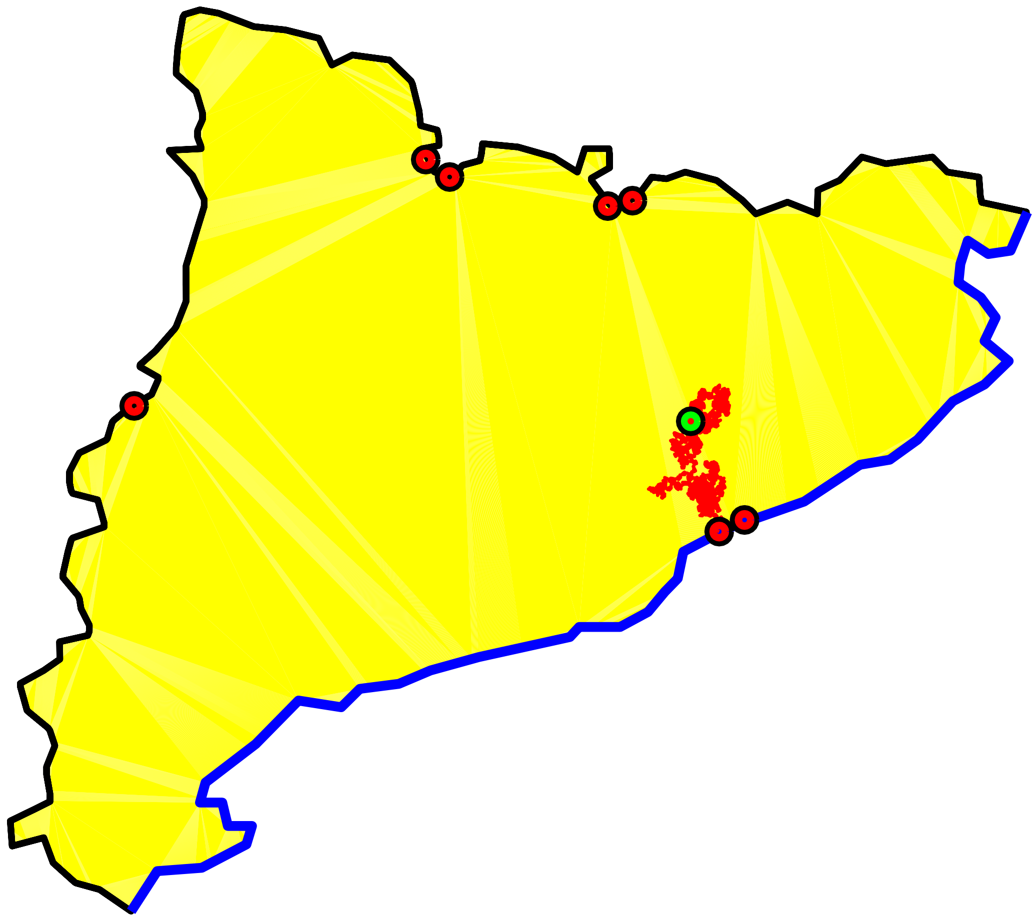

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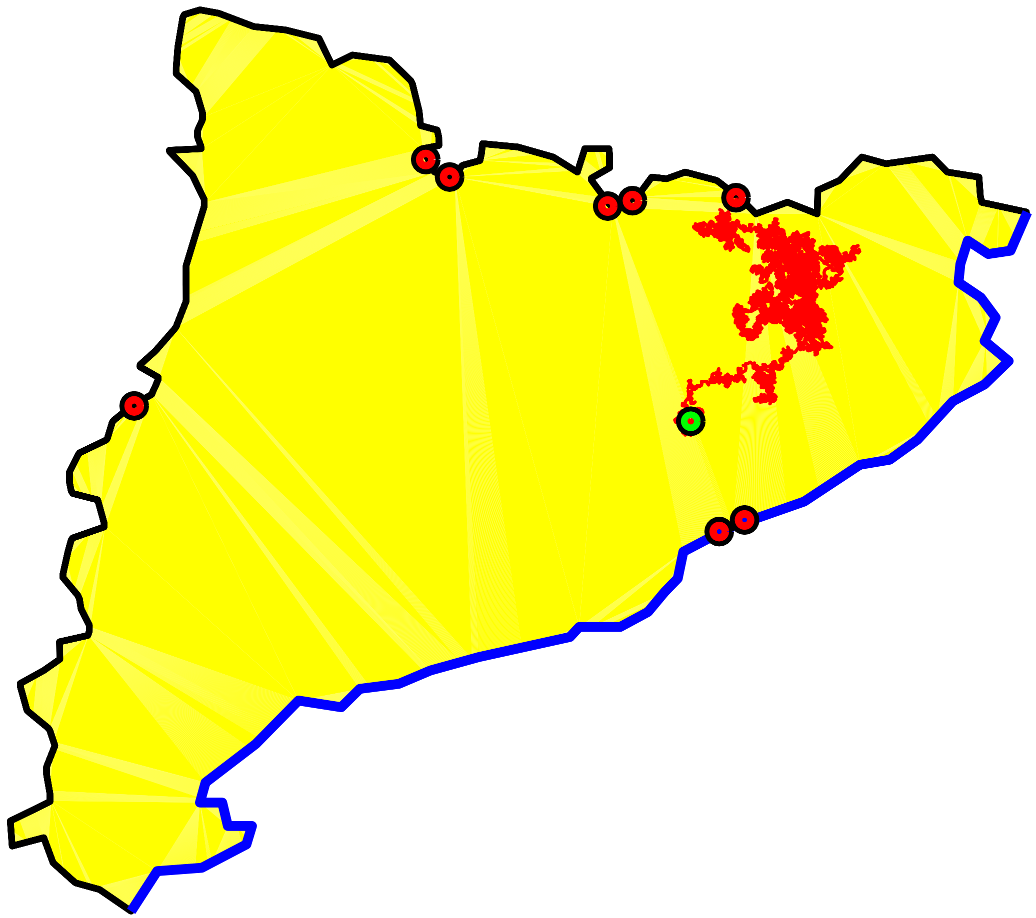

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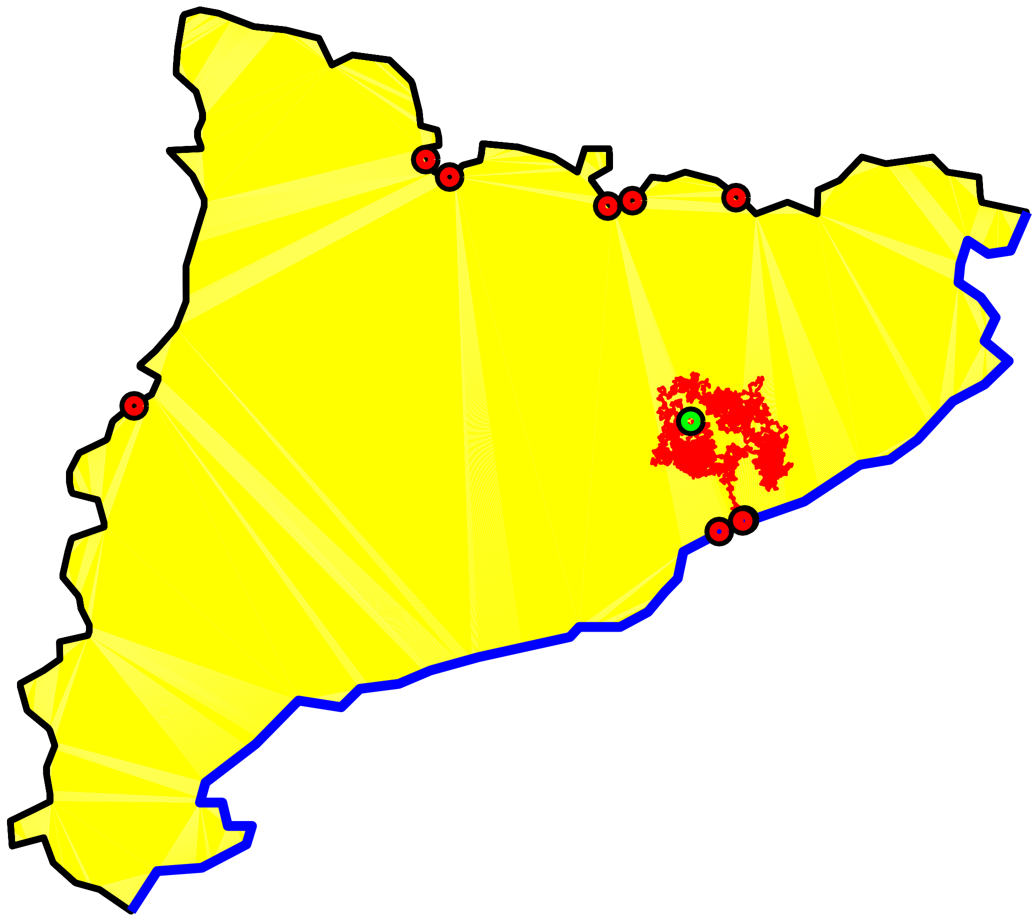

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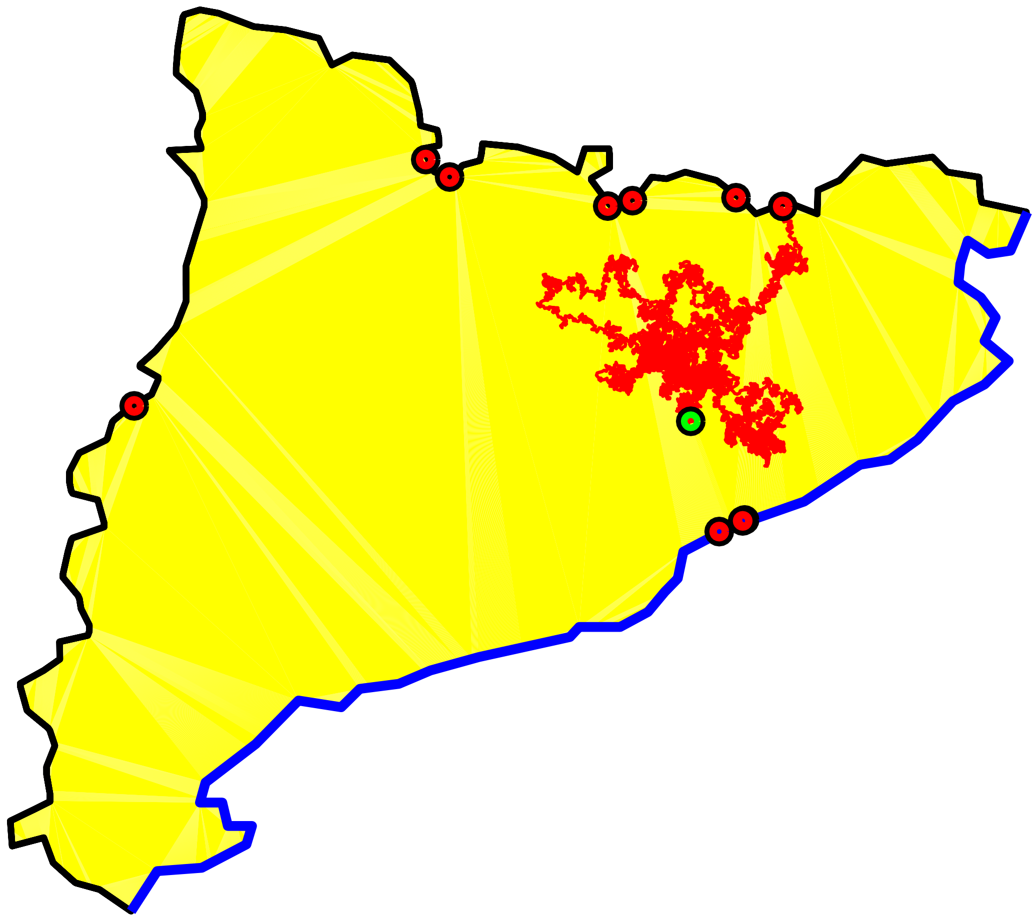
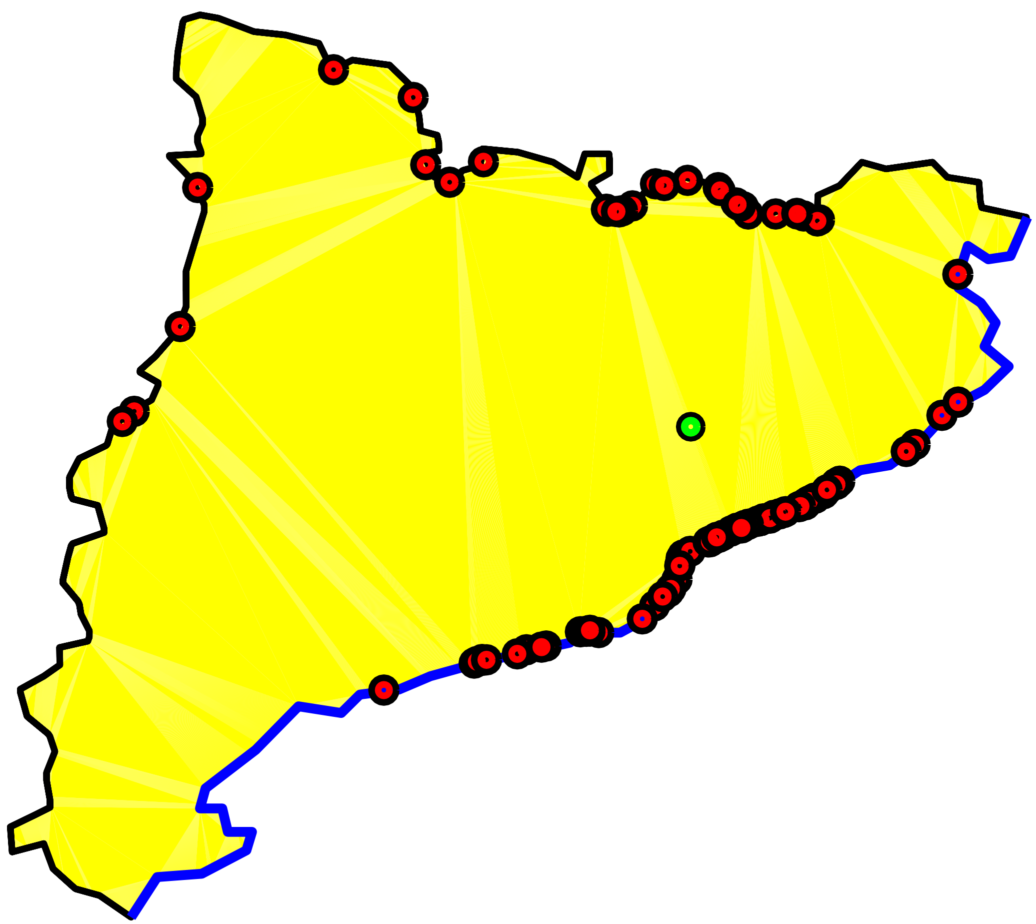


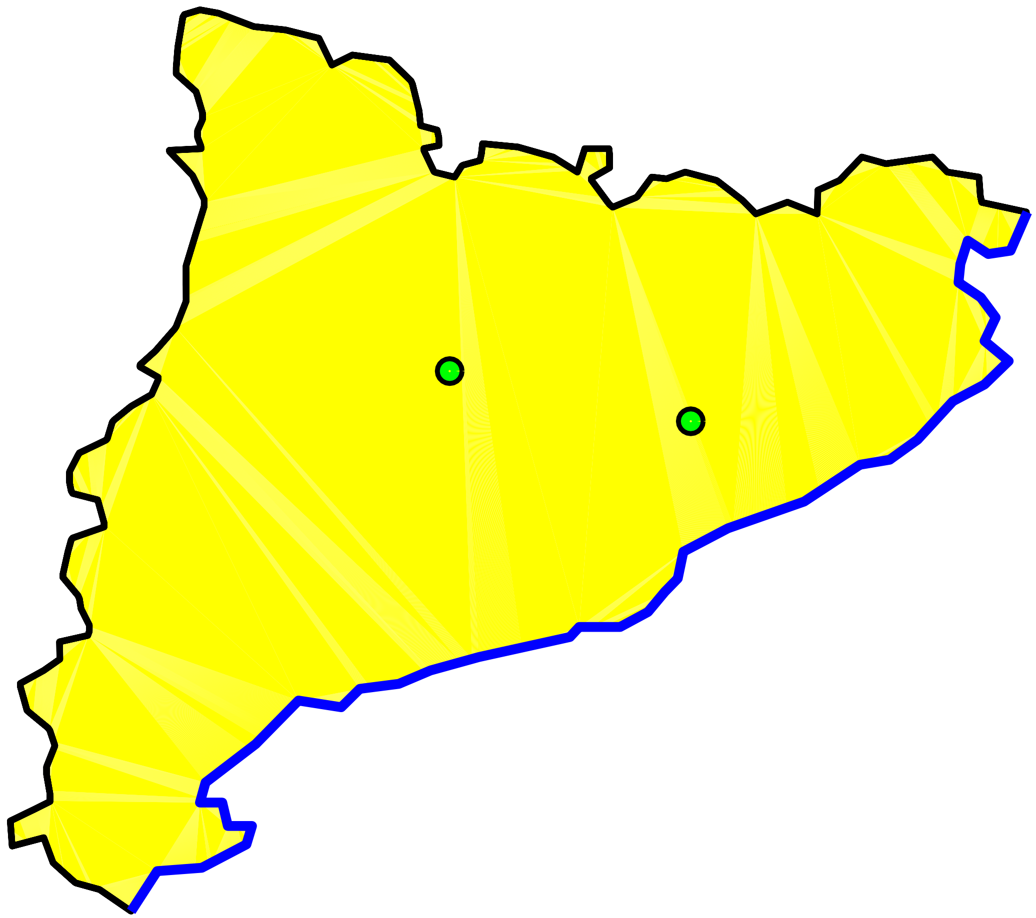
$$\omega(z_1, E, \Omega) \approx 3/10.$$

Harmonic measure = hitting distribution of Brownian motion



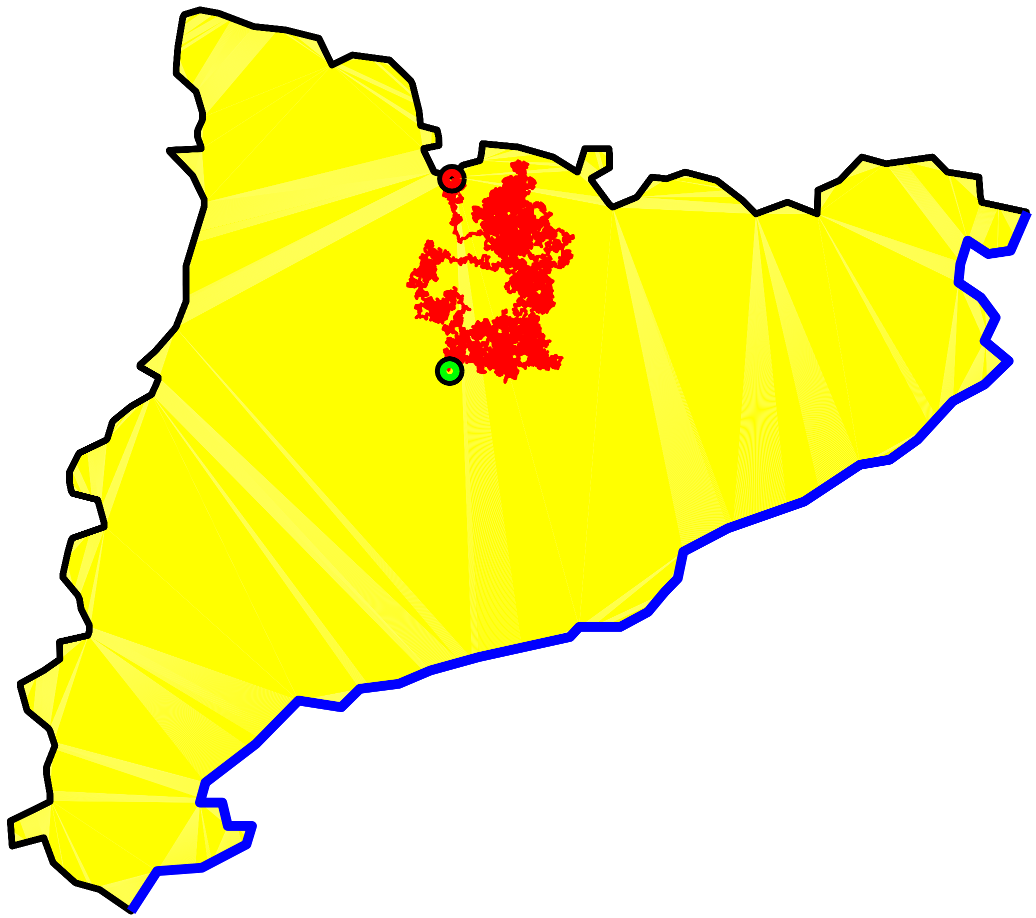
$$\omega(z_1, E, \Omega) \approx 61/100.$$

Harmonic measure = hitting distribution of Brownian motion



What if we move the starting point?

Harmonic measure = hitting distribution of Brownian motion



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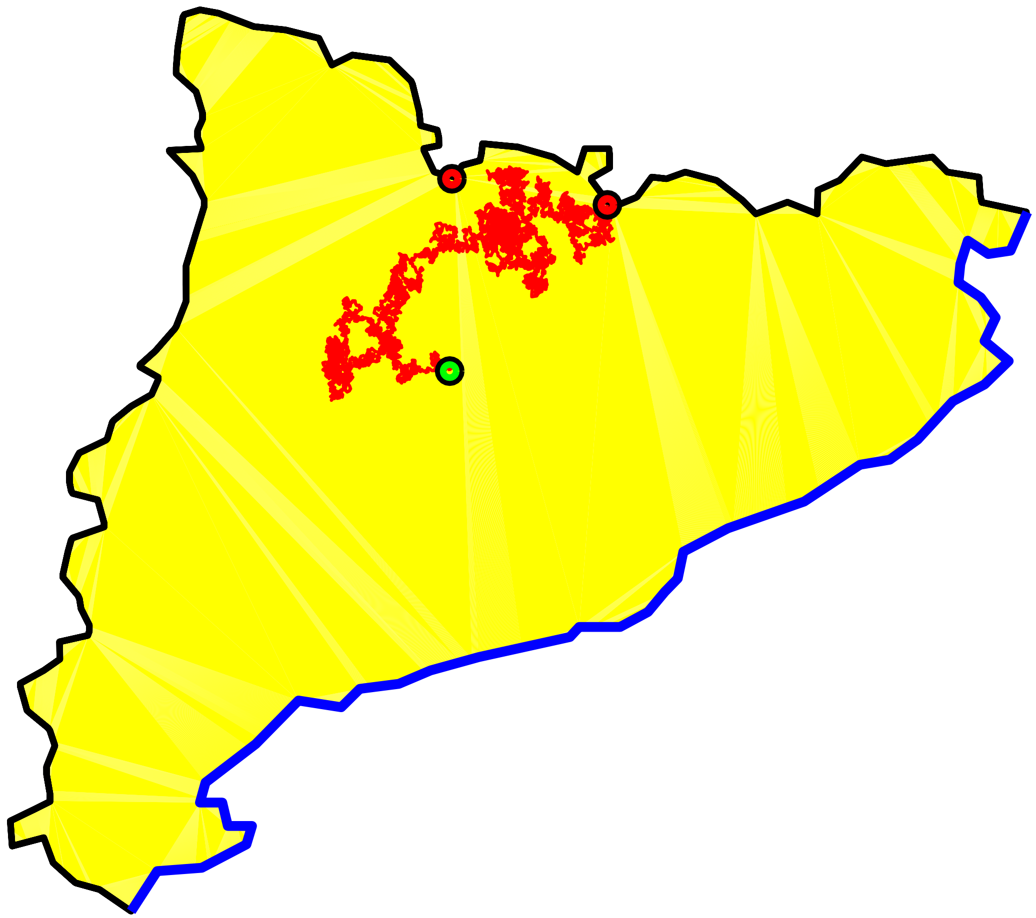

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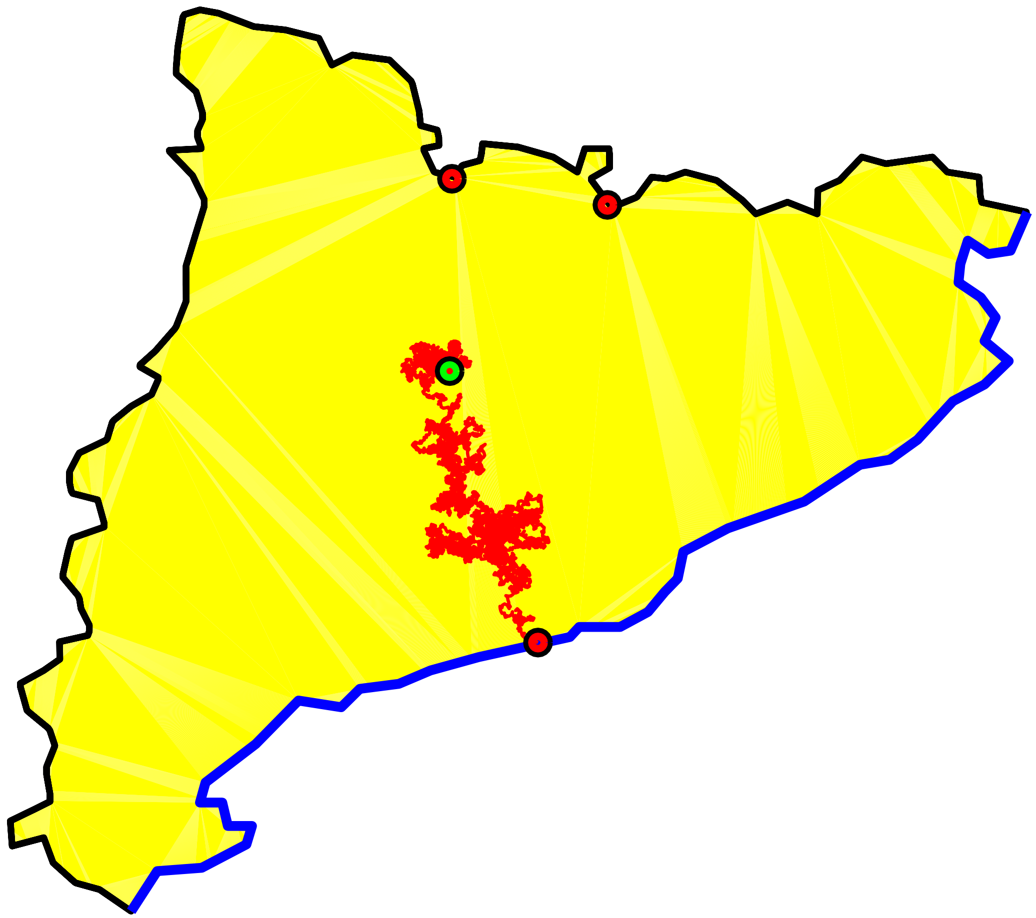

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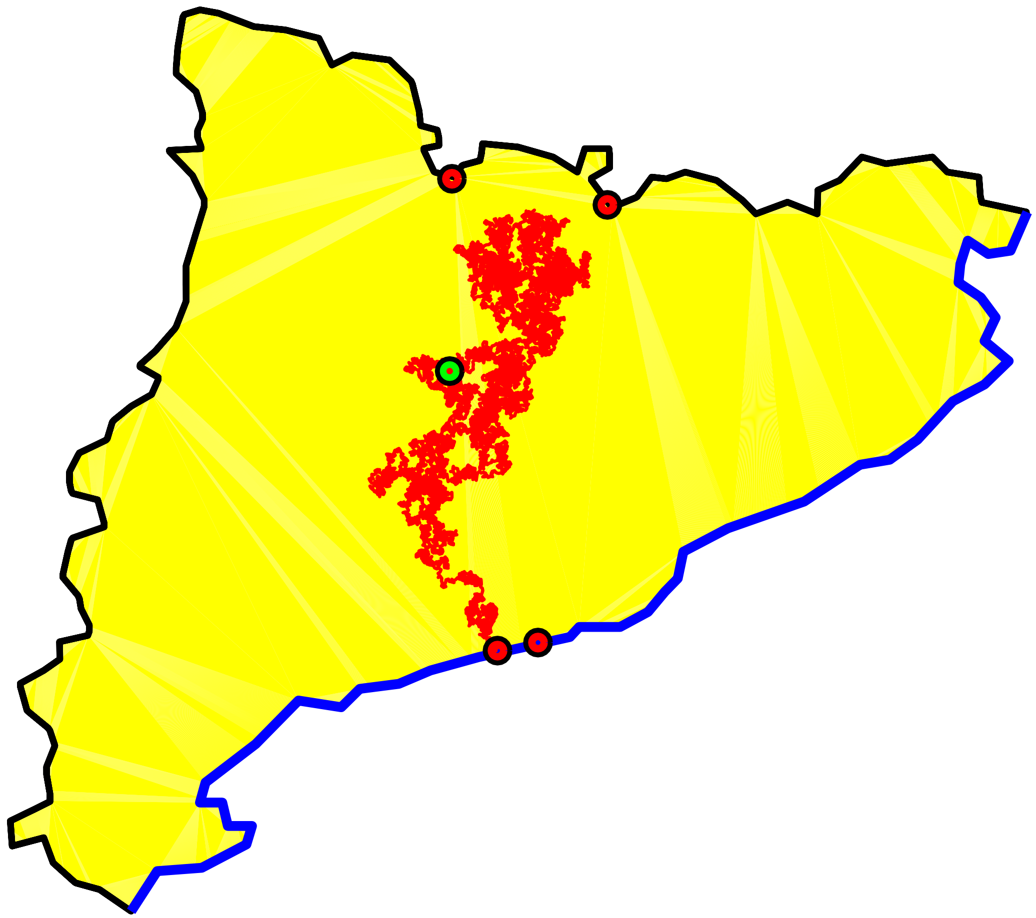

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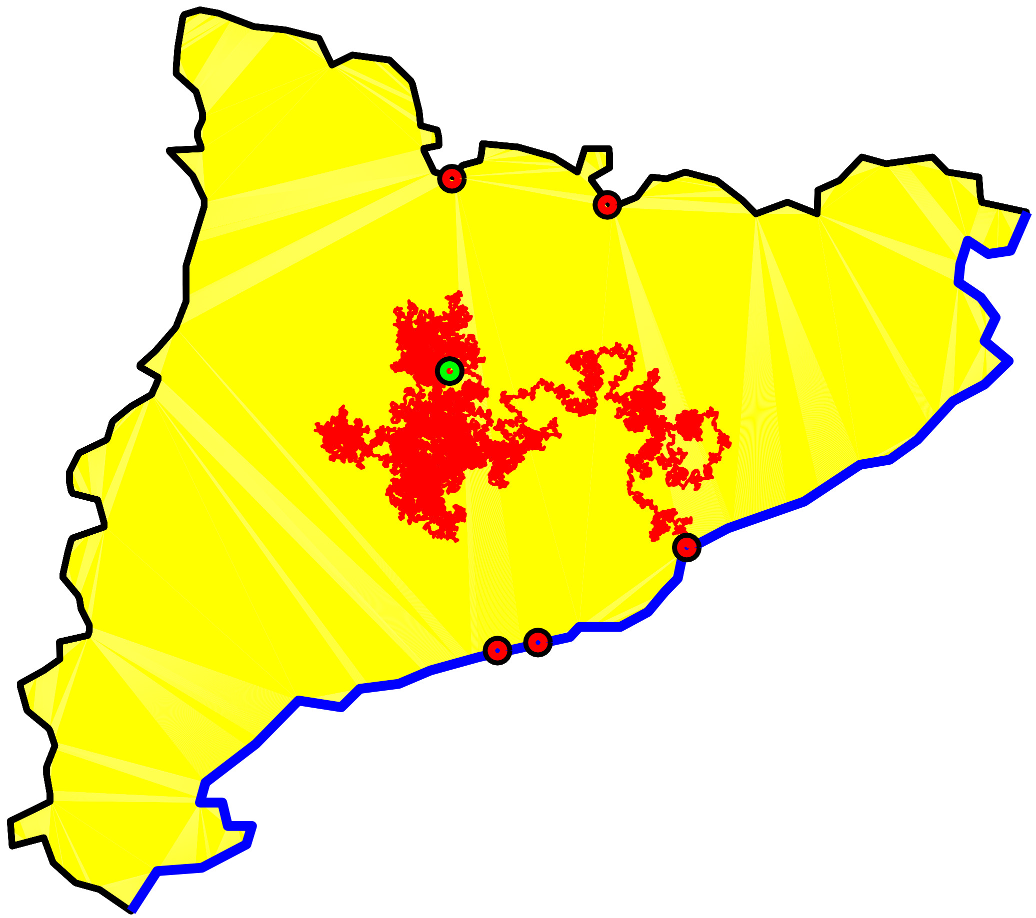

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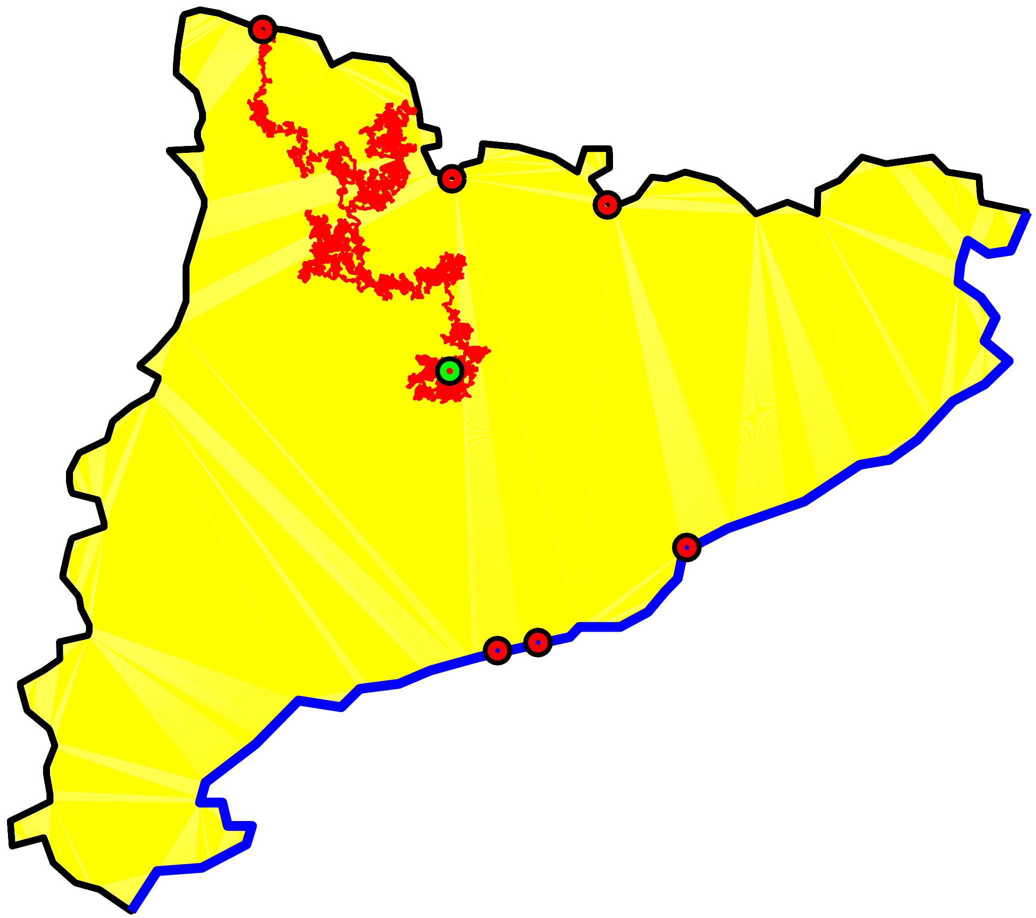

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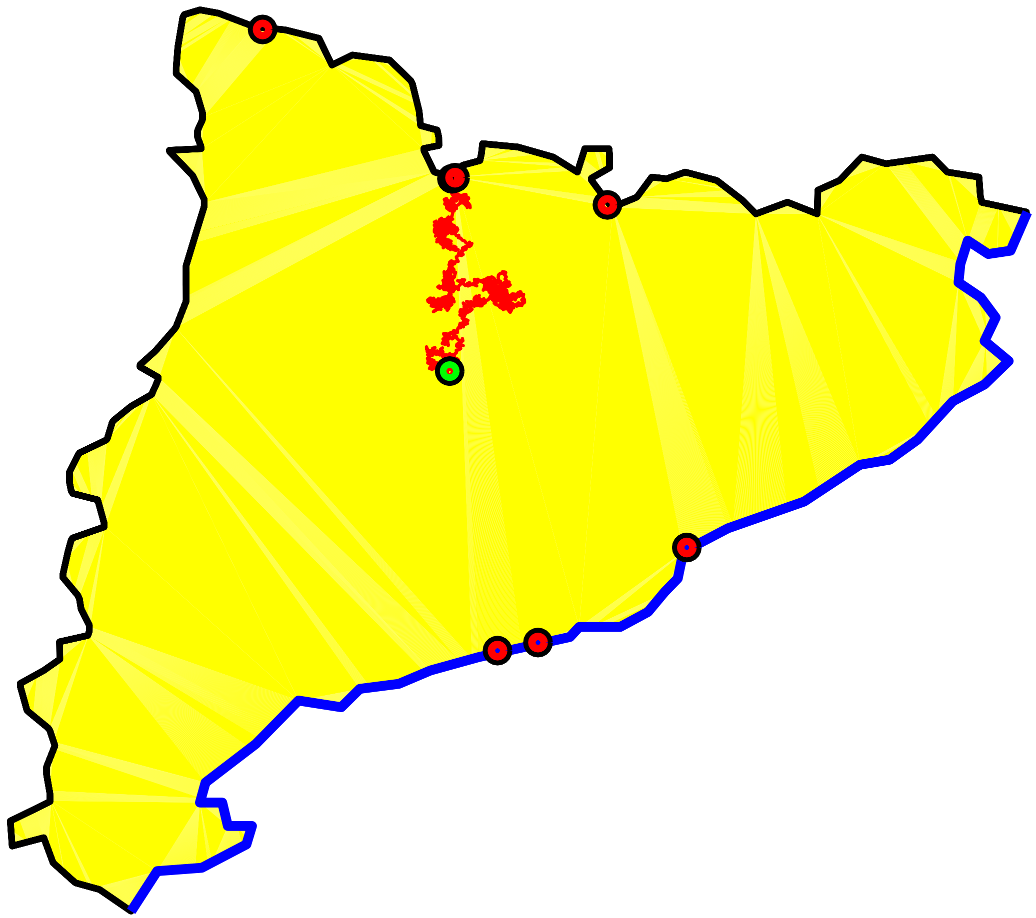

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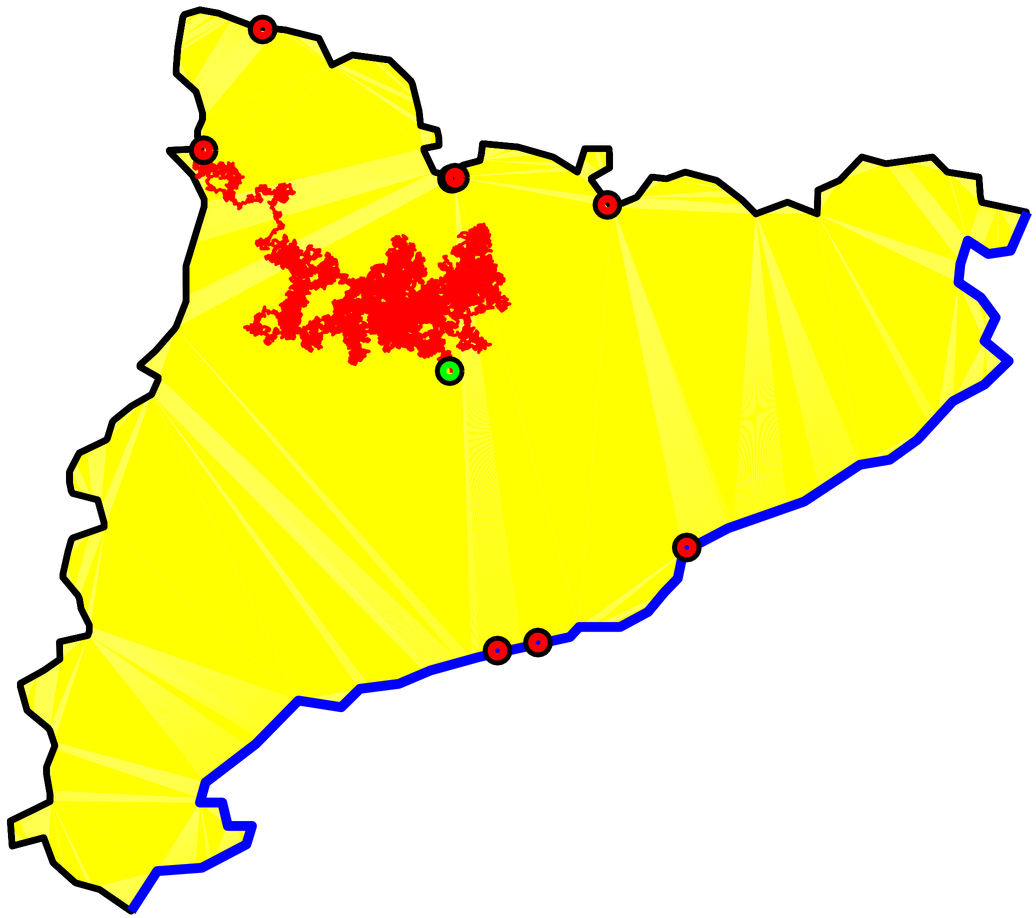

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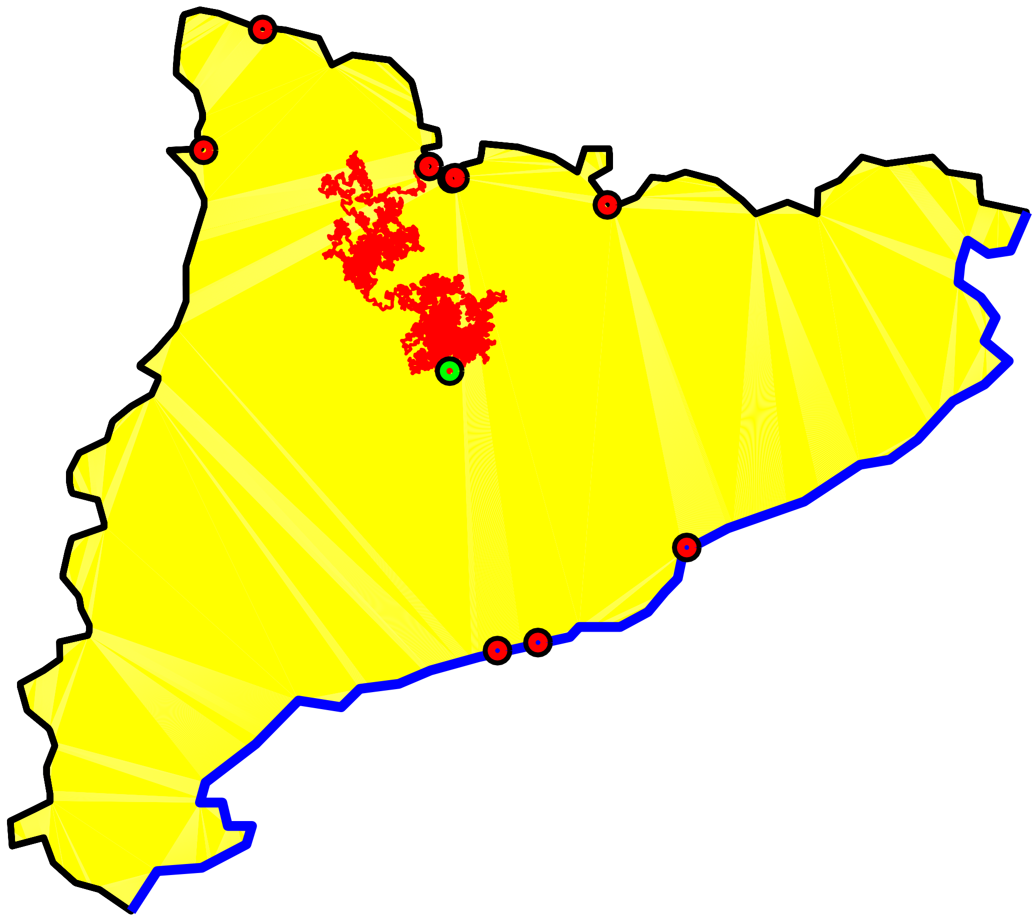

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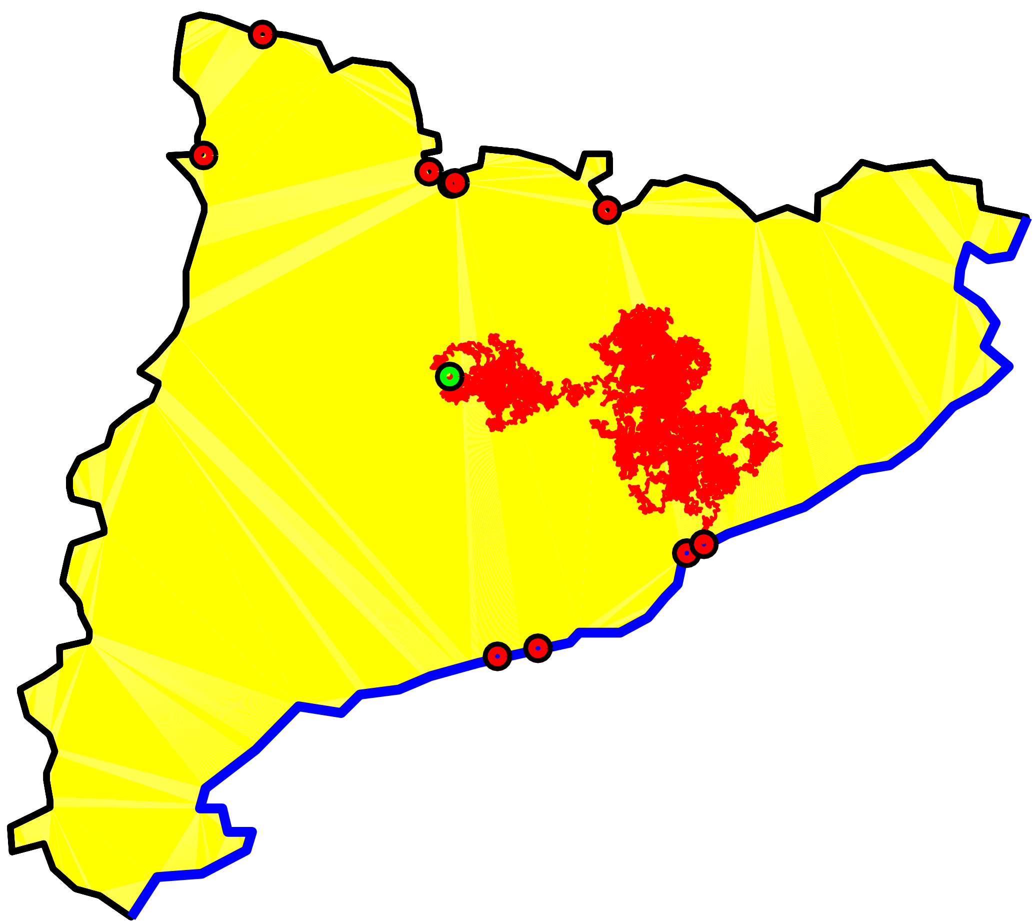
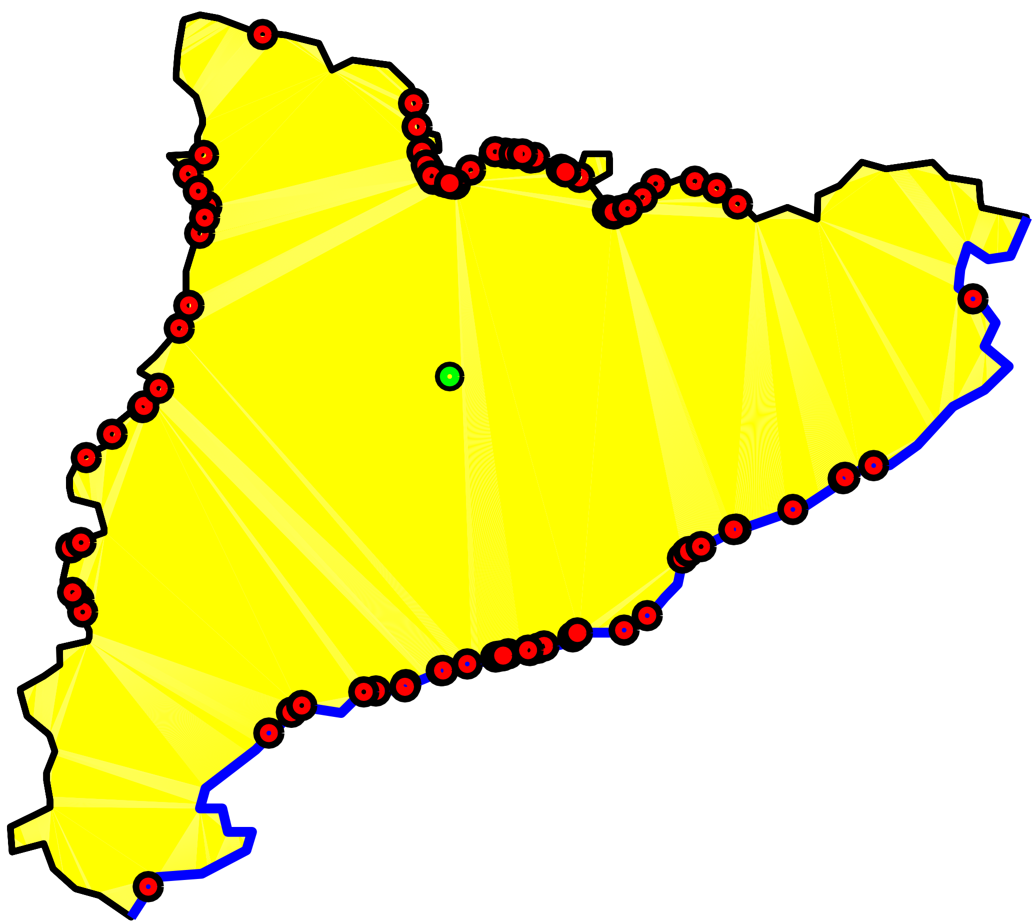


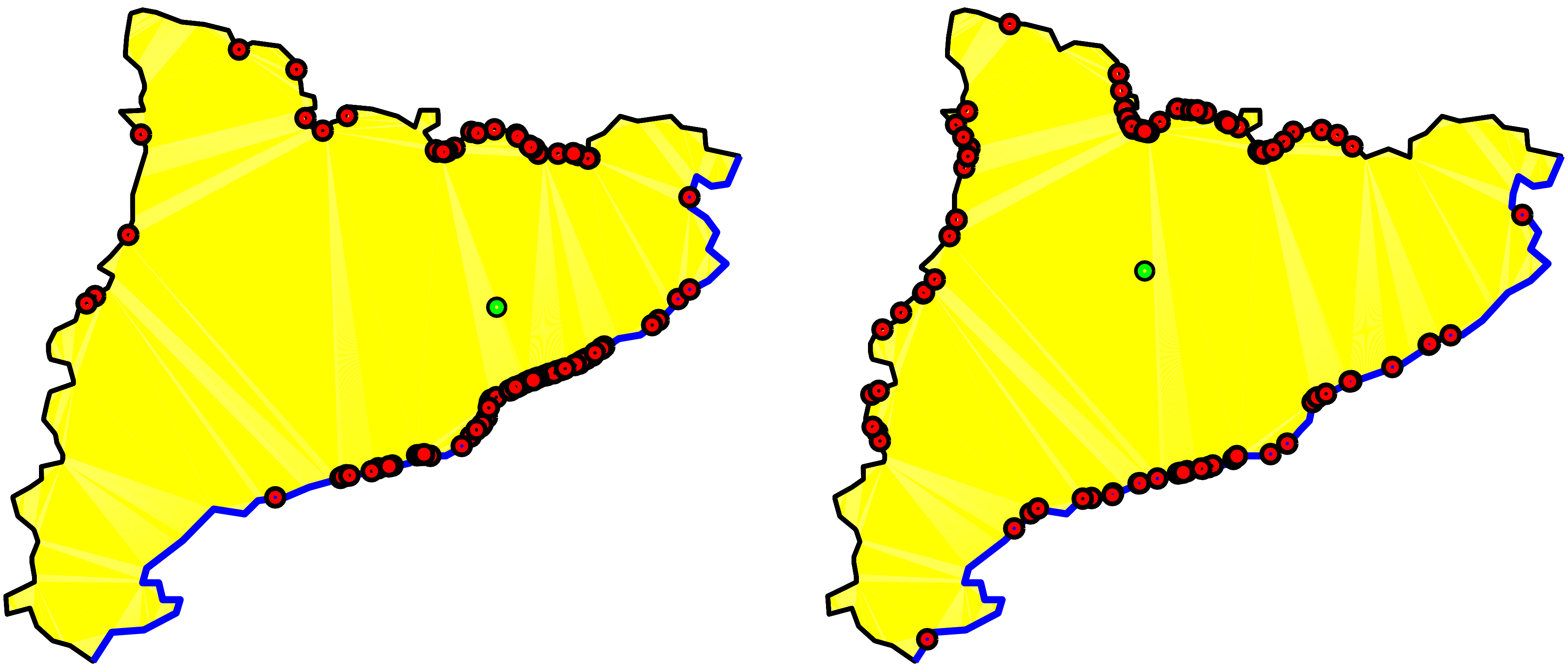
$$\omega(z_2, E, \Omega) \approx 4/10.$$

Harmonic measure = hitting distribution of Brownian motion



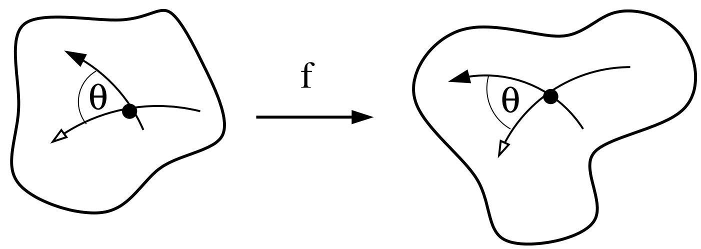
$$\omega(z_2, E, \Omega) \approx 34/100.$$

Harmonic measure = hitting distribution of Brownian motion

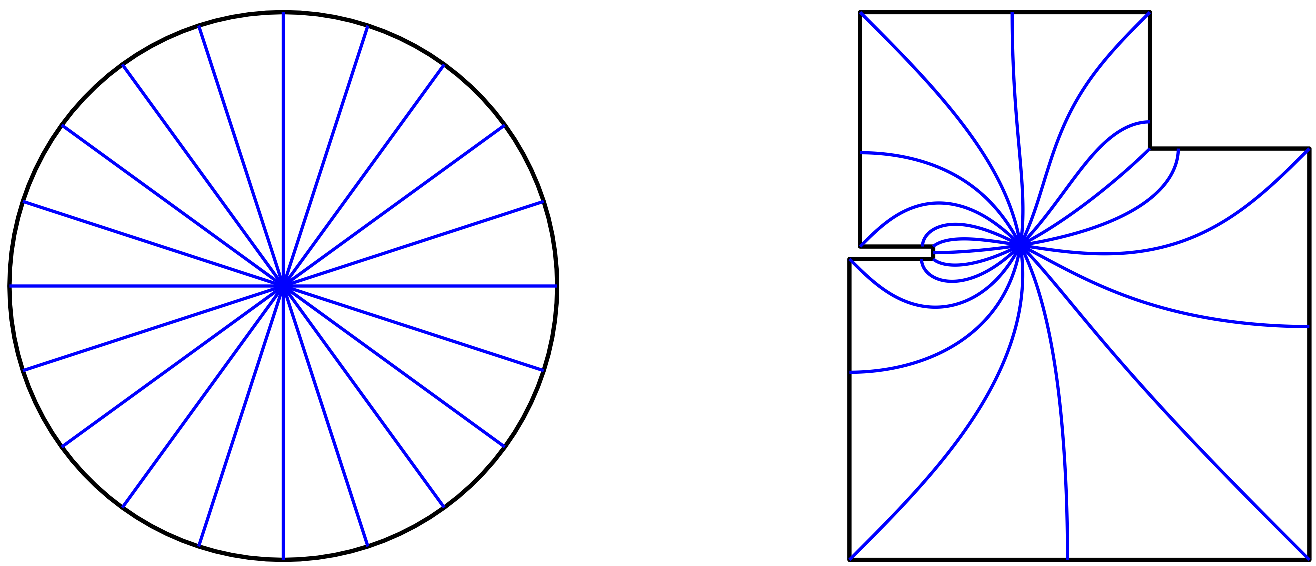


61/100 versus 34/100.

Riemann Mapping Theorem: If $\Omega \subsetneq \mathbb{R}^2$ is simply connected, then there is a conformal map $f : \mathbb{D} \to \Omega$. (conformal = angle preserving)



Brownian motion is conformally invariant, so normalized length measure maps to harmonic measure. Fastest way to compute harmonic measure.



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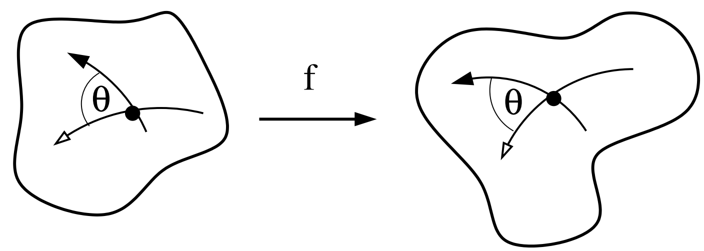

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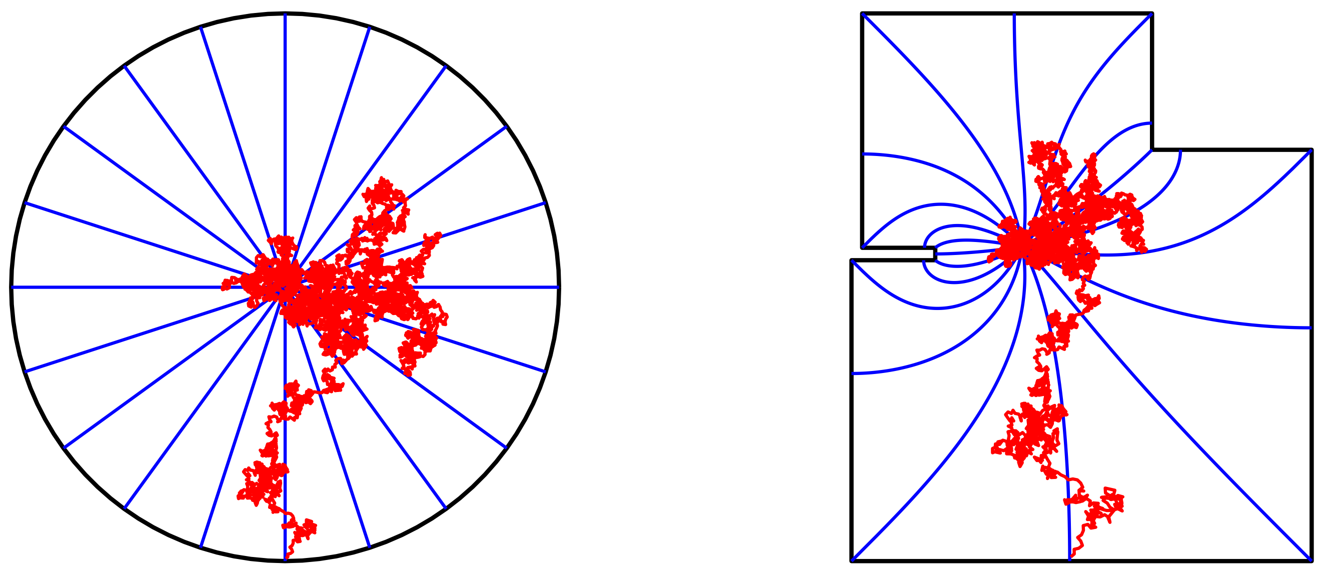

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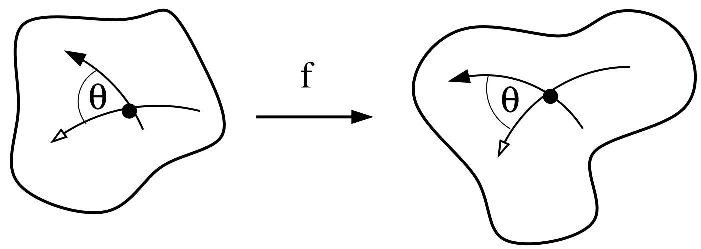

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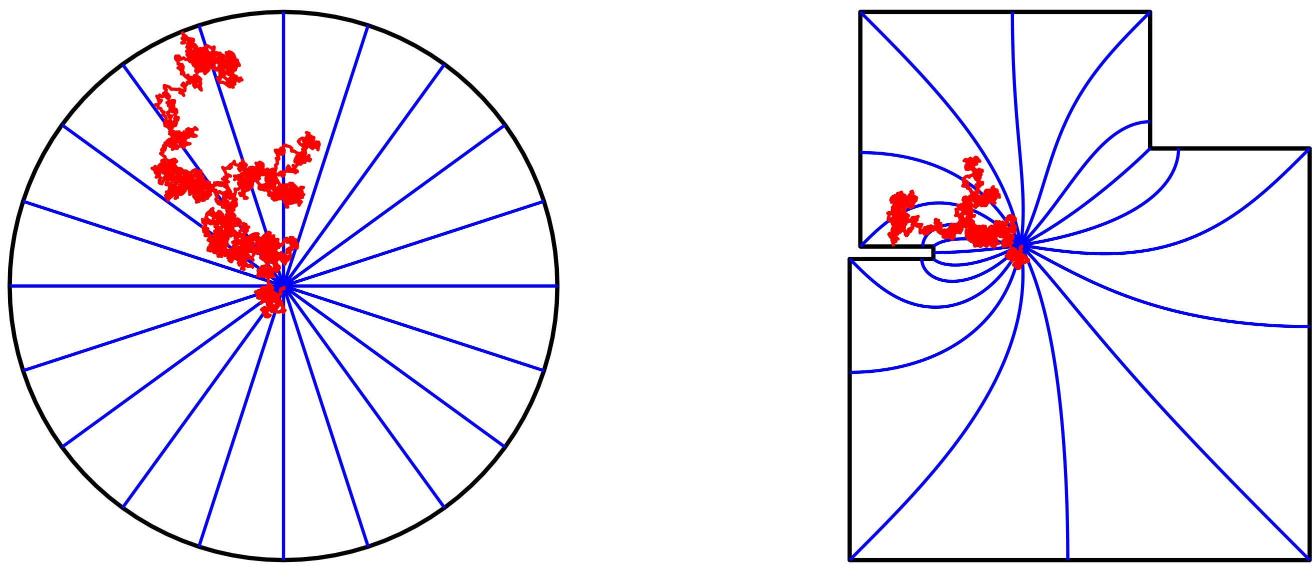

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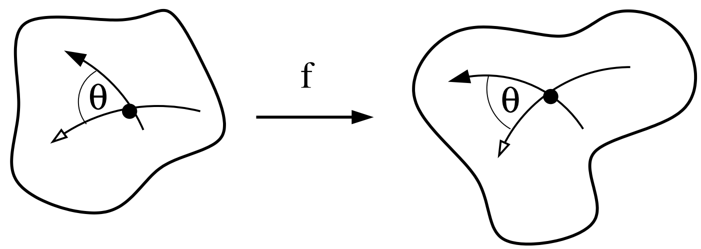

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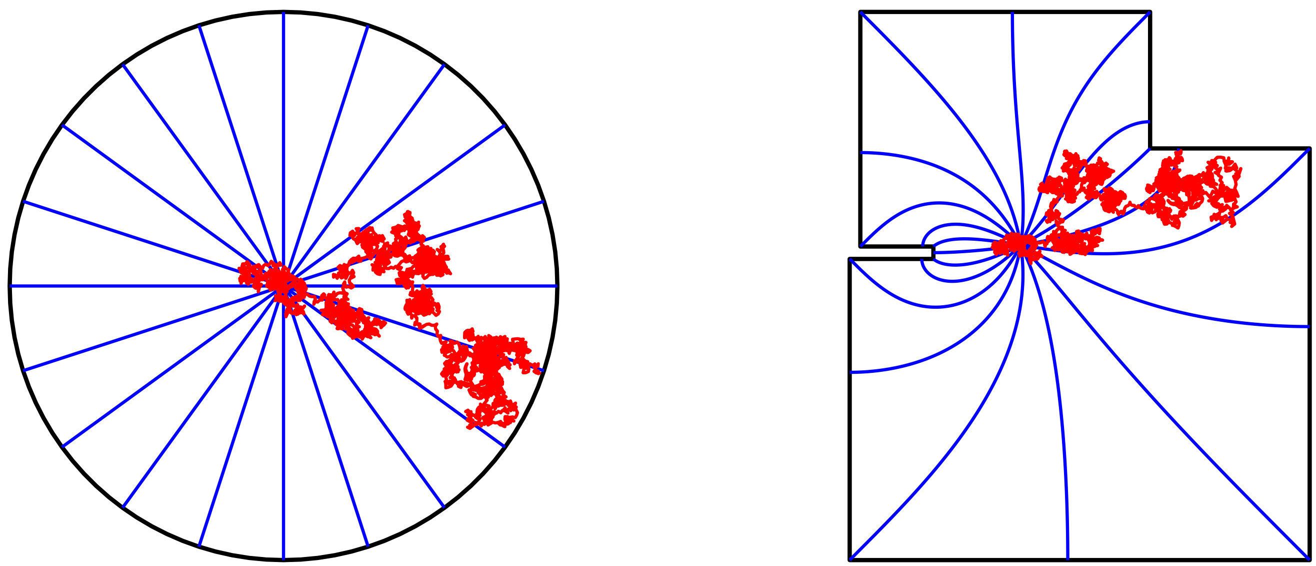
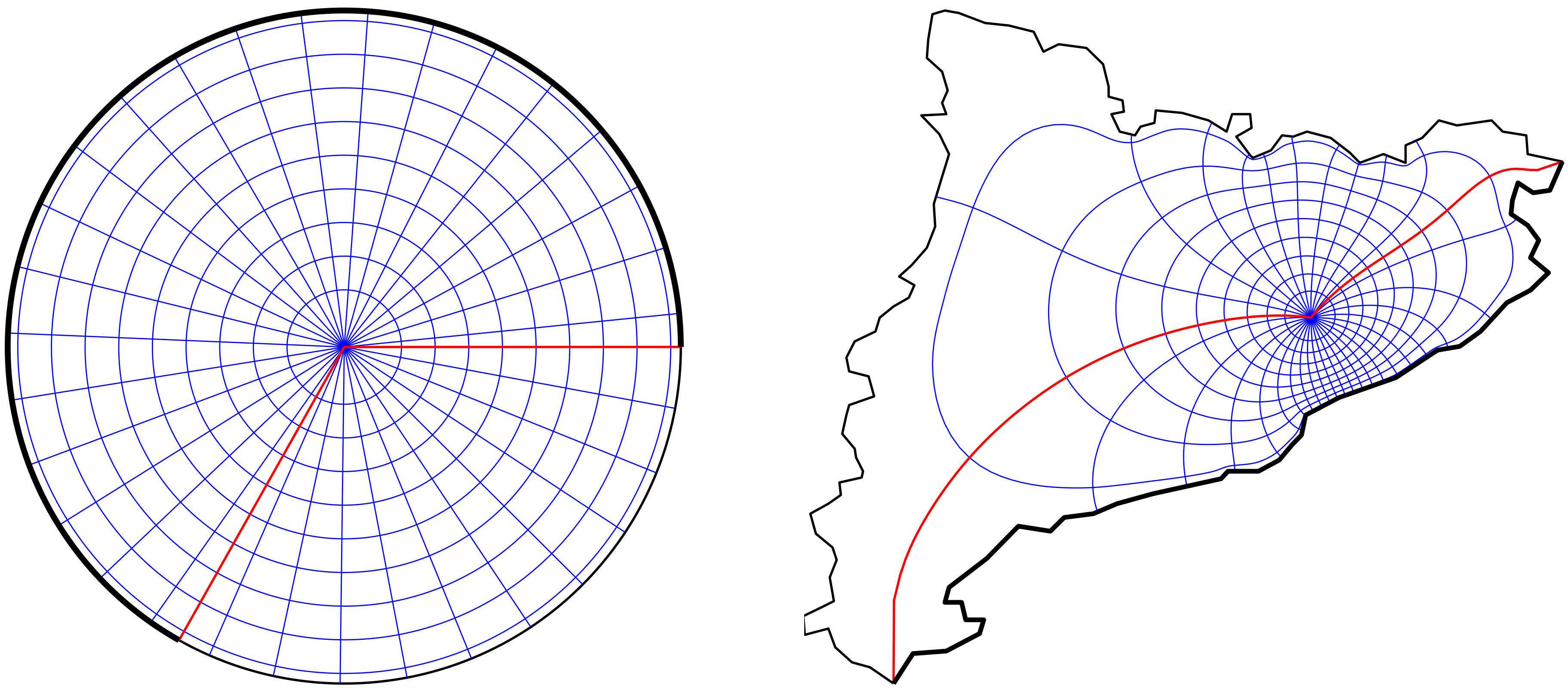
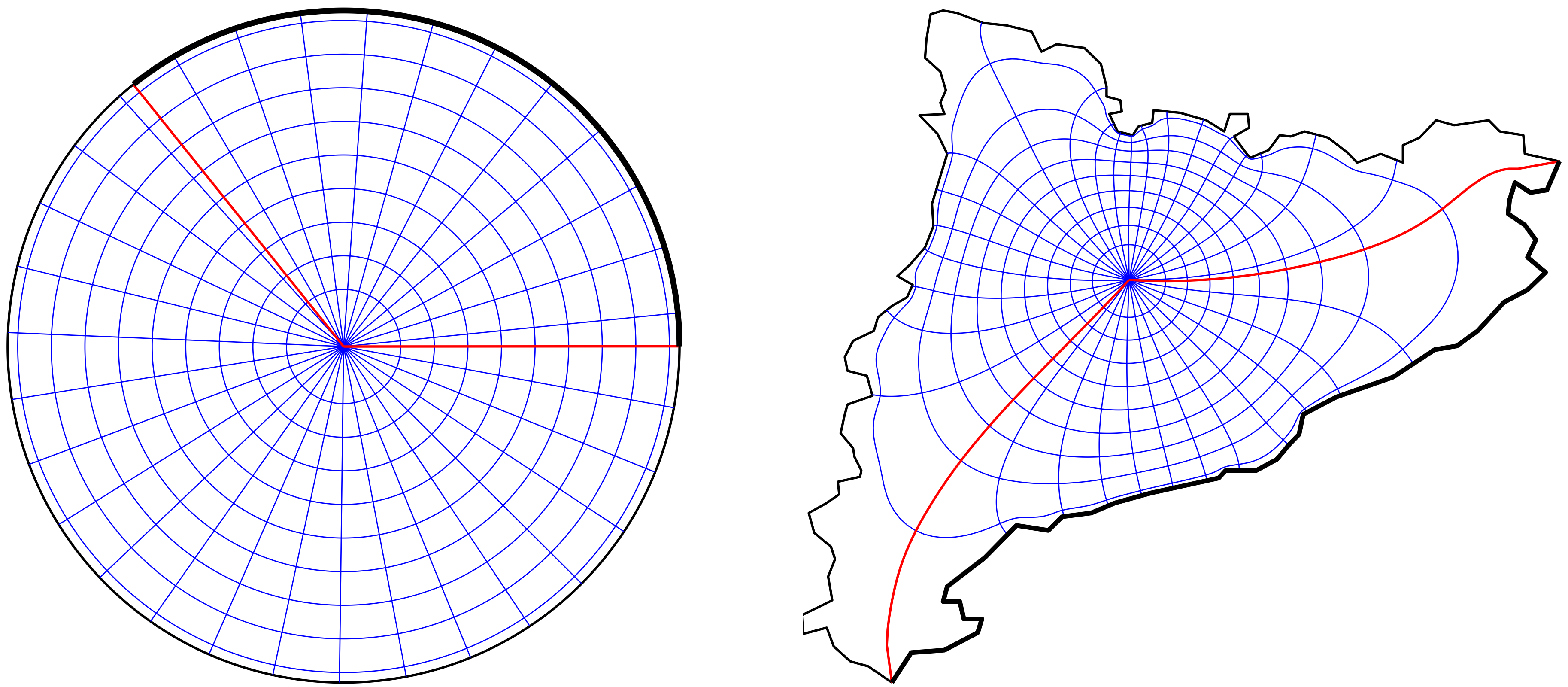
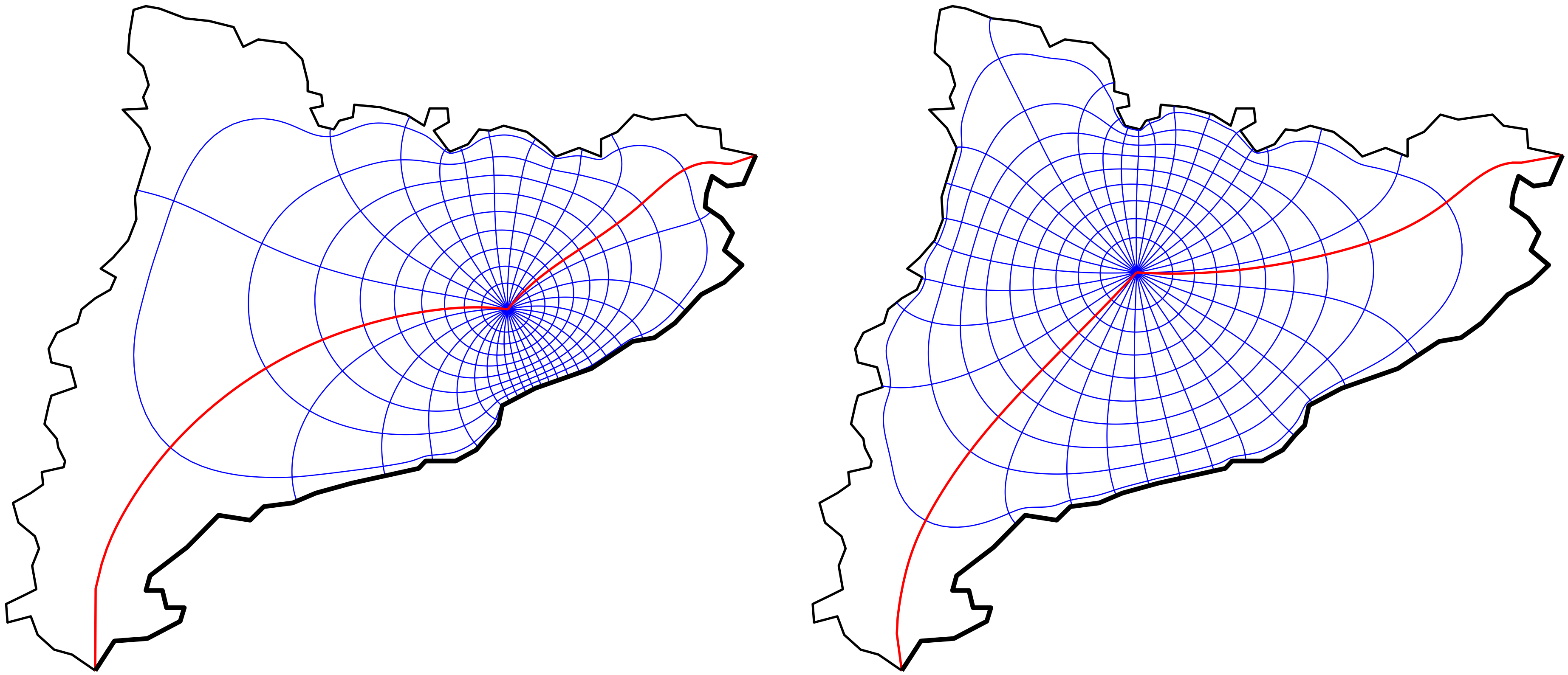




harmonic measure = .6683



harmonic measure = .3575



harmonic measure: .6683 versus .3575

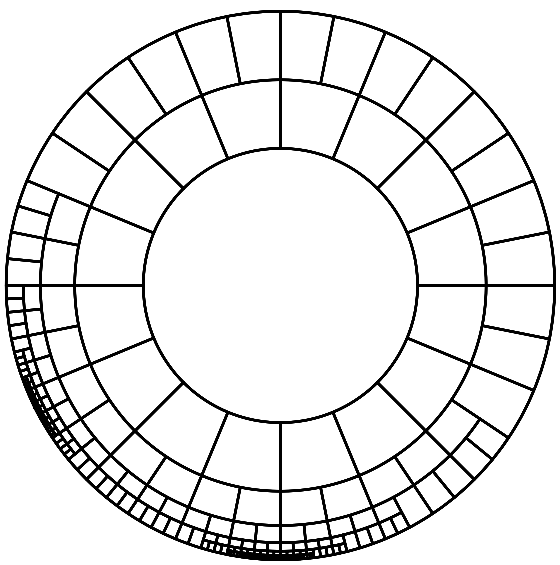
Some basic properties of harmonic measure:

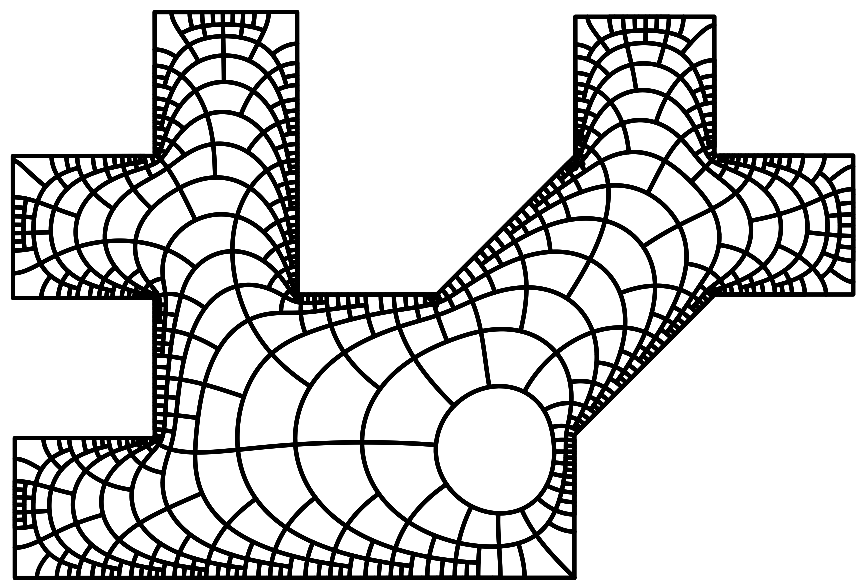
$\omega(z, E, \Omega)$ is the harmonic measure of $E \subset \partial\Omega$.

ω is harmonic in z and $0 \le \omega \le 1$

Harnack $\Rightarrow$ different base points give comparable measures.

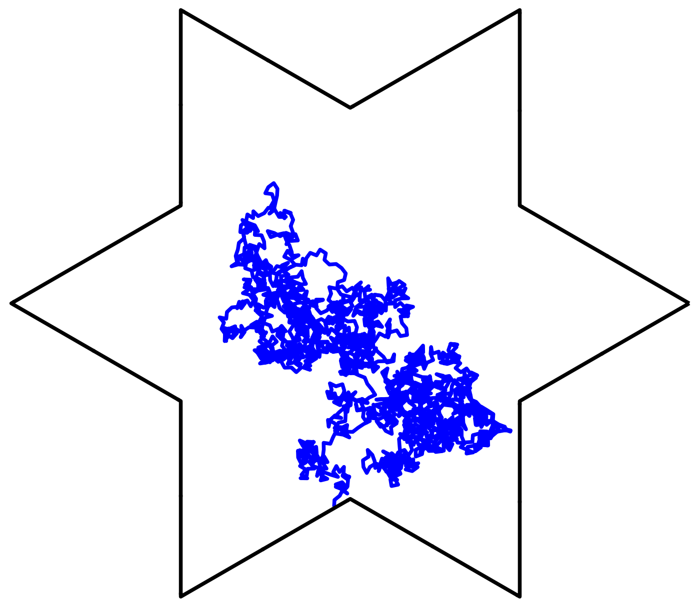
(If base points are on same side of curve.)





Thm (F & M Riesz 1916):

For rectifiable boundaries, $\omega(E) = 0$ iff E has zero length.



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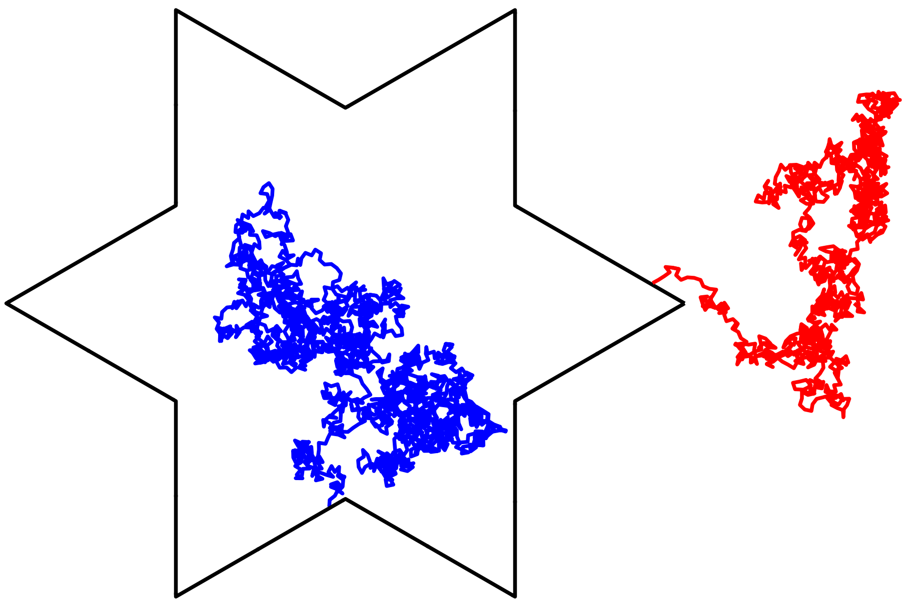
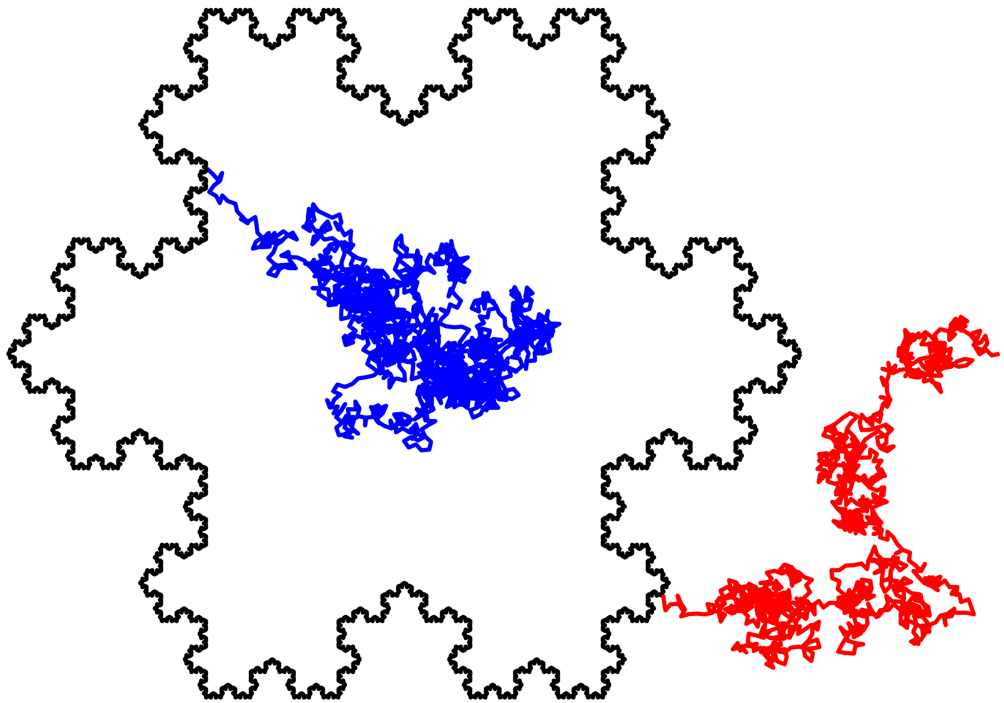


$\Rightarrow$ “Inside” and “outside” harmonic measures have same null sets.

$\Rightarrow \omega_1, \omega_2$ are mutually absolutely continuous.

Recent deep generalizations to $\mathbb{R}^n$ by Tolsa and others.

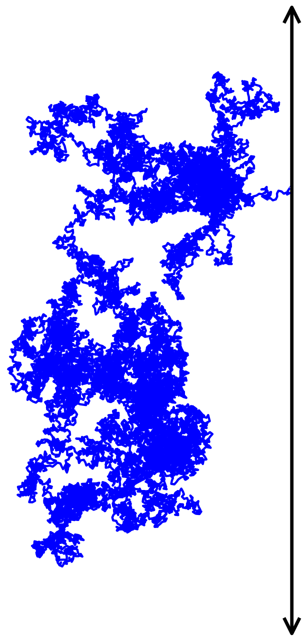
For a fractal curve, inside and outside harmonic measures are singular.



$\omega_1 \perp \omega_2$ iff tangents points have zero length.

For which curves is $\omega_1 = \omega_2$?

For which curves is $\omega_1 = \omega_2$? True for lines:



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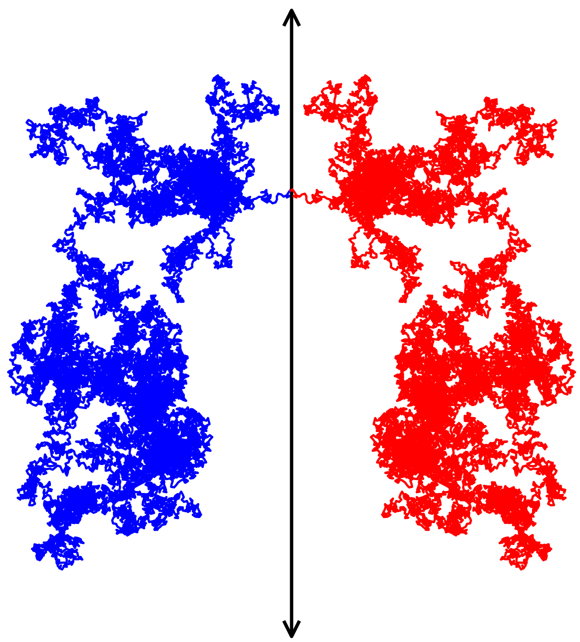

Also for circles (= lines conformally).

Converse? If $\omega_1 = \omega_2$ must Γ be a circle/line?

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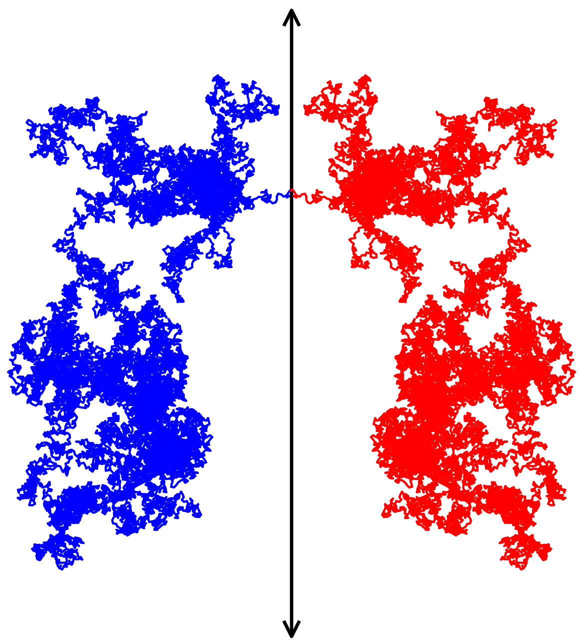
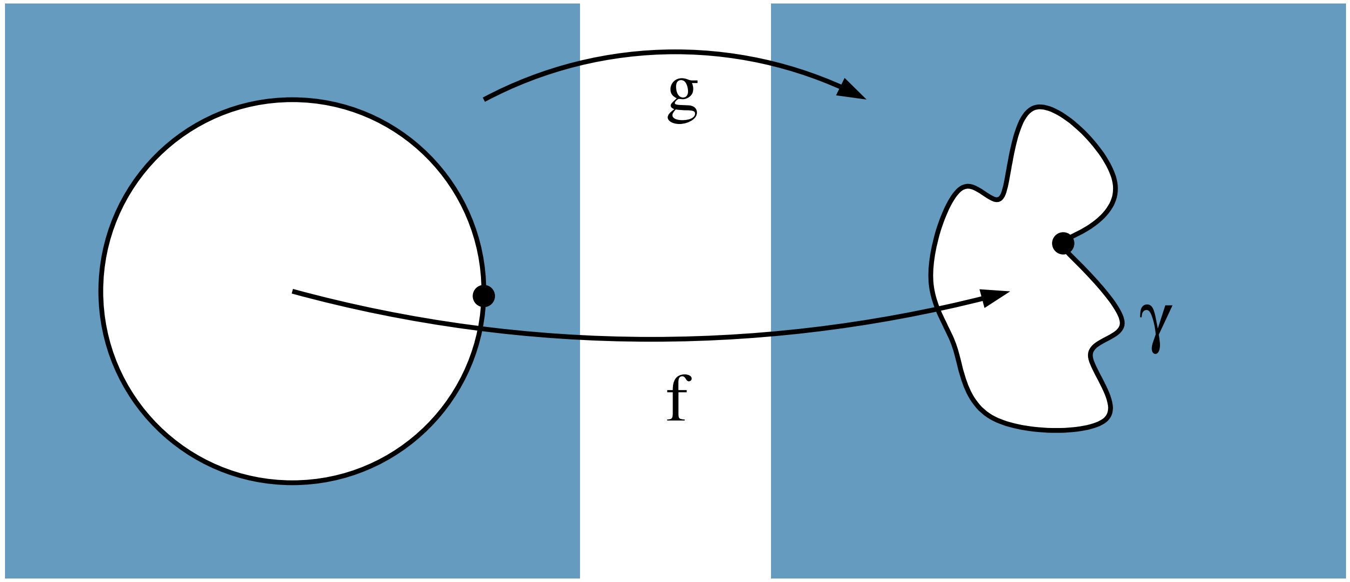


Also for circles (= lines conformally).

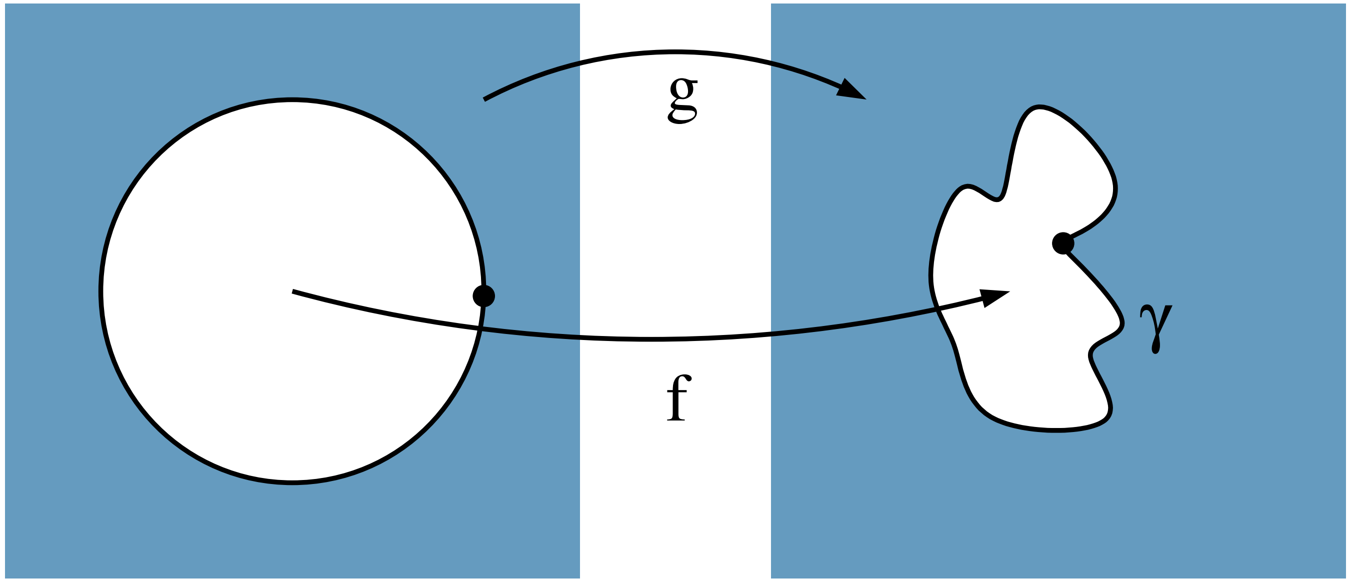
Converse? If $\omega_1 = \omega_2$ must Γ be a circle/line? **Yes**



Suppose $\omega_1 = \omega_2$ for a curve γ.

Conformally map two sides of circle to two sides of γ so $f(1) = g(1)$.

$\omega_1 = \omega_2$ implies maps agree on whole boundary.



Suppose $\omega_1 = \omega_2$ for a curve γ.

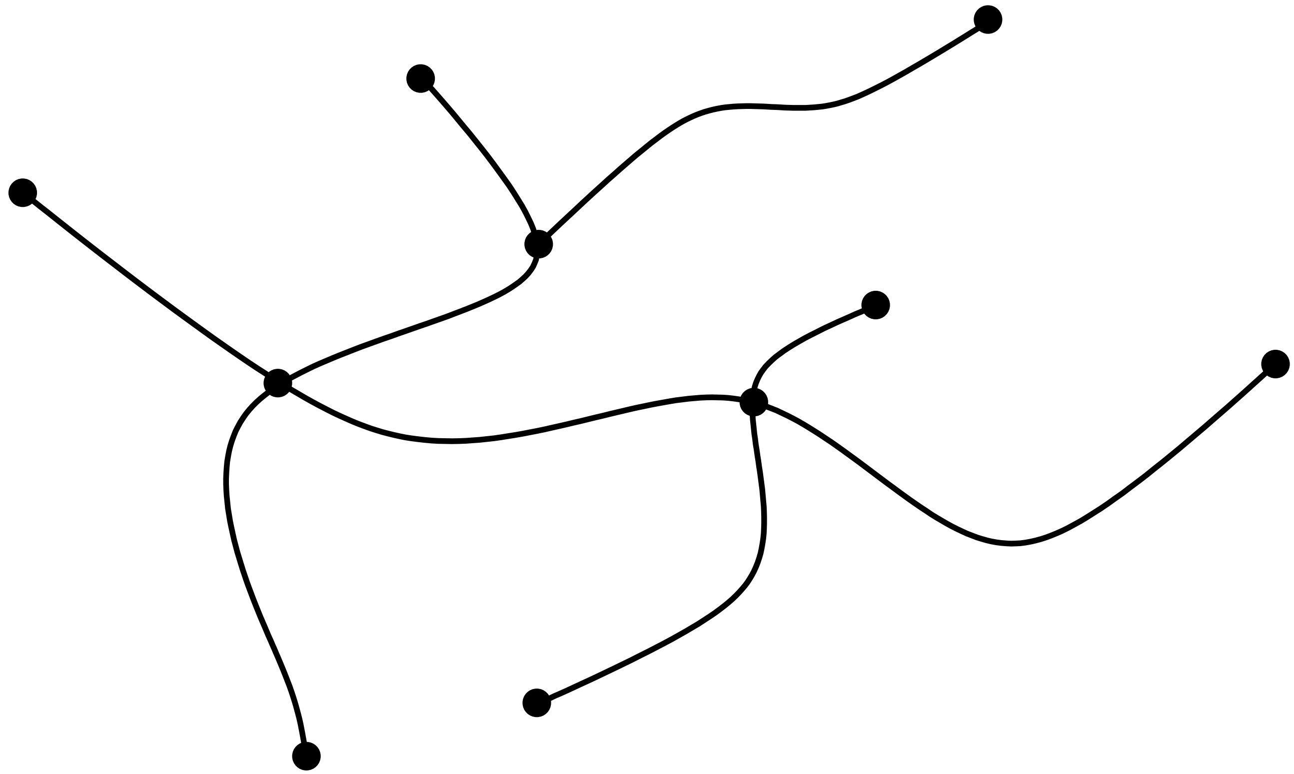
Conformally map two sides of circle to two sides of γ so $f(1) = g(1)$.

$\omega_1 = \omega_2$ implies maps agree on whole boundary.

So f, g define homeomorphism h of plane holomorphic off circle.

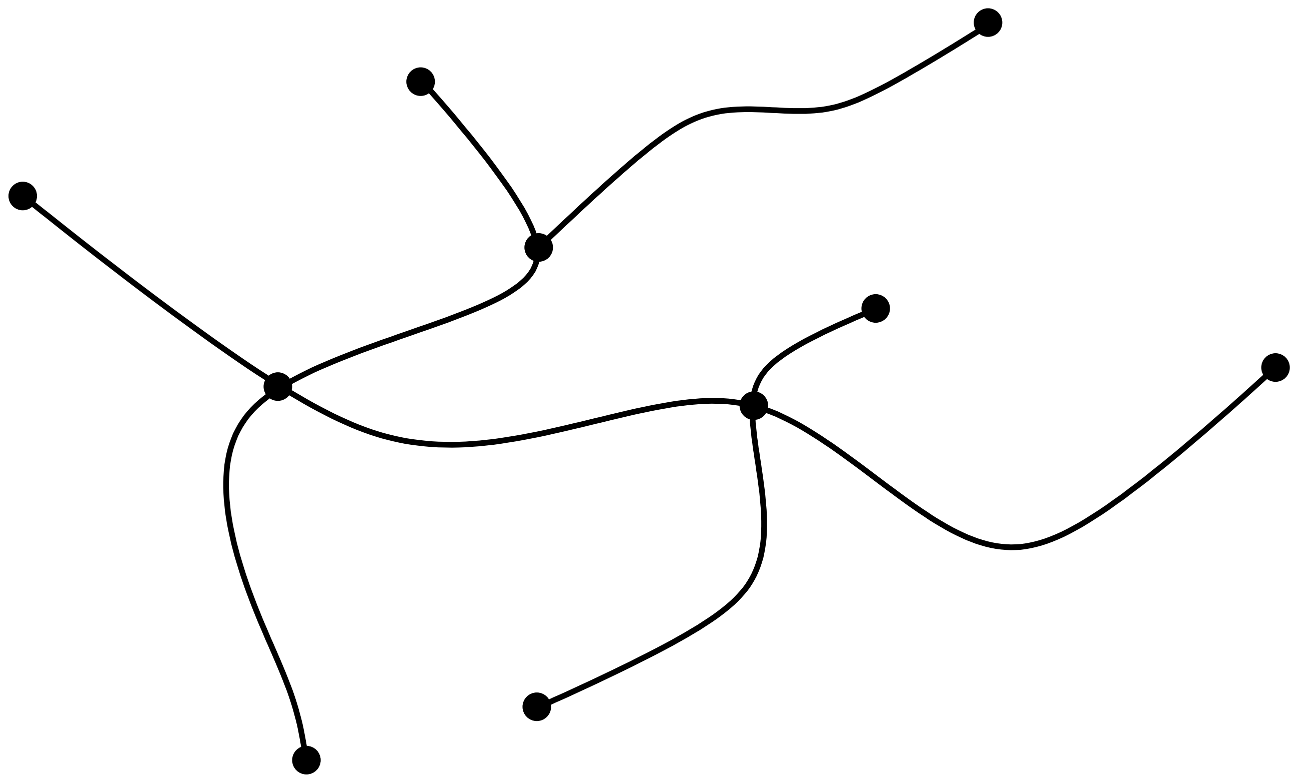
Then h is entire by Morera's theorem.

Entire and 1-1 implies h is linear (Liouville's thm), so γ is a circle.



A planar graph is a finite set of points connected by non-crossing edges.

It is a tree if there are no closed loops.



A planar tree is **conformally balanced** if

- every edge has equal harmonic measure from ∞
- edge subsets have same measure from both sides

This is also called a "**true tree**" (true form of a tree).

A line segment is an example.

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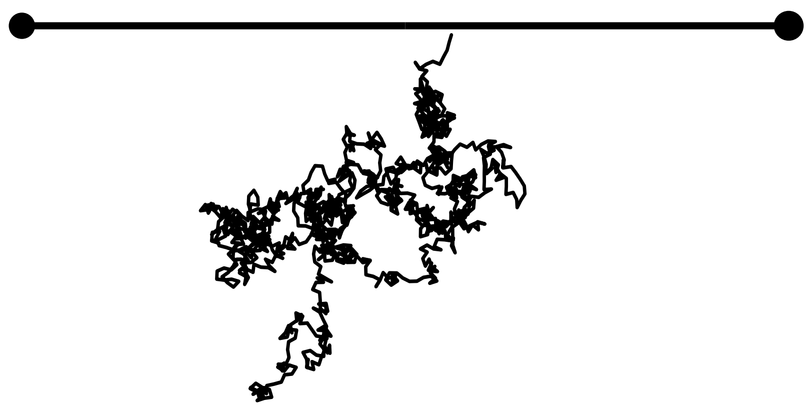
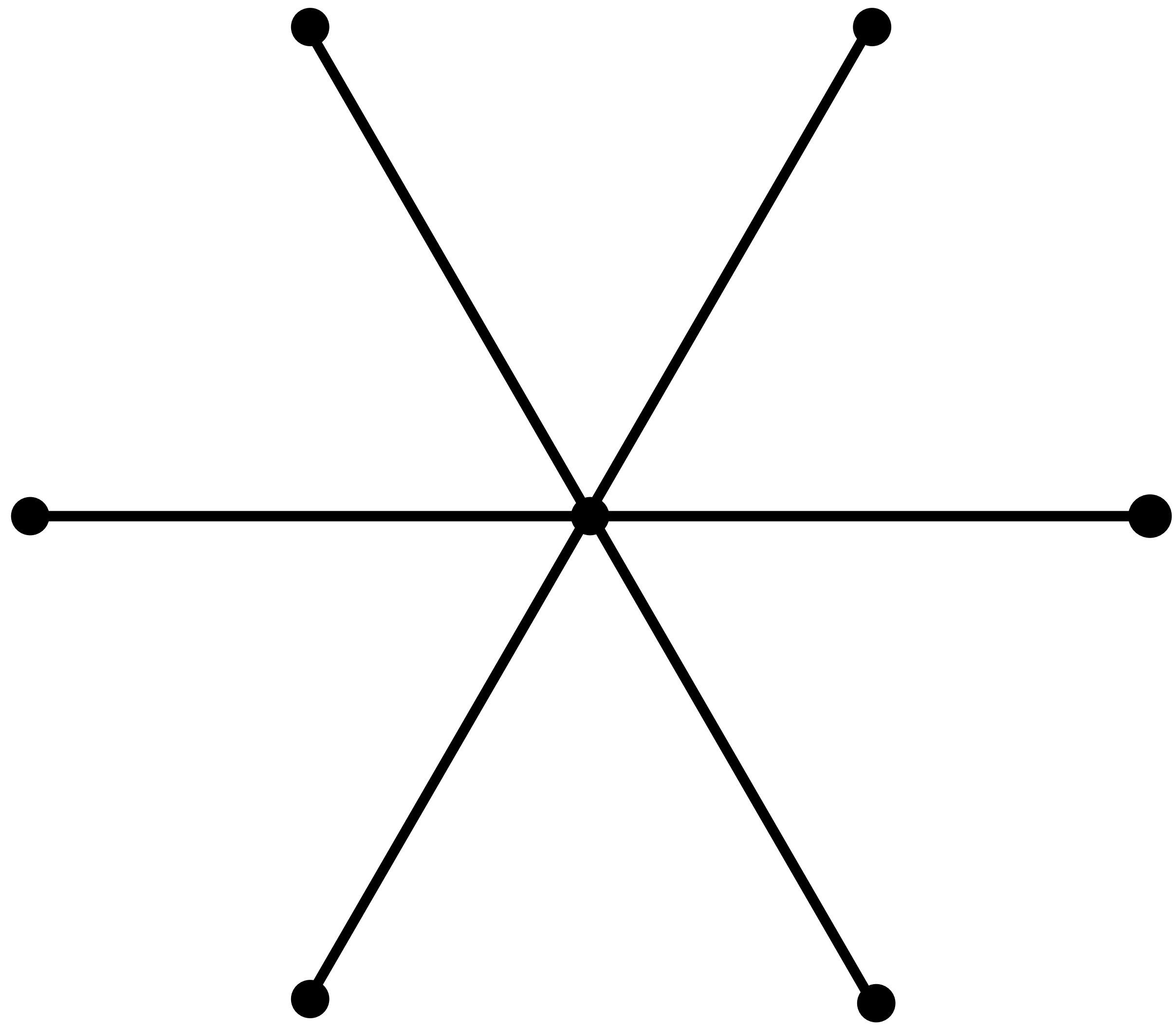
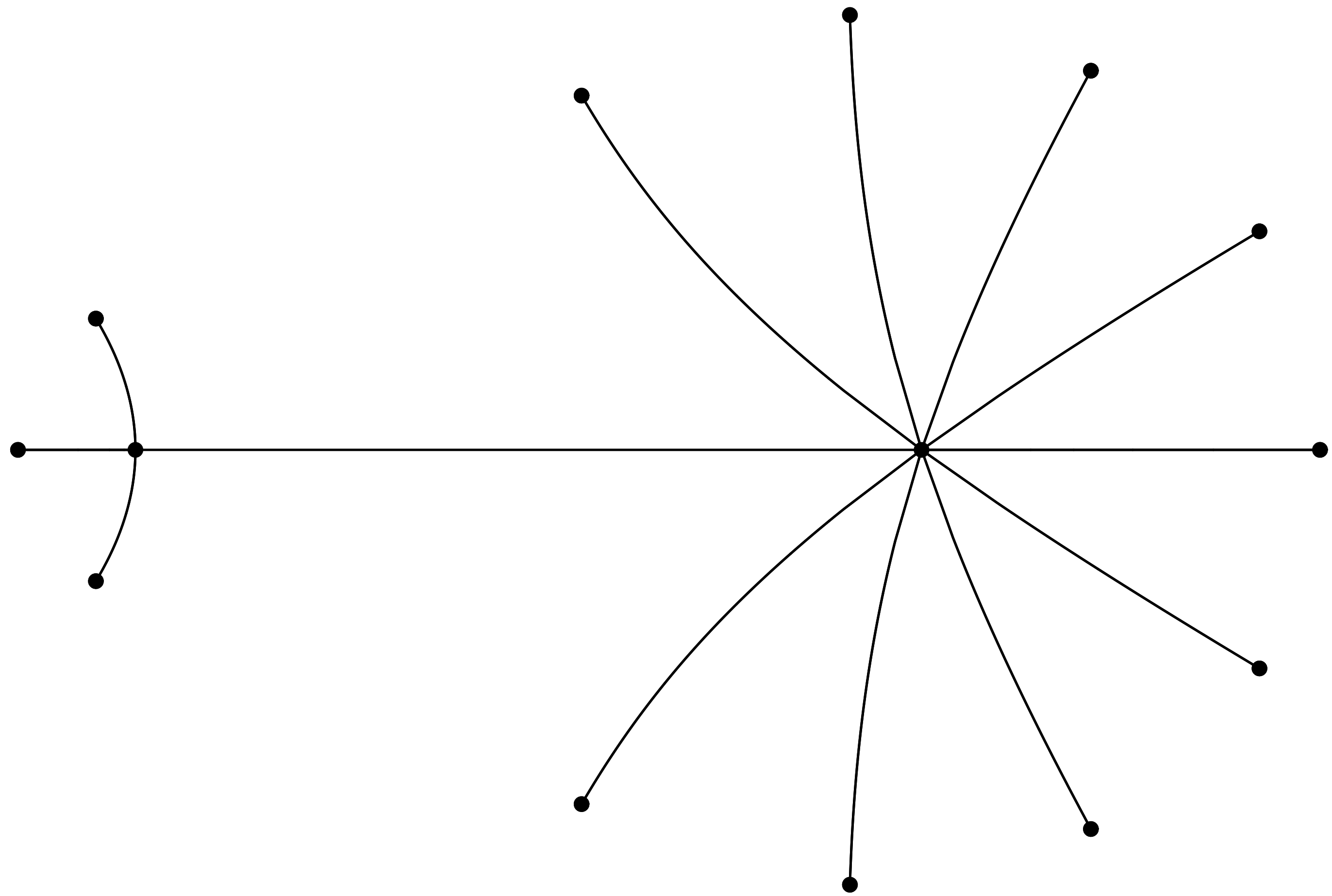
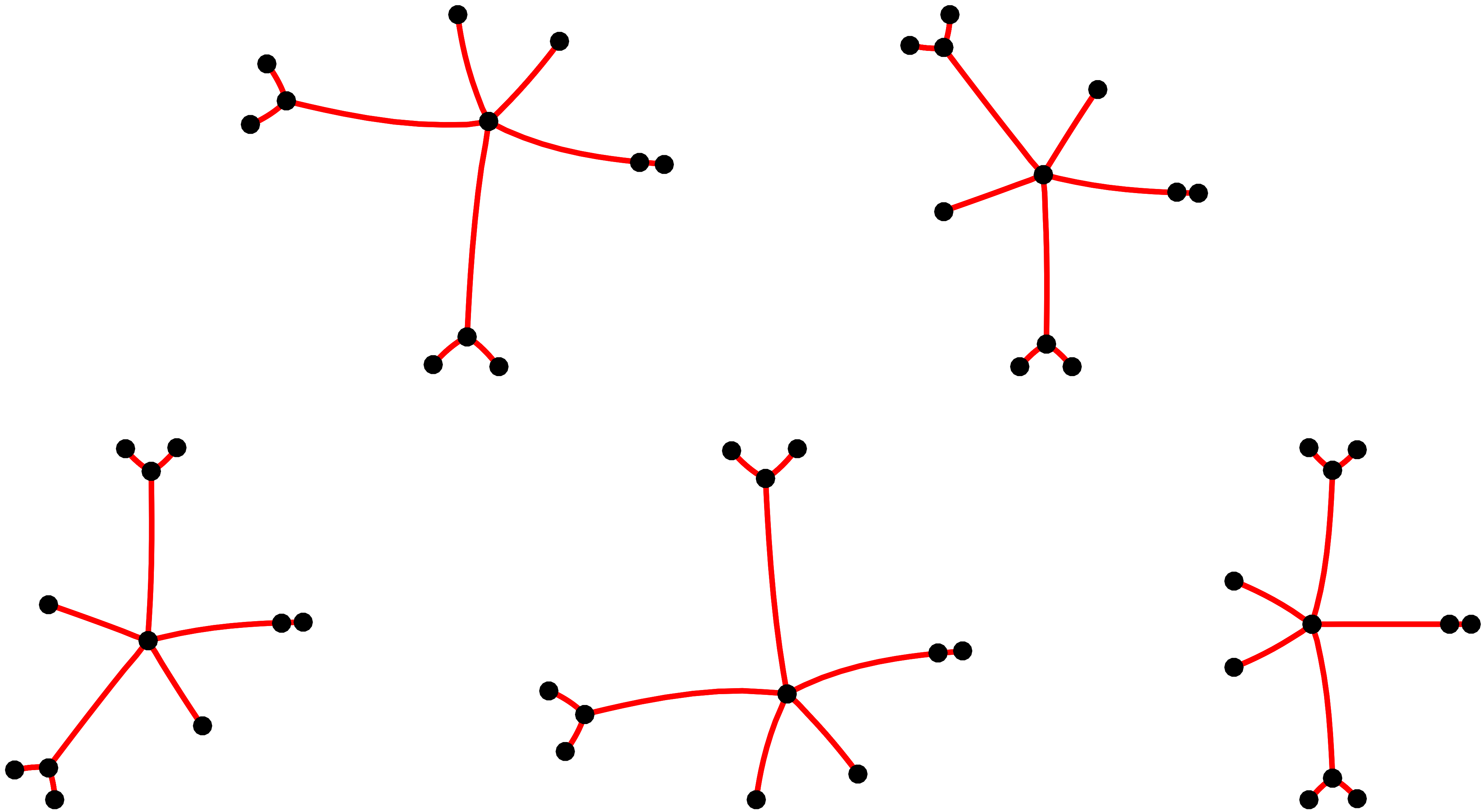


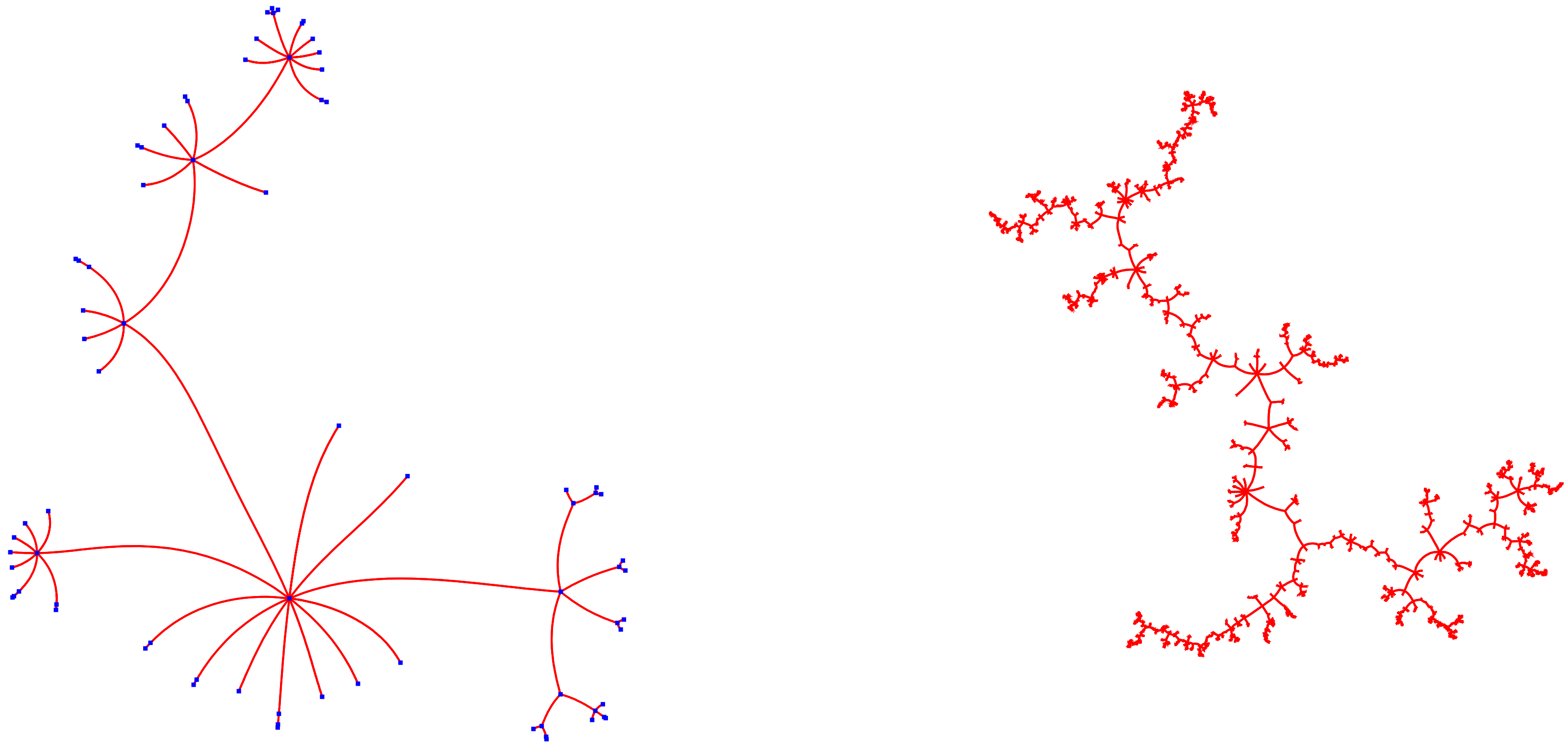


Trivially true by symmetry



Non-obvious true tree



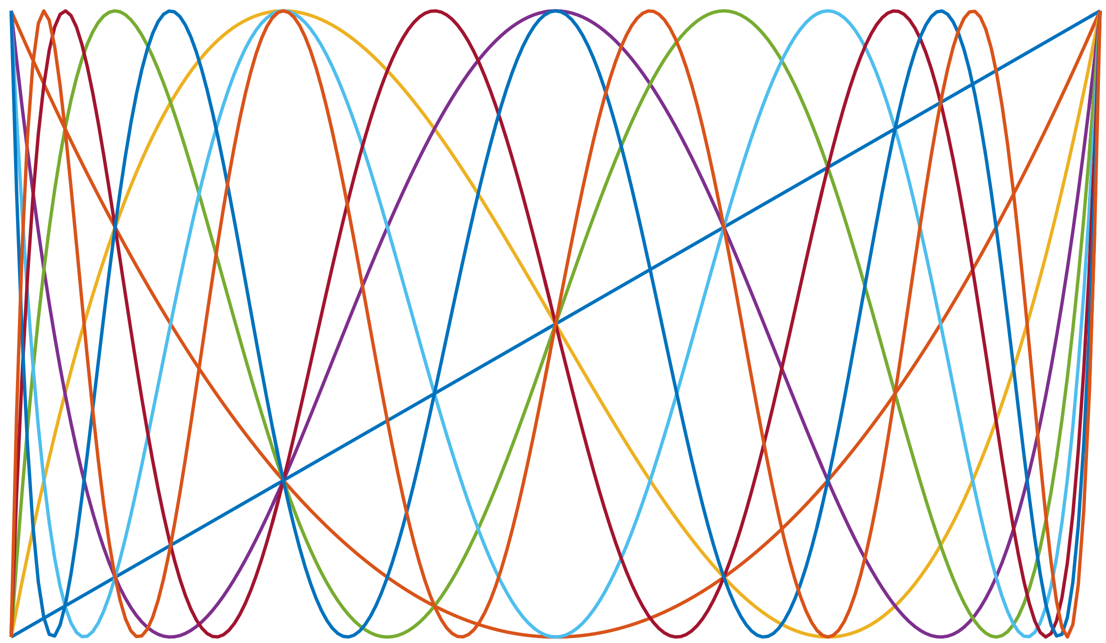


Some random true trees by Don Marshall and Steffen Rohde

Definition of critical value: if p = polynomial, then

$$\mathrm{CV}(p) = \{p(z) : p'(z) = 0\} = \text{critical values}$$

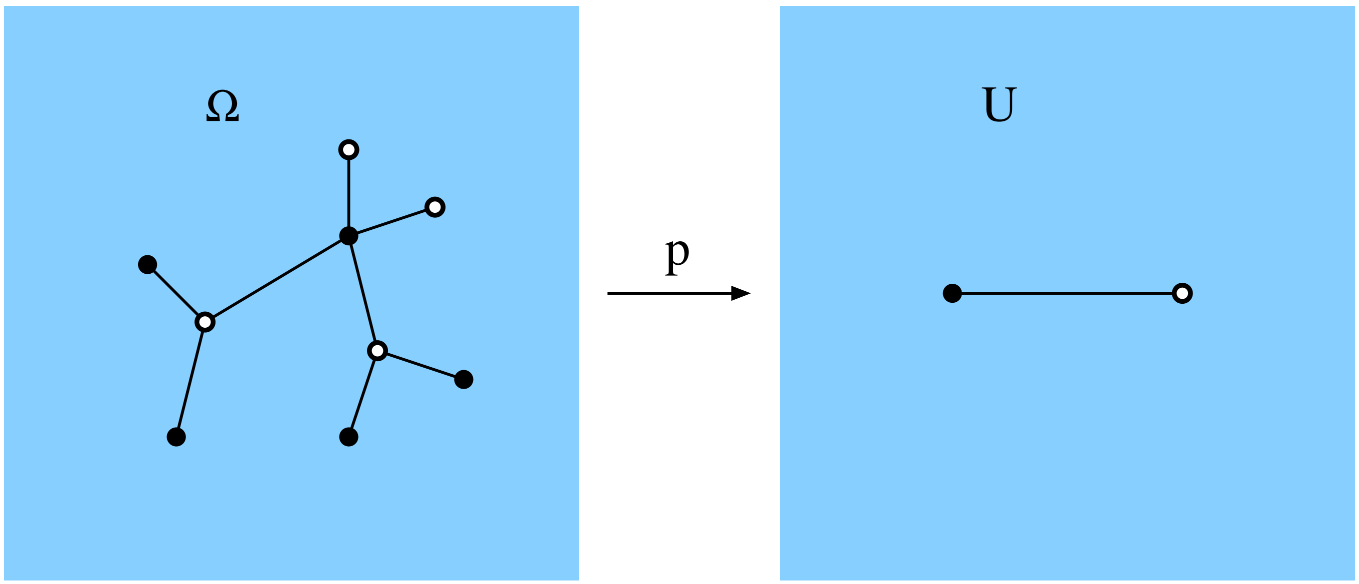
If $\mathrm{CV}(p) = \pm 1$, p is called **generalized Chebyshev** or **Shabat**.



10 classical Chebyshev polynomials

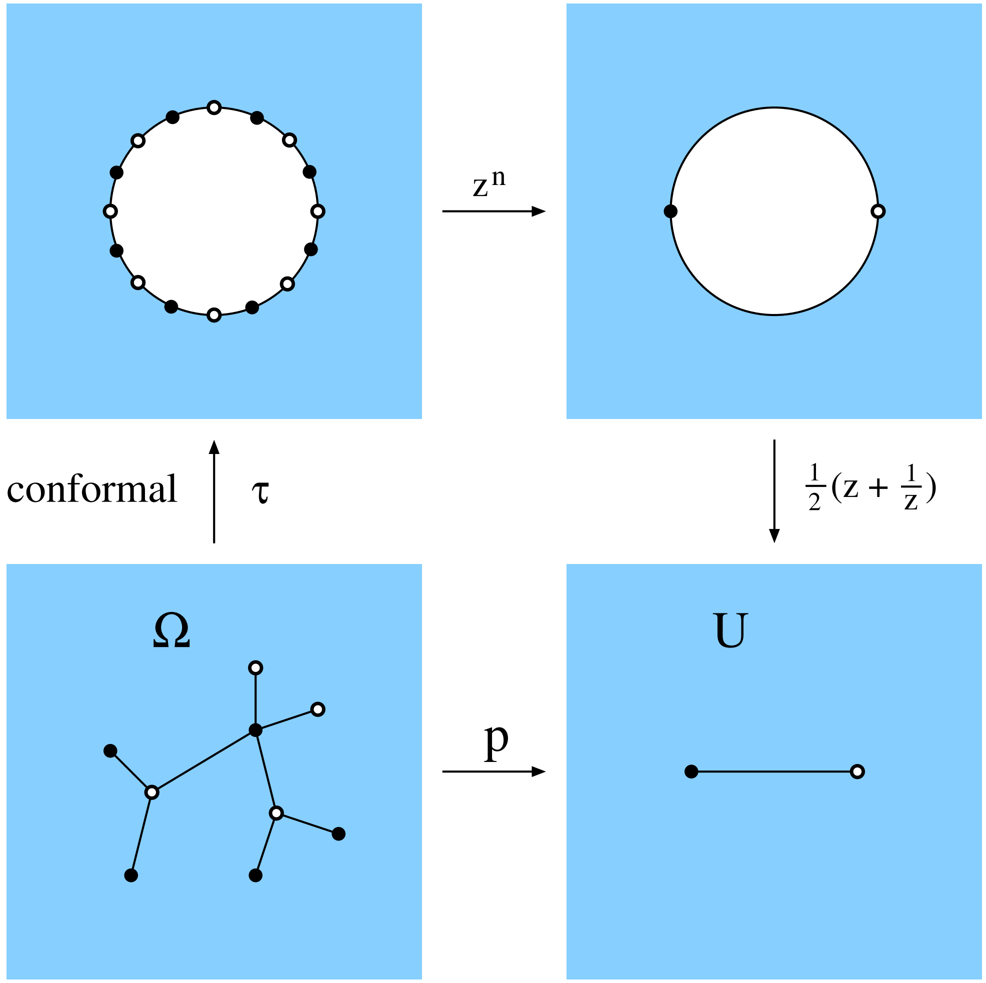
Balanced trees $\leftrightarrow$ Shabat polynomials

Fact: T is balanced iff $T = p^{-1}([-1, 1])$, $p =$ Shabat.



$\Omega = \mathbb{C} \setminus T$ $\qquad$ $U = \mathbb{C} \setminus [-1, 1]$

T conformally balanced $\Leftrightarrow$ p Shabat.

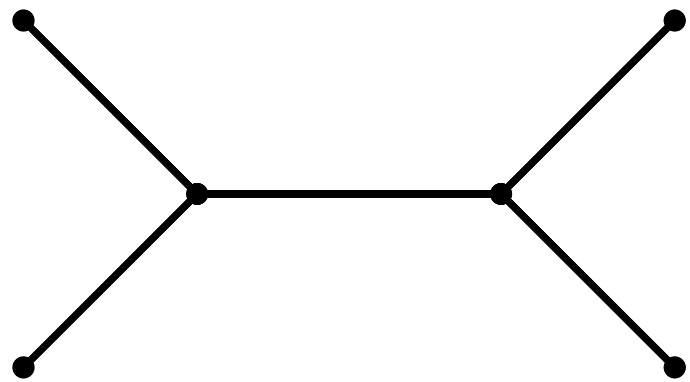


p is entire and n-to-1 $\Leftrightarrow$ p = polynomial.

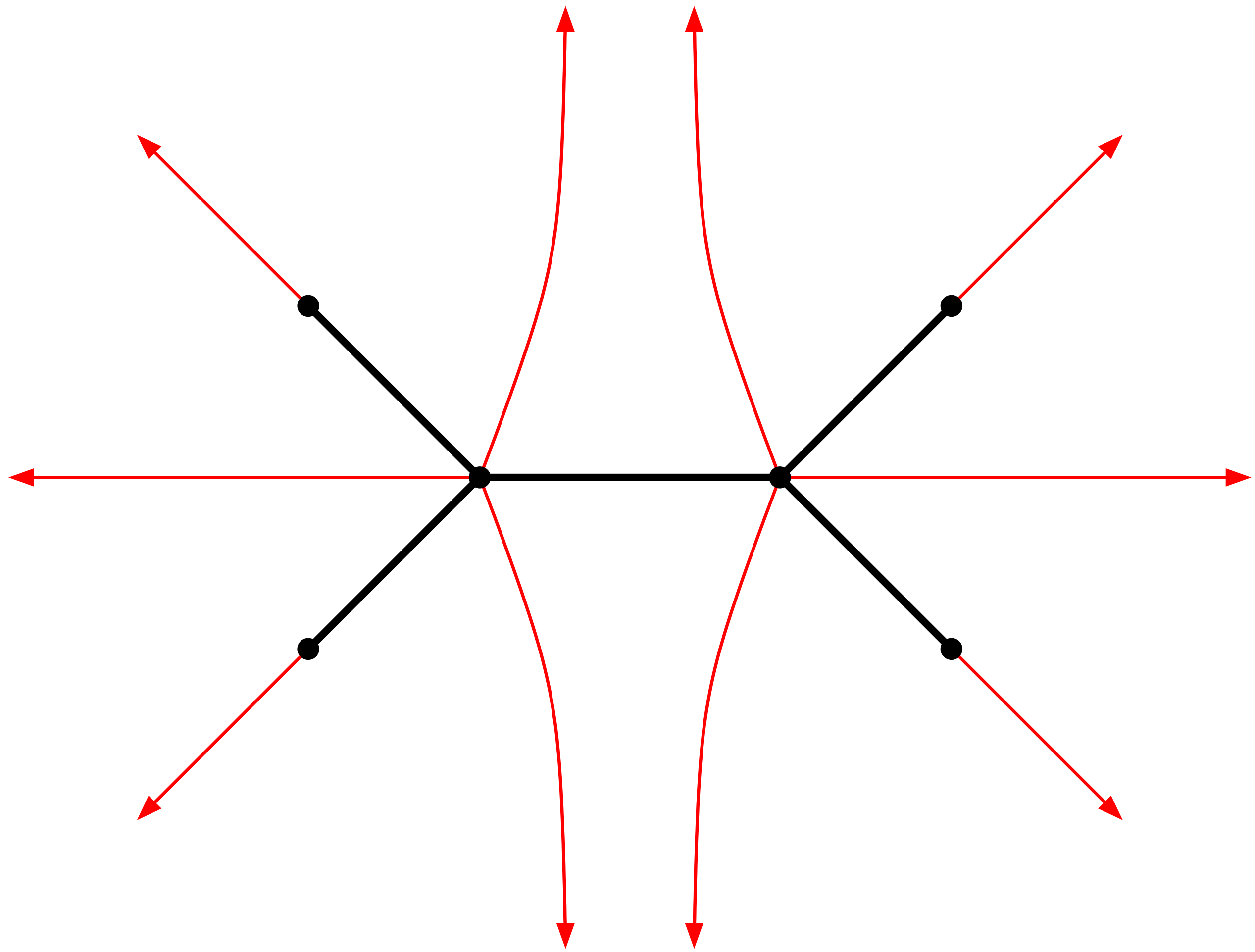
$\mathrm{CV}(p) \notin U \Leftrightarrow p : \Omega \to U$ is covering map.

Theorem: Every finite tree has a true form.

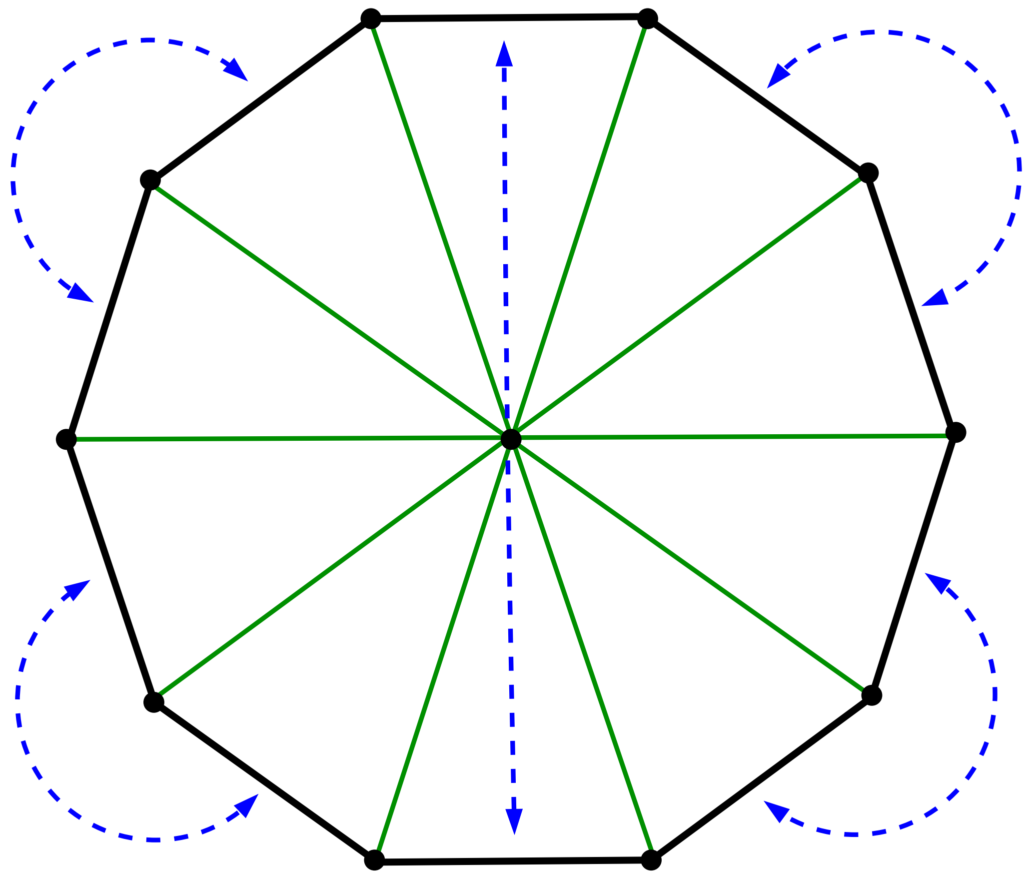
Standard proof uses the uniformization theorem.



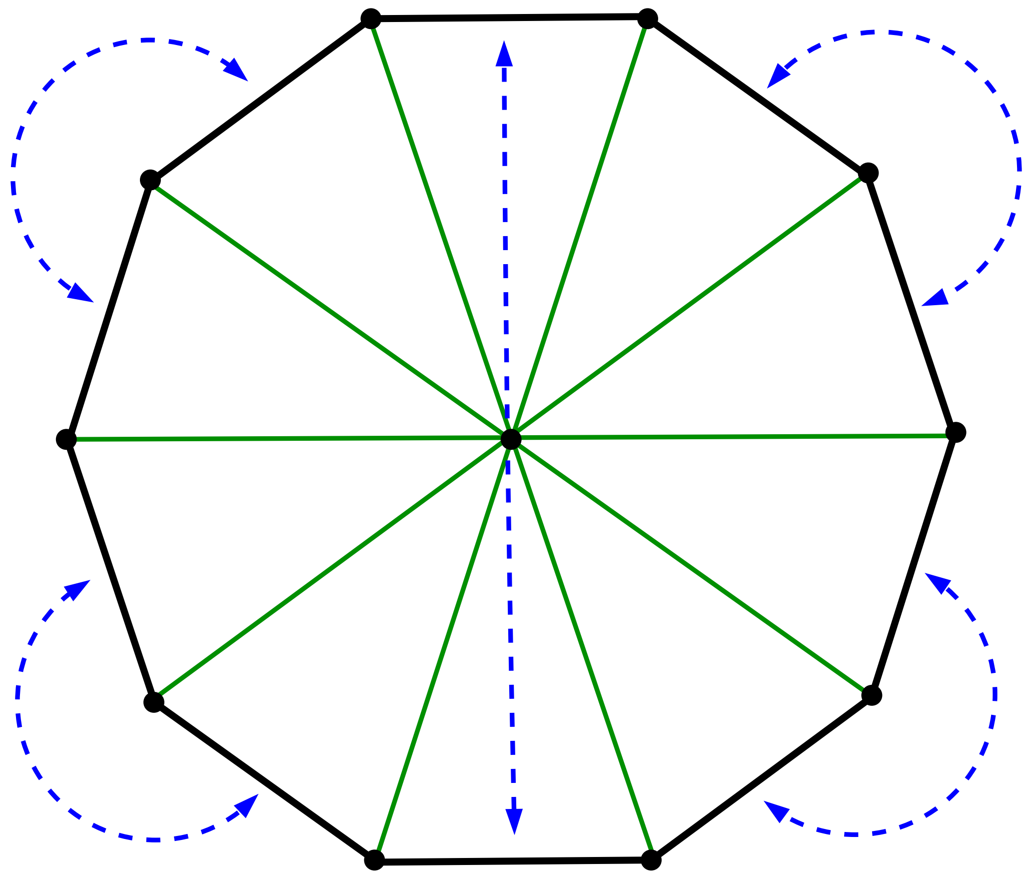
- start with a finite tree.



- connect vertices of T to infinity; gives finite triangulation of sphere.
- Defines adjacencies between triangles.



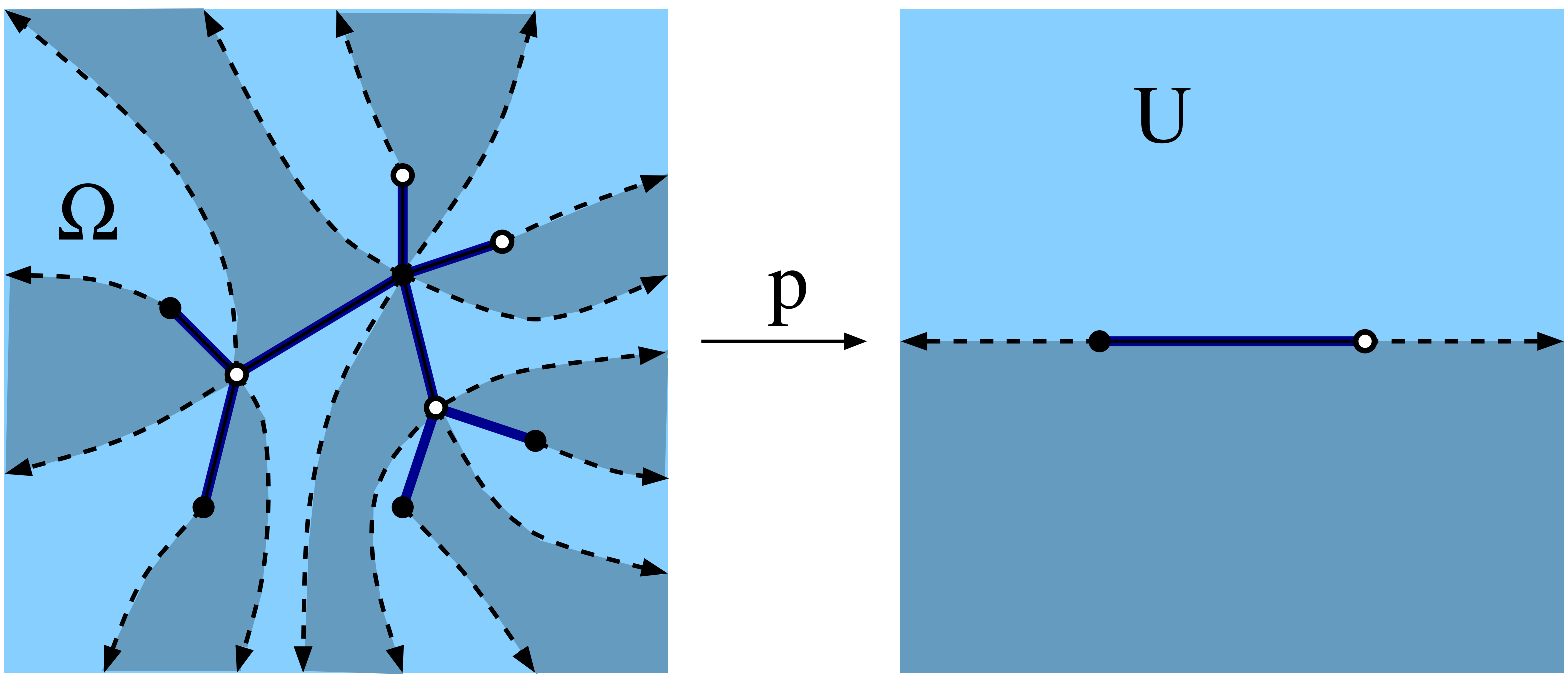
- Glue equilateral triangles using adjacencies. Get conformal 2-sphere.
- By uniformization theorem, equivalent to Riemann sphere.
- Tree maps to balanced tree. Shape unclear.



- Note: conformal 2-sphere has many "equilateral triangulations".
- Which other Riemann surfaces can be obtained in this way?

Belyi function = holomorphic map from Riemann surface with 3 critical values (usually $0, 1, \infty$).

Theorem: A Riemann surface can constructed from equilateral triangles iff it has a Belyi function.



Triangles are inverse images of upper and lower half-planes.

Belyi function $=$ holomorphic map from Riemann surface with 3 critical values (usually $0, 1, \infty$).

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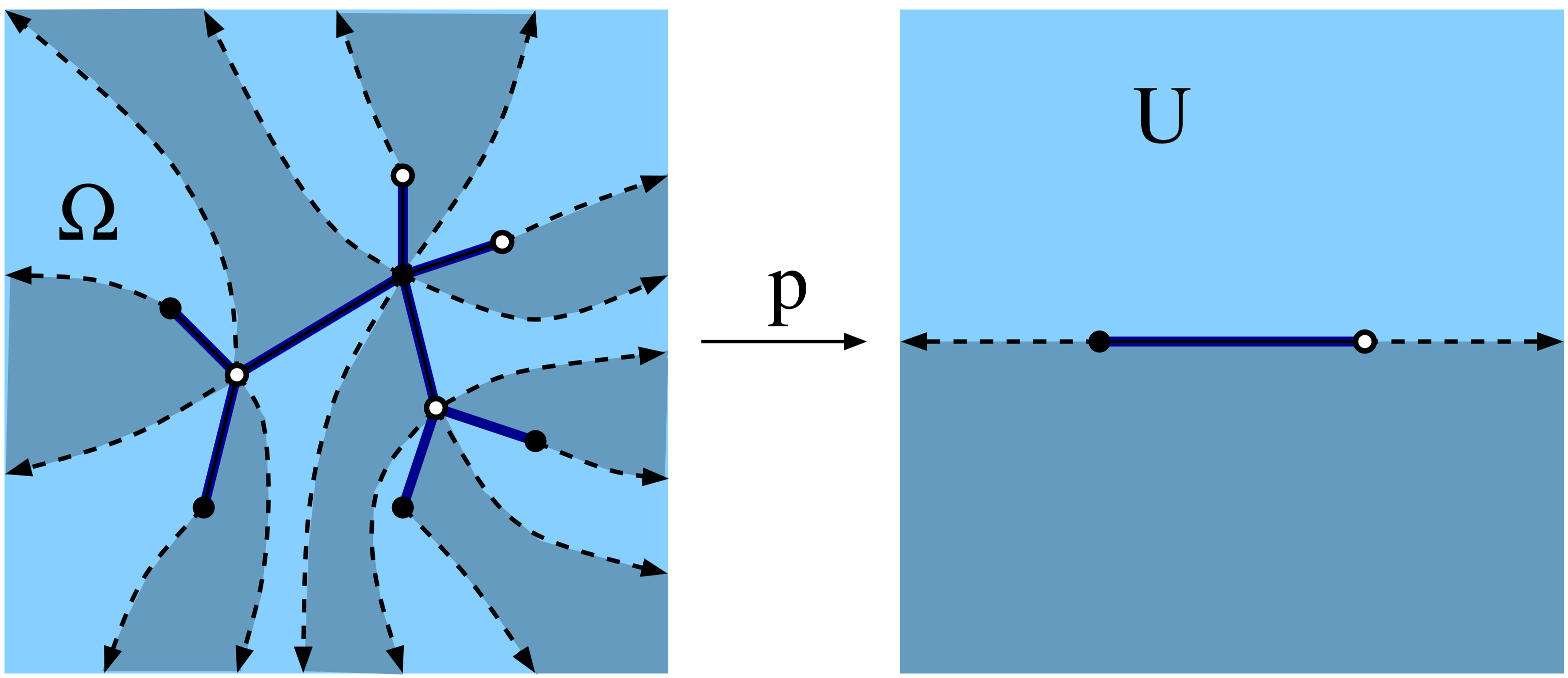

A triangulation is equilateral iff every edge has an anti-holomorphic reflection fixing it pointwise and swapping the two adjacent triangles.

Belyi function $=$ holomorphic map from Riemann surface with 3 critical values (usually $0, 1, \infty$).

Theorem: A Riemann surface can constructed from equilateral triangles iff it has a Belyi function.

There are only finitely many ways to glue together n triangles.

$\Rightarrow$ only countably many compact surfaces have a Belyi function.

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There are only finitely many ways to glue together n triangles.

$\Rightarrow$ only countably many compact surfaces have a Belyi function.

Belyi's Thm: A compact surface has a Belyi function iff it is algebraic.

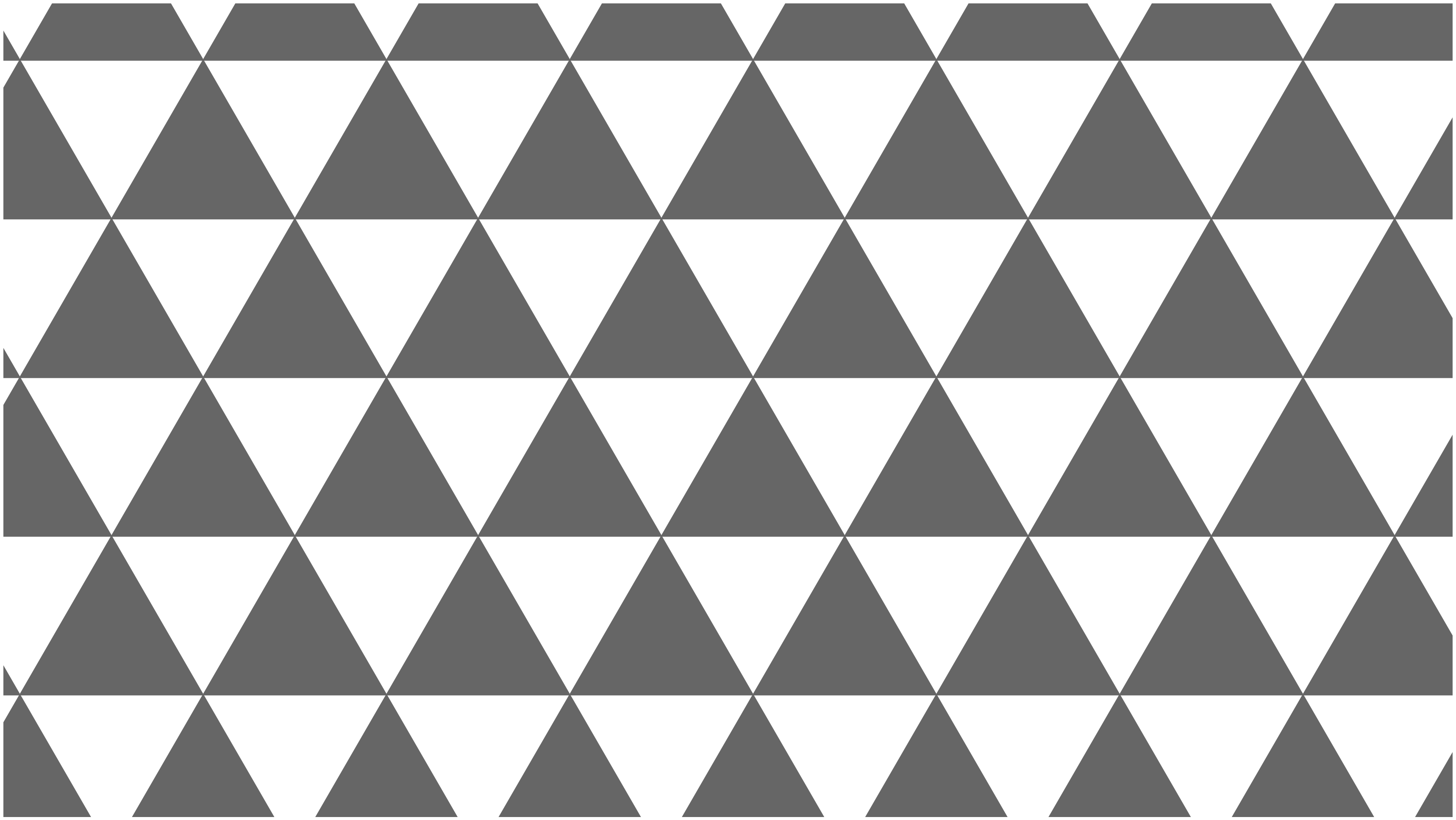
(Zero set of $P(z, w)$ with algebraic integers as coefficients).

Motivated Grothendieck's theory of *dessins d'enfant*.

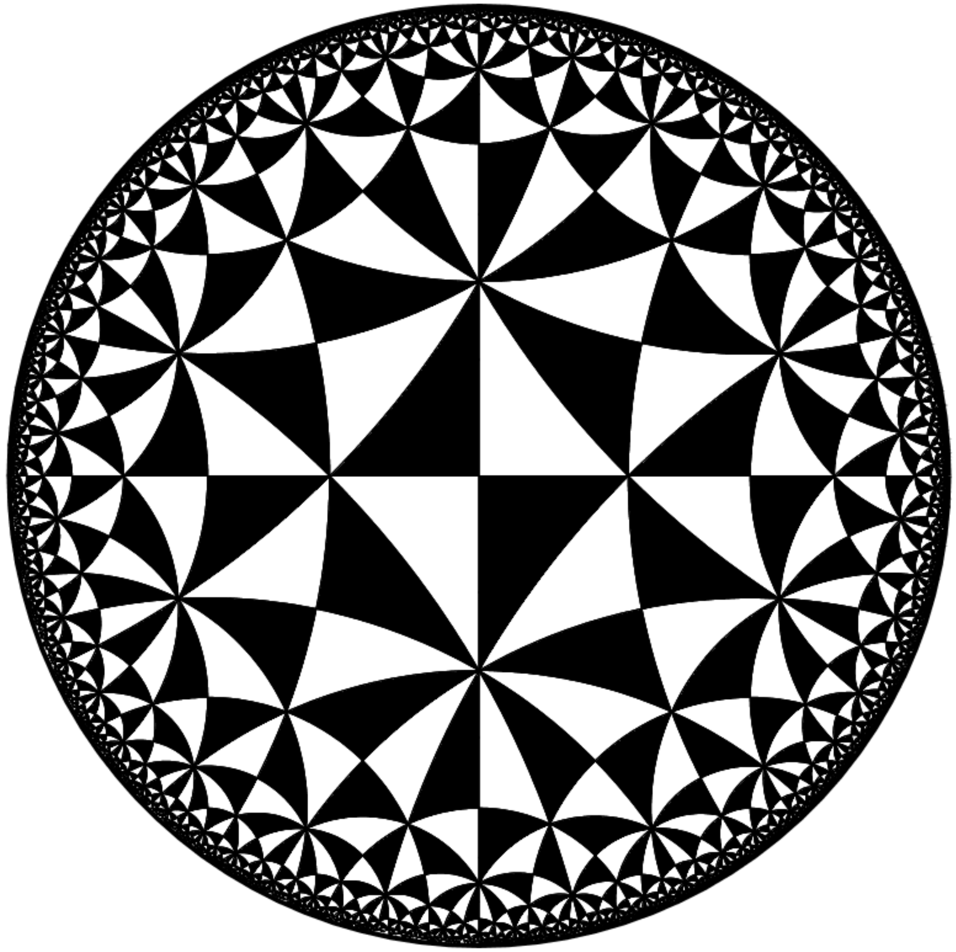
Which non-compact Riemann surfaces are Belyi?

= has a Belyi function

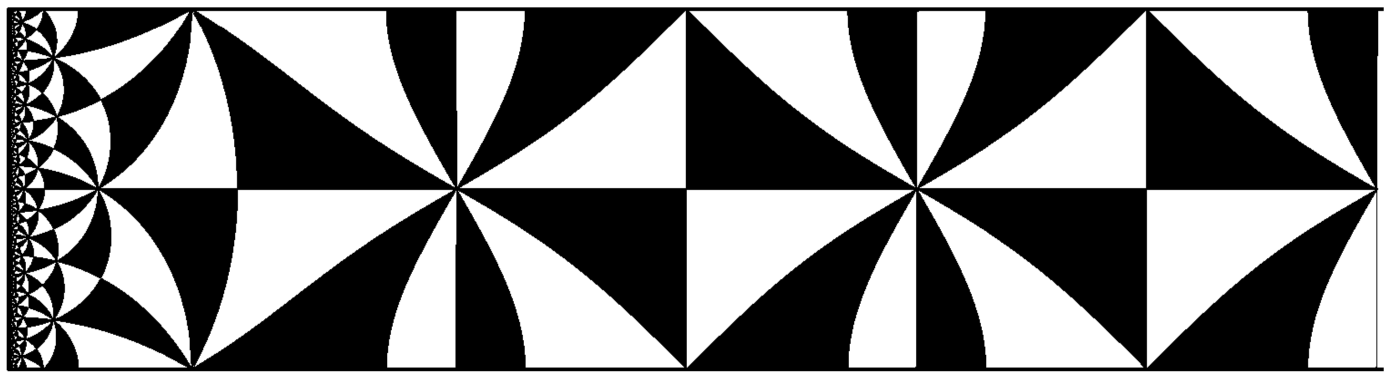
= has an equilateral triangulation



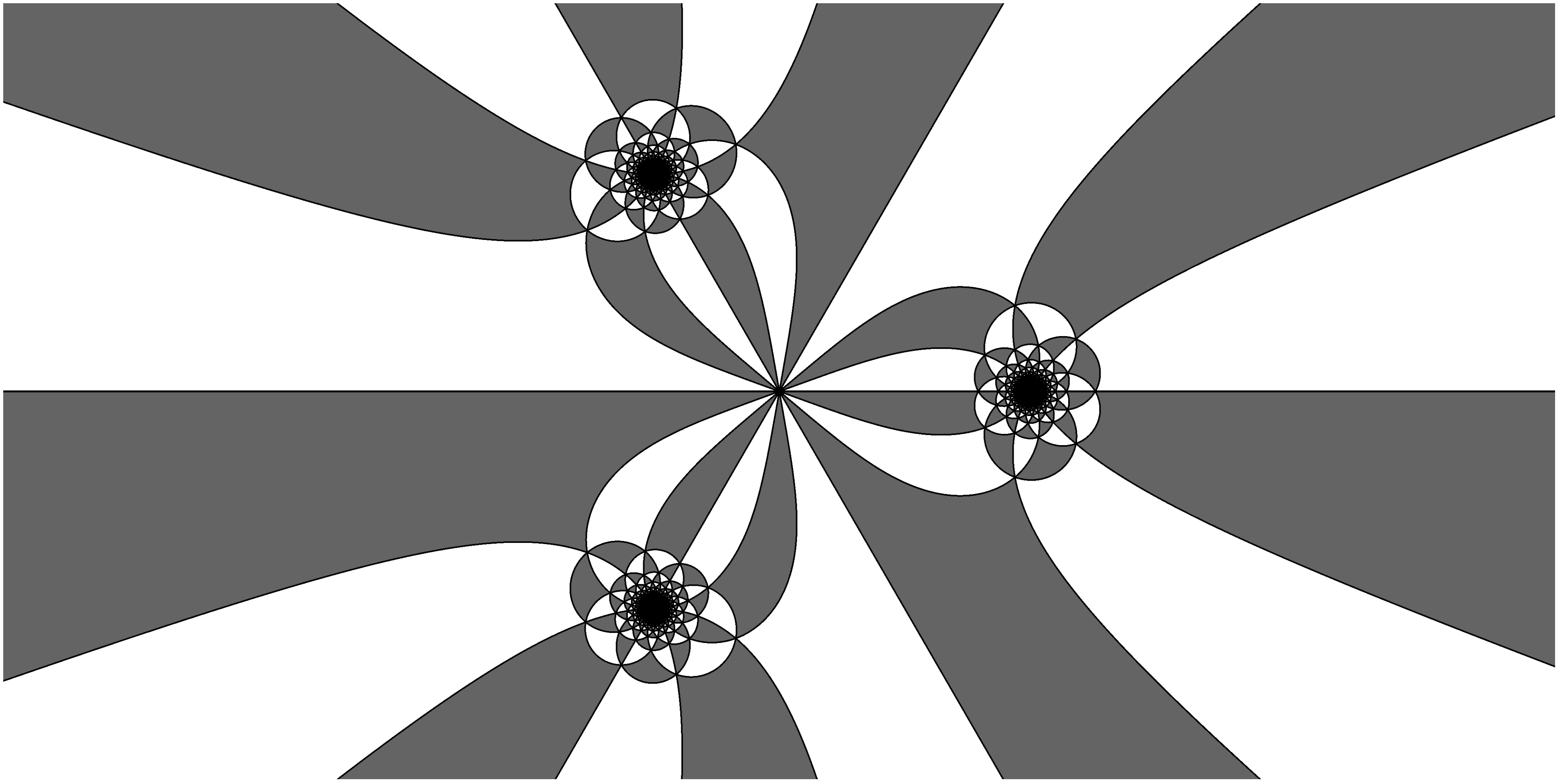
The plane



The disk



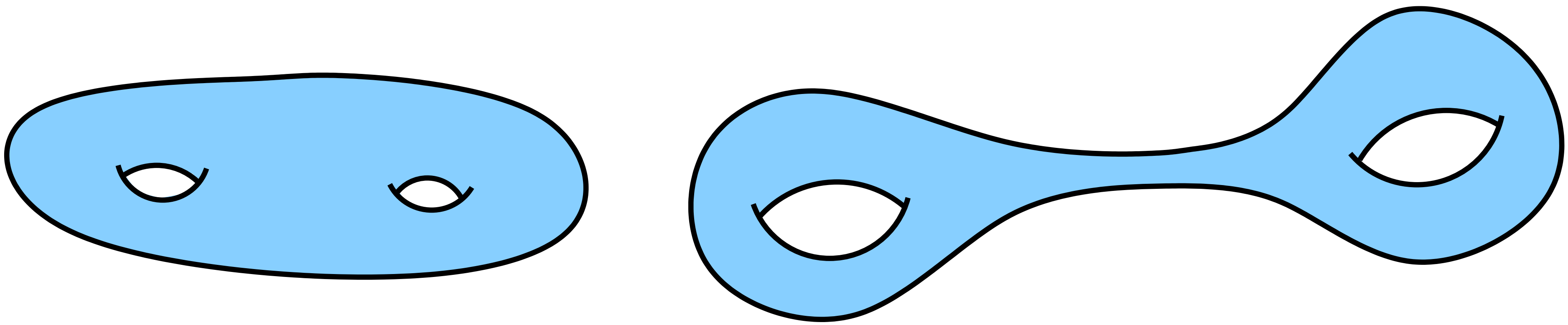
The punctured disk (identity top and bottom sides)



The trice punctured sphere

Most surfaces have many conformal structures, e.g., disk $\neq$ plane.

Higher genus surfaces have multi-dimensional moduli spaces.

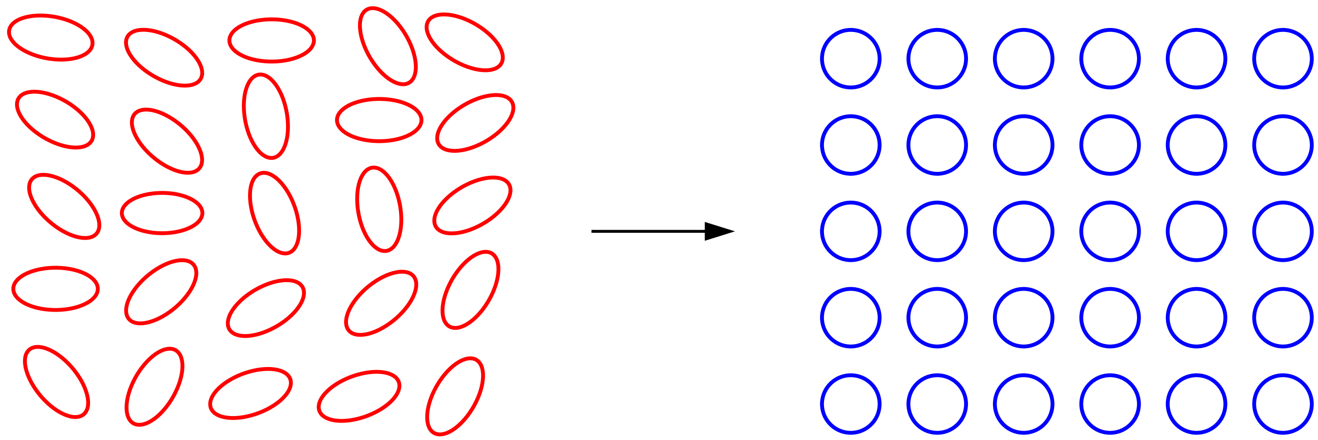


Every Riemann surface can be triangulated.

$\Rightarrow$ every surface is homeomorphic a Belyi surface.

Which conformal structures occur? Need QC maps to answer this.

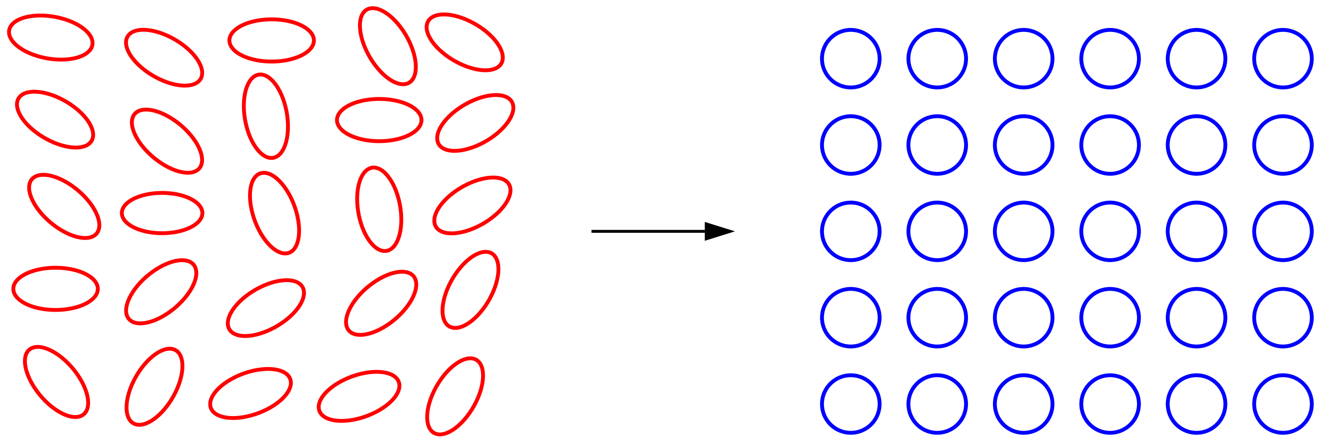
Quasiconformal (QC) maps send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

For K-QC maps, ellipses have eccentricity $\leq K$

Quasiconformal (QC) maps send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

For K-QC maps, ellipses have eccentricity $\leq K$

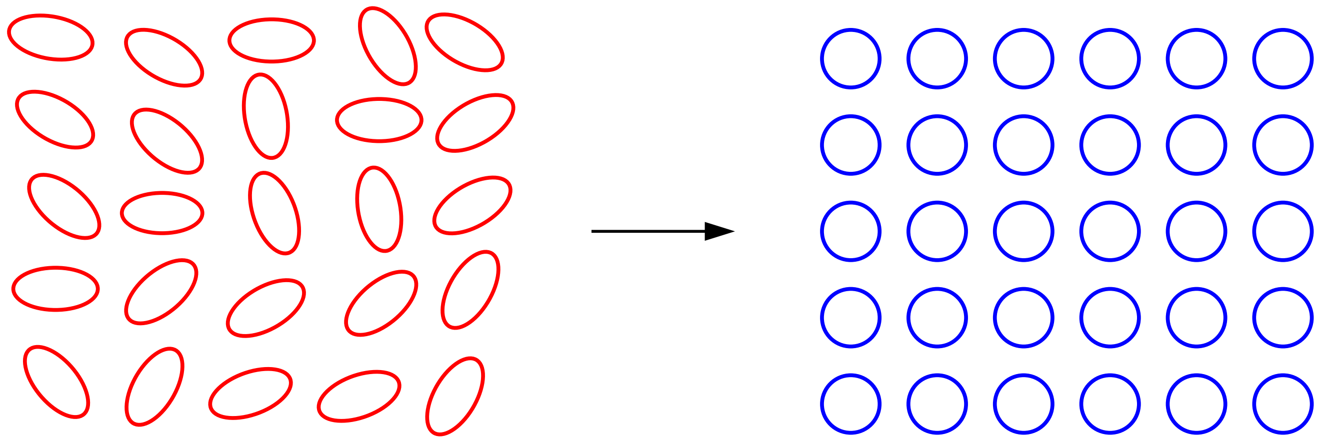
Ellipses determined a.e. by measurable dilatation $\mu = f_{\overline{z}}/f_z$ with

$$f_z = f_x - i f_y \qquad f_{\overline{z}} = f_x + i f_y.$$

$$|\mu| \leq \frac{K-1}{K+1} < 1.$$

Conversely, ...

Quasiconformal (QC) maps send infinitesimal ellipses to circles.



Mapping theorem: any such μ comes from some QC map f.

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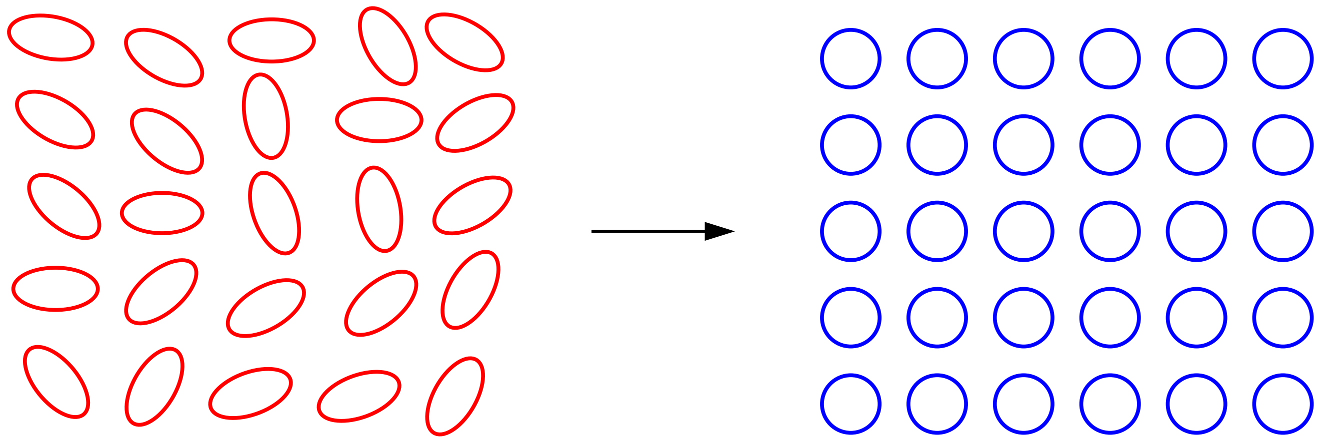
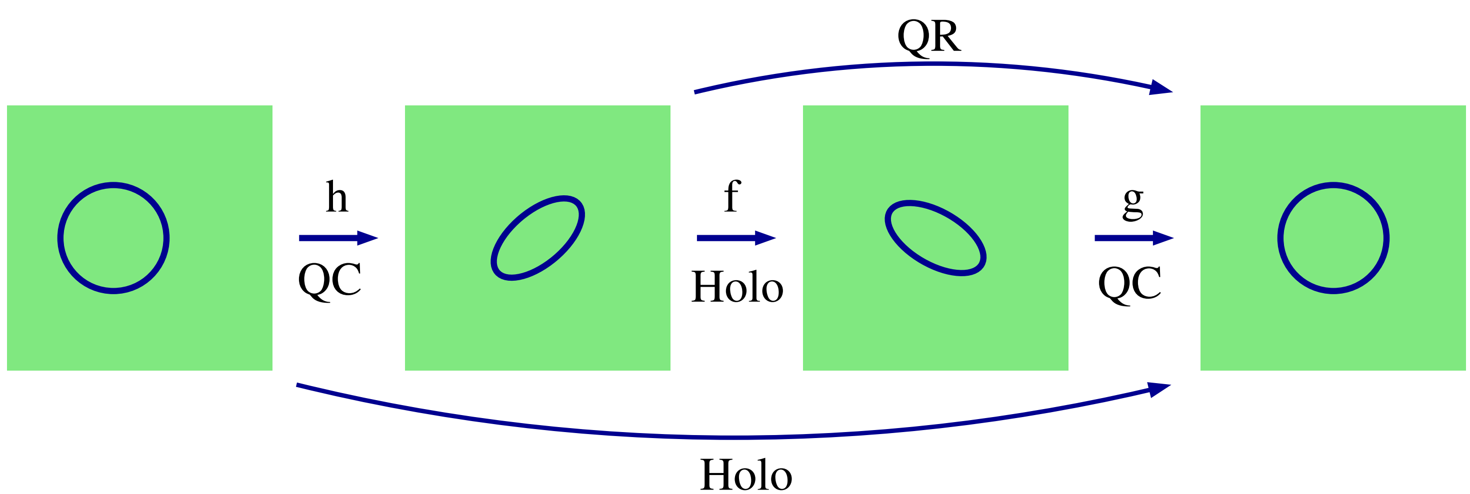
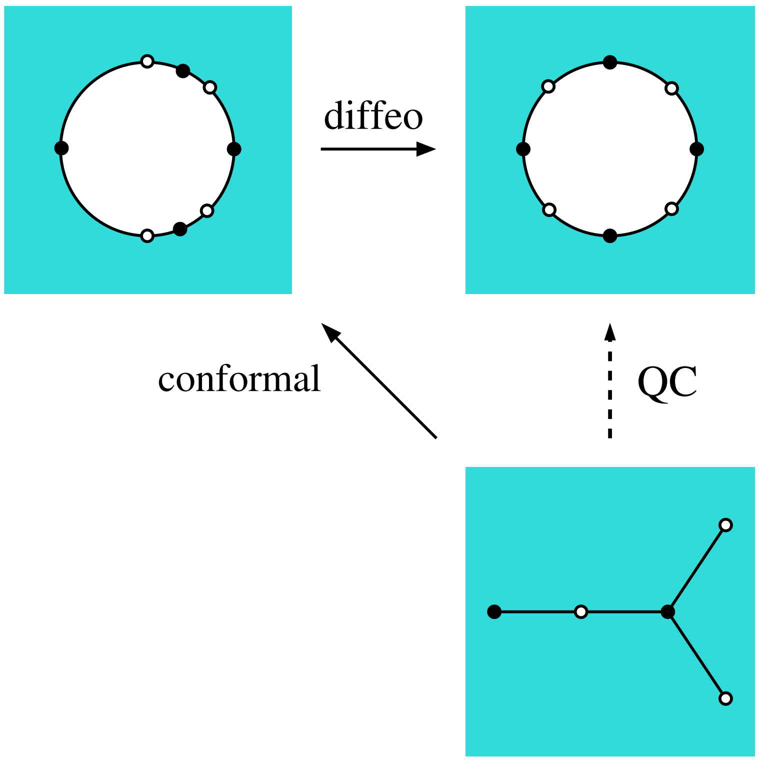


Mapping theorem: any such μ comes from some QC map f.

Cor: If f is holomorphic and g is QC, then there is a QC map h so that $F = g \circ f \circ h$ is also holomorphic. ($g \circ f$ is quasiregular.)



QC proof that every finite tree has a true form:



"Equalize intervals" by diffeomorphism. Composition is quasiconformal.

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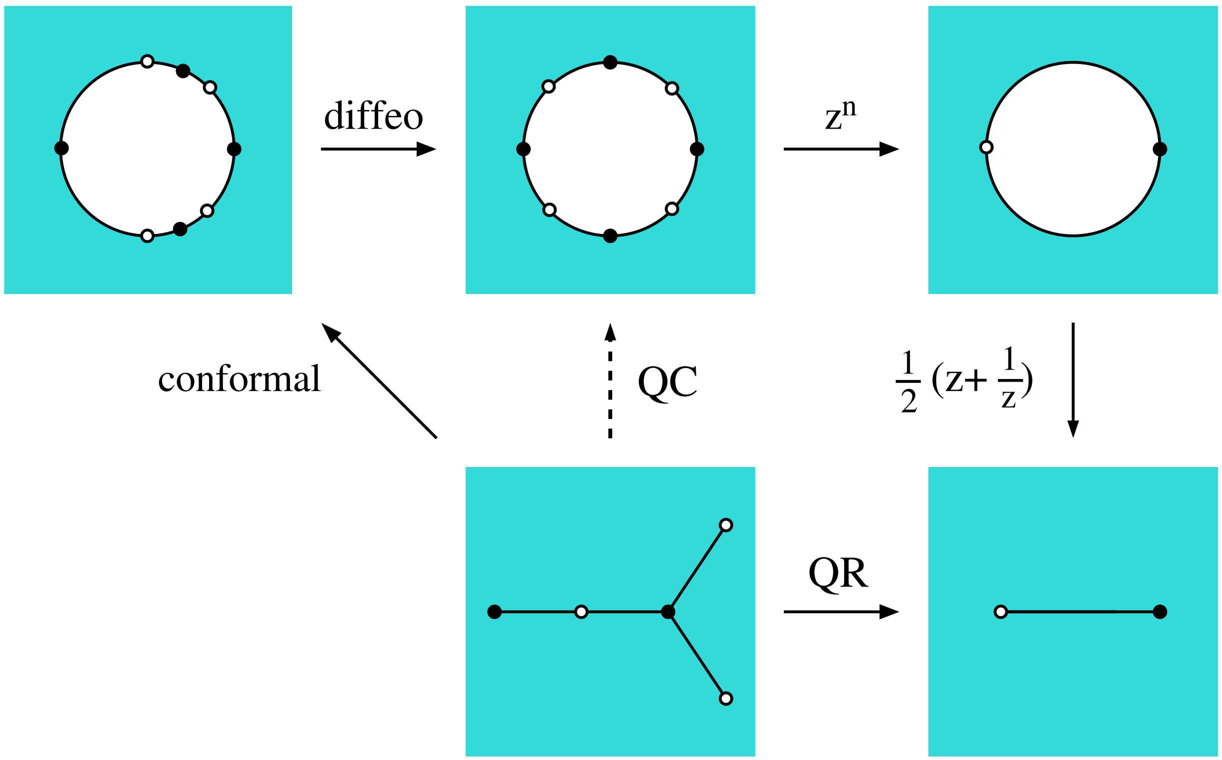

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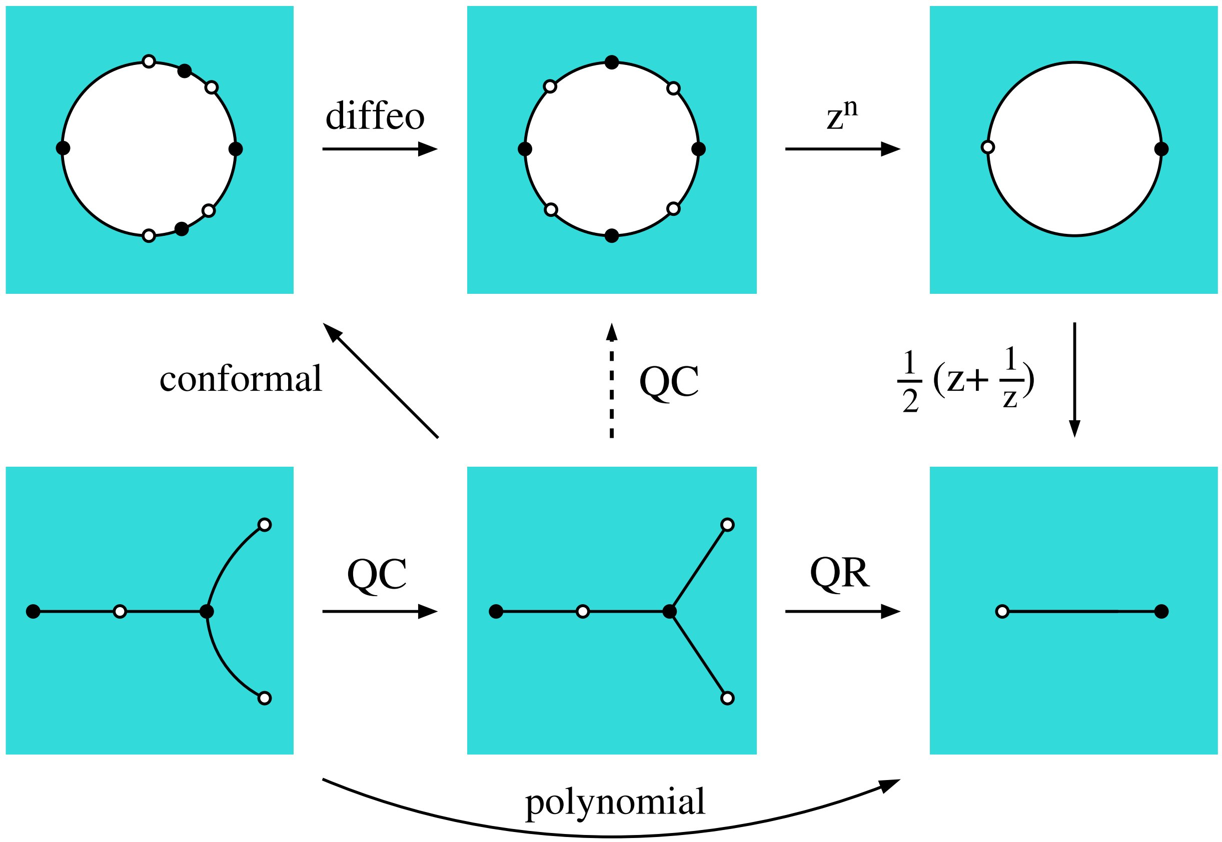
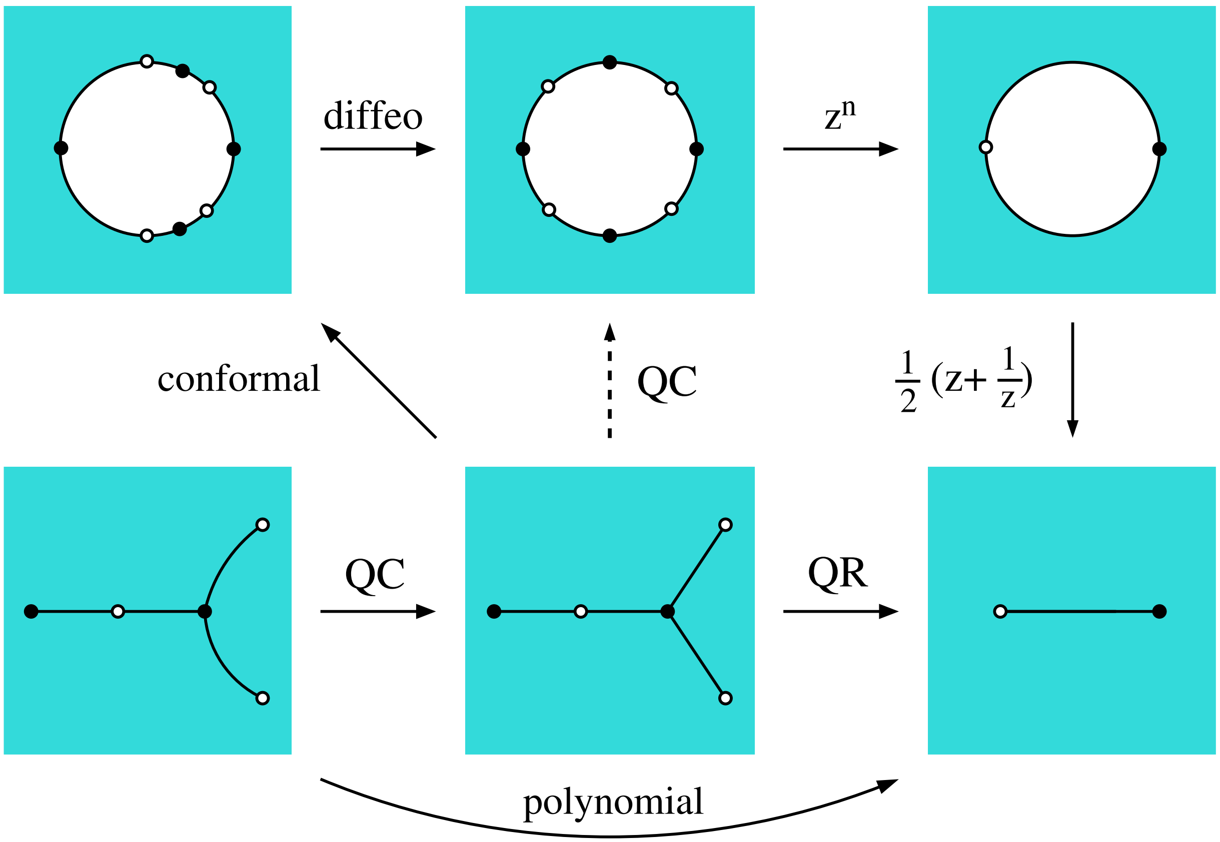


Mapping theorem implies there is a QC φ so $p = q \circ \varphi$ is a polynomial.

Only possible critical points are vertices of tree; these map to ± 1.

QC proof that every finite tree has a true form:



Constructive: can estimate QC correction, get shape of true tree.

Eremenko: do true trees approximate all possible shapes?

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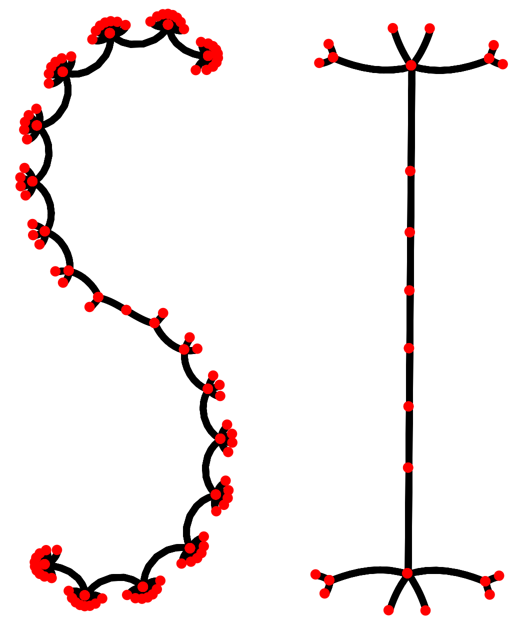
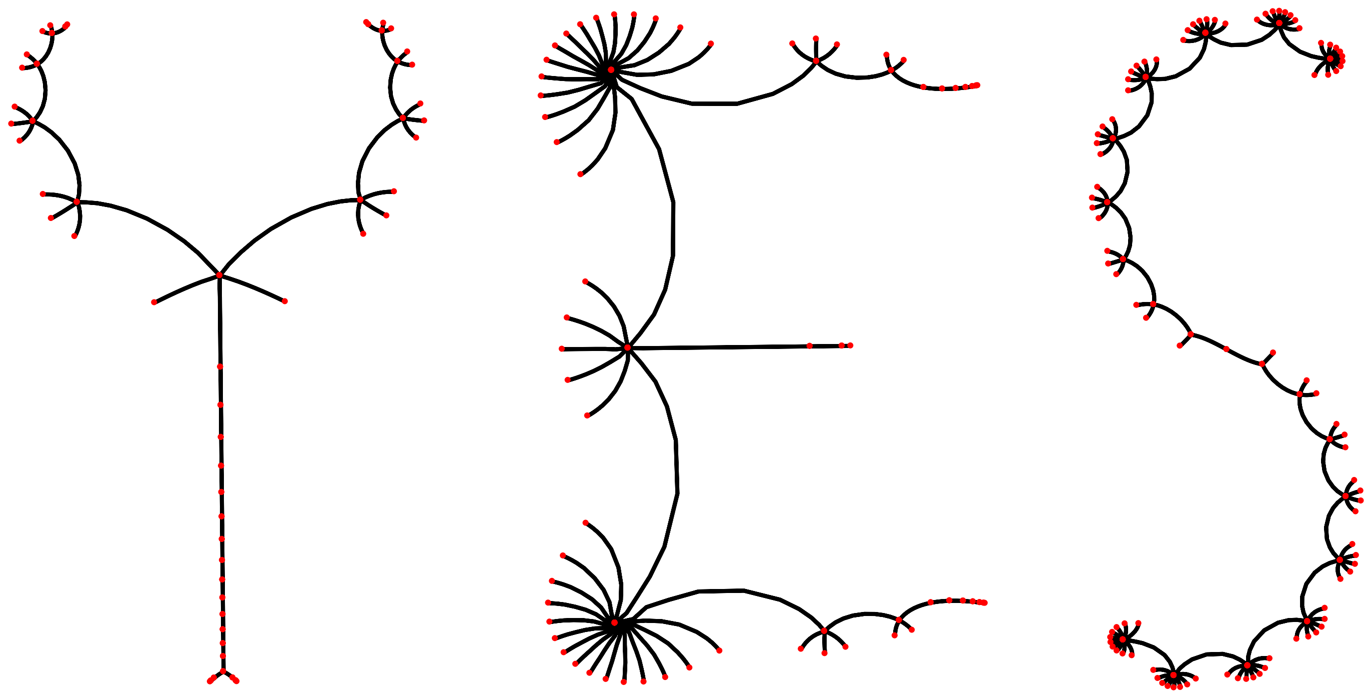


Eremenko: do true trees approximate all possible shapes?



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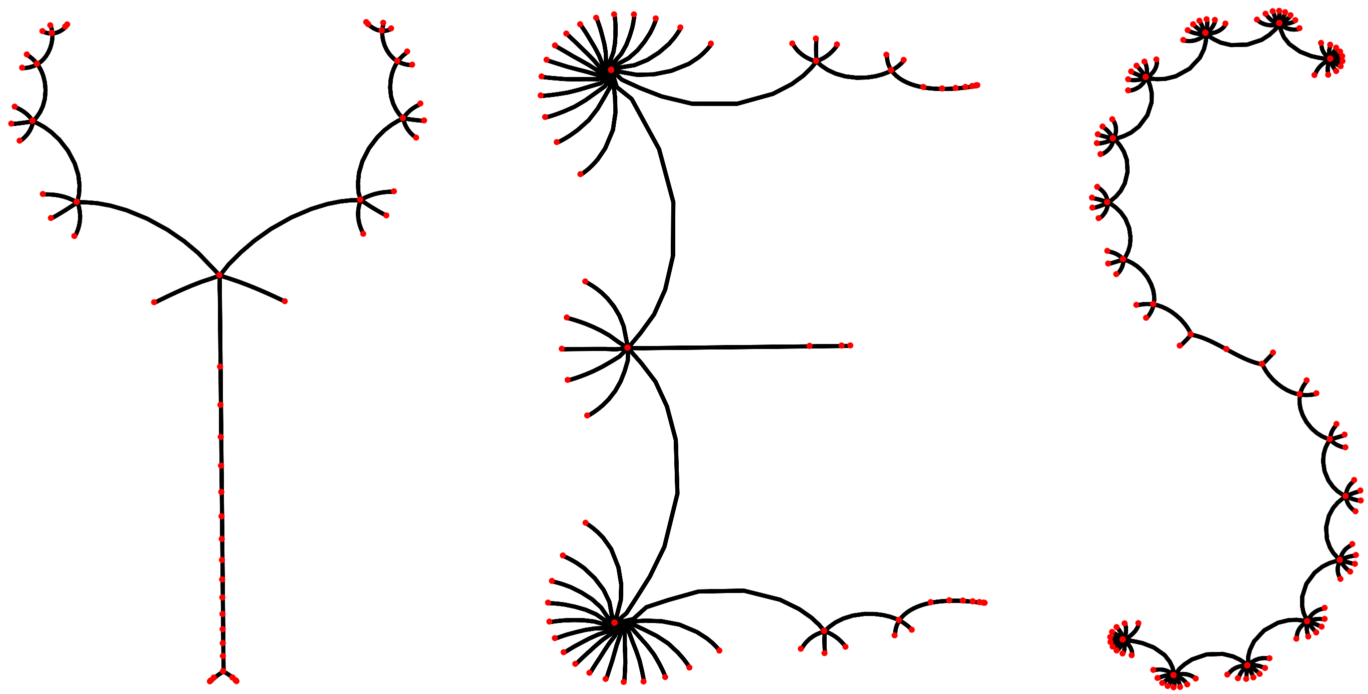
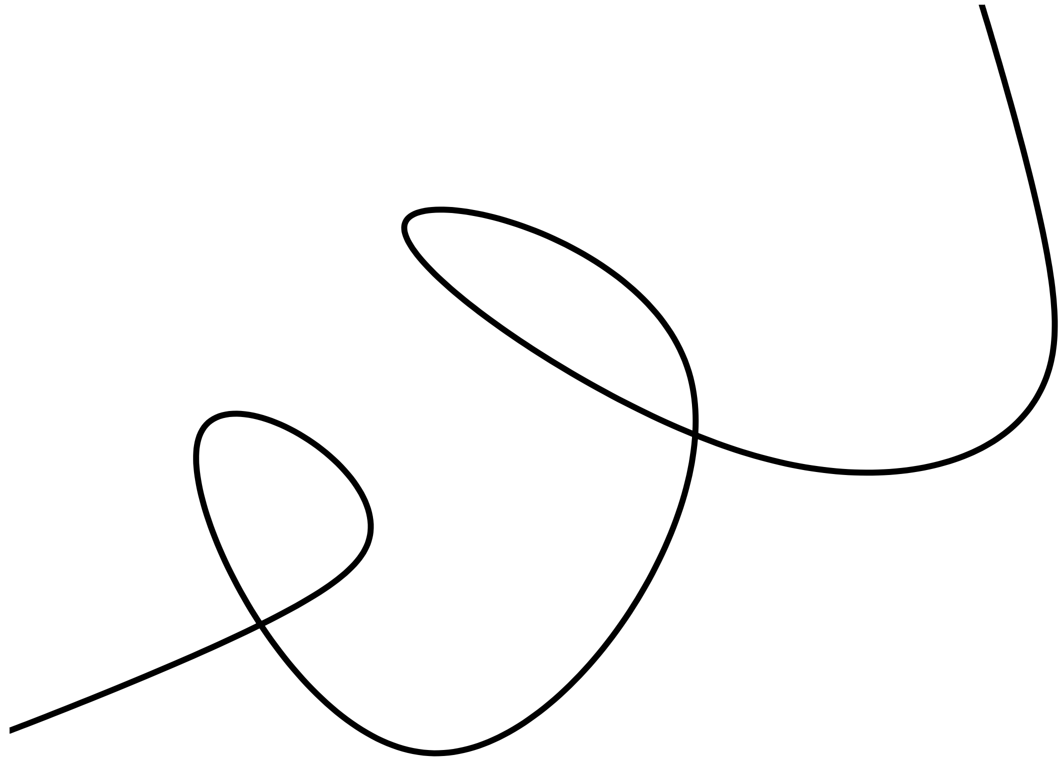
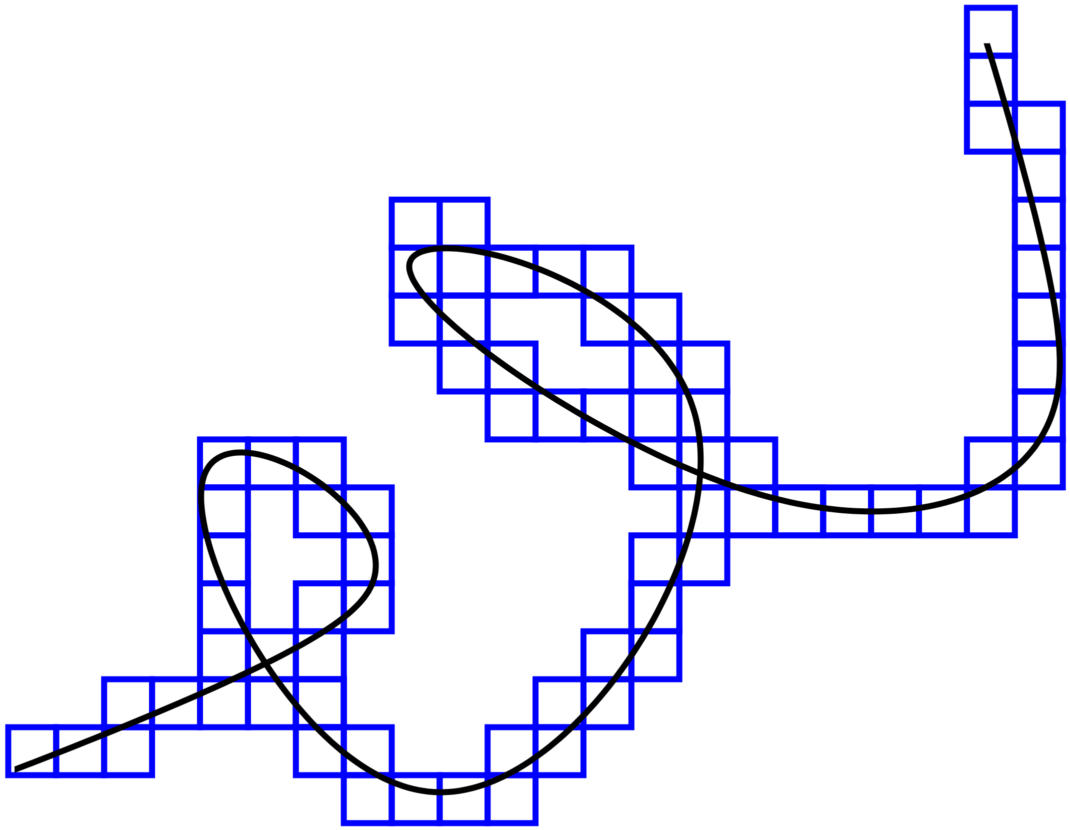


Thm: Every planar continuum is Hausdorff limit of true trees.

E, F are distance $\leq \epsilon$ apart if each is in an ϵ-neighborhood of the other.



Suffices to approximate subtrees of a grid.



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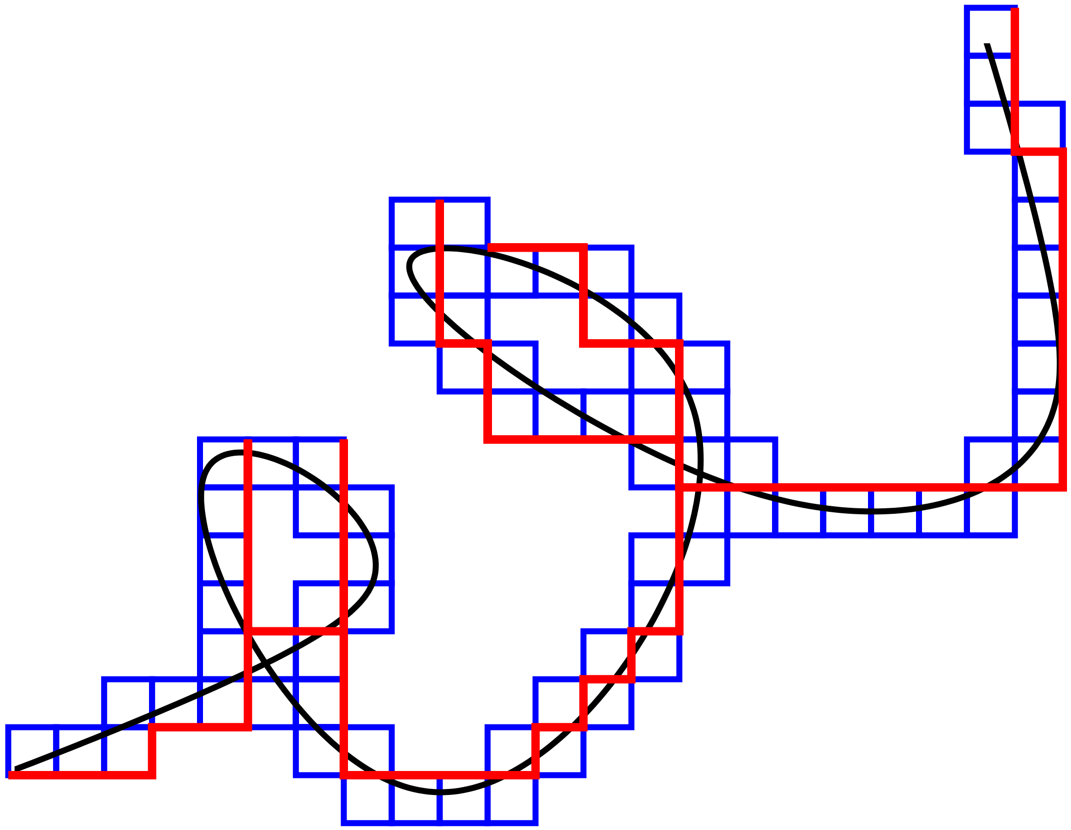

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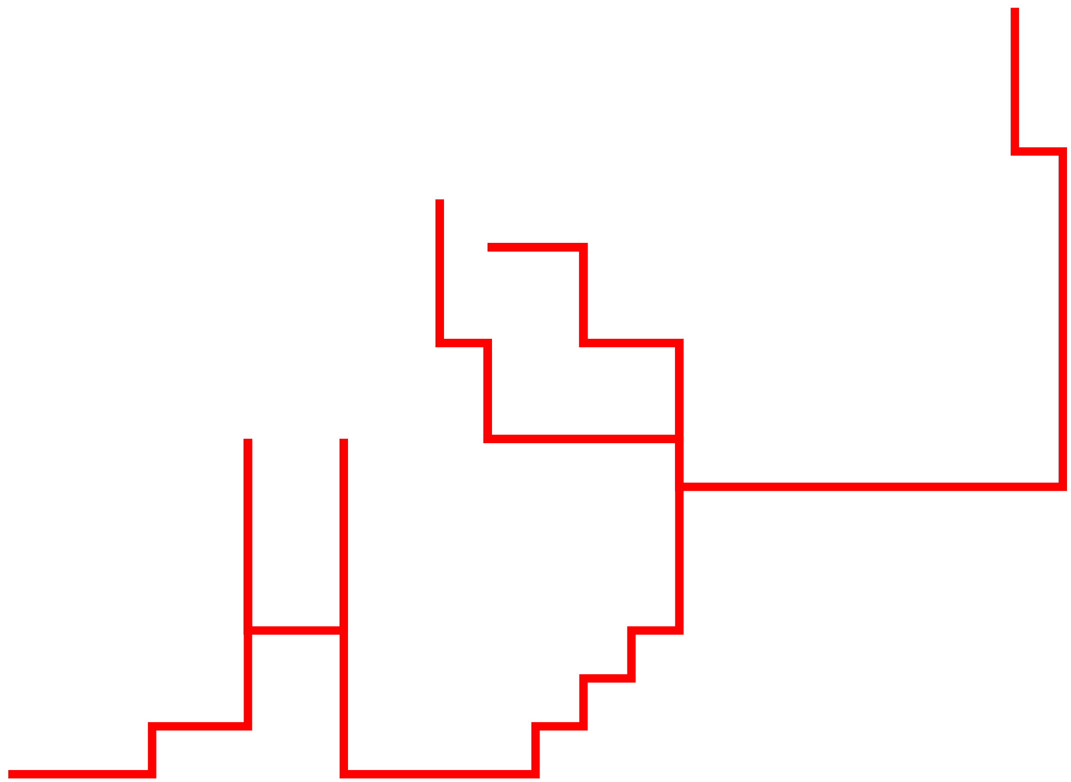
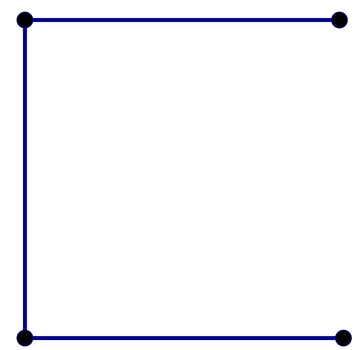


Suffices to approximate subtrees of a grid.

Theorem: Every planar continuum is a limit of true trees.

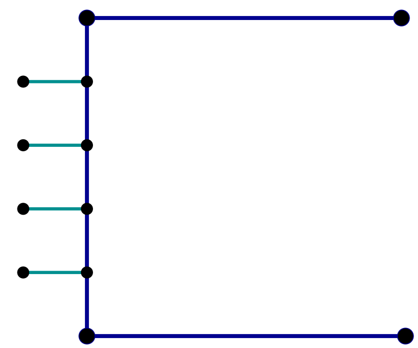
Idea of Proof: reduce harmonic measure ratio by adding edges.



Vertical side has much larger harmonic measure from left.

Theorem: Every planar continuum is a limit of true trees.

Idea of Proof: reduce harmonic measure ratio by adding edges.



"Left" harmonic measure is reduced (roughly 3-to-1).

New edges are approximately balanced (universal constant).

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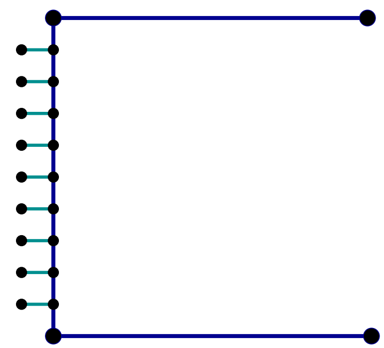


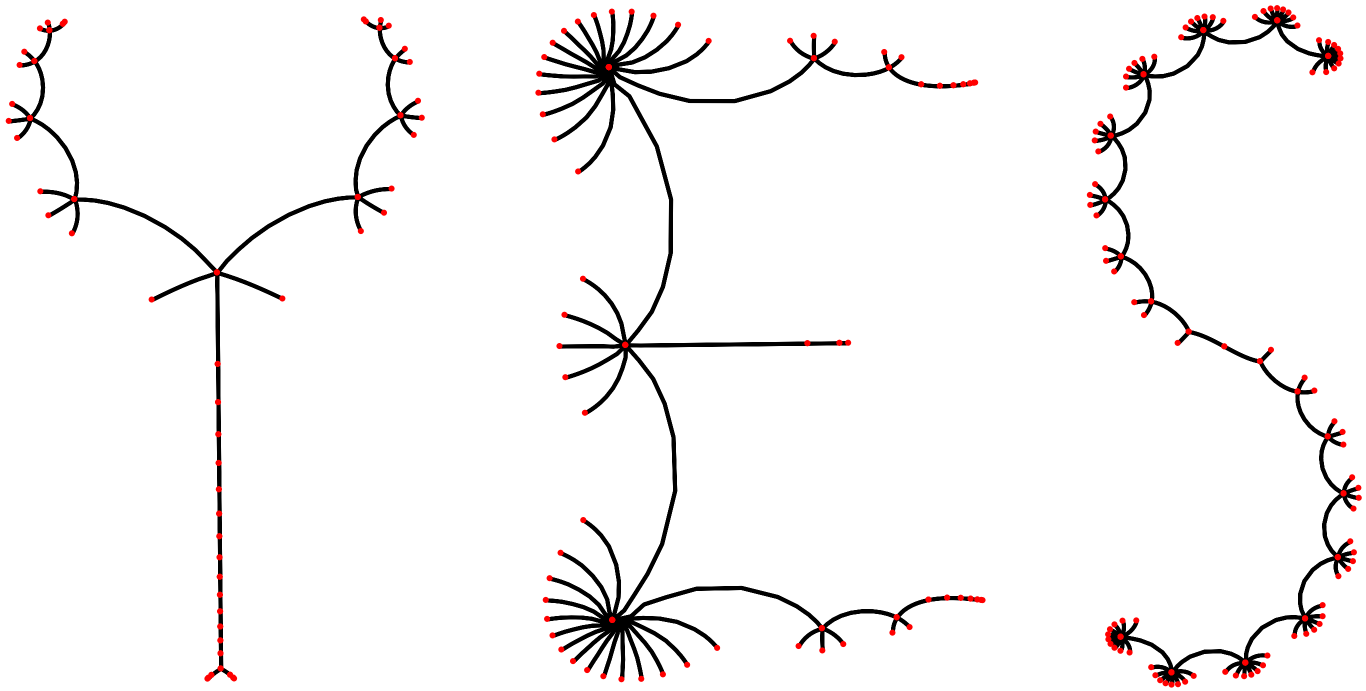
"Left" harmonic measure is reduced (roughly 3-to-1).

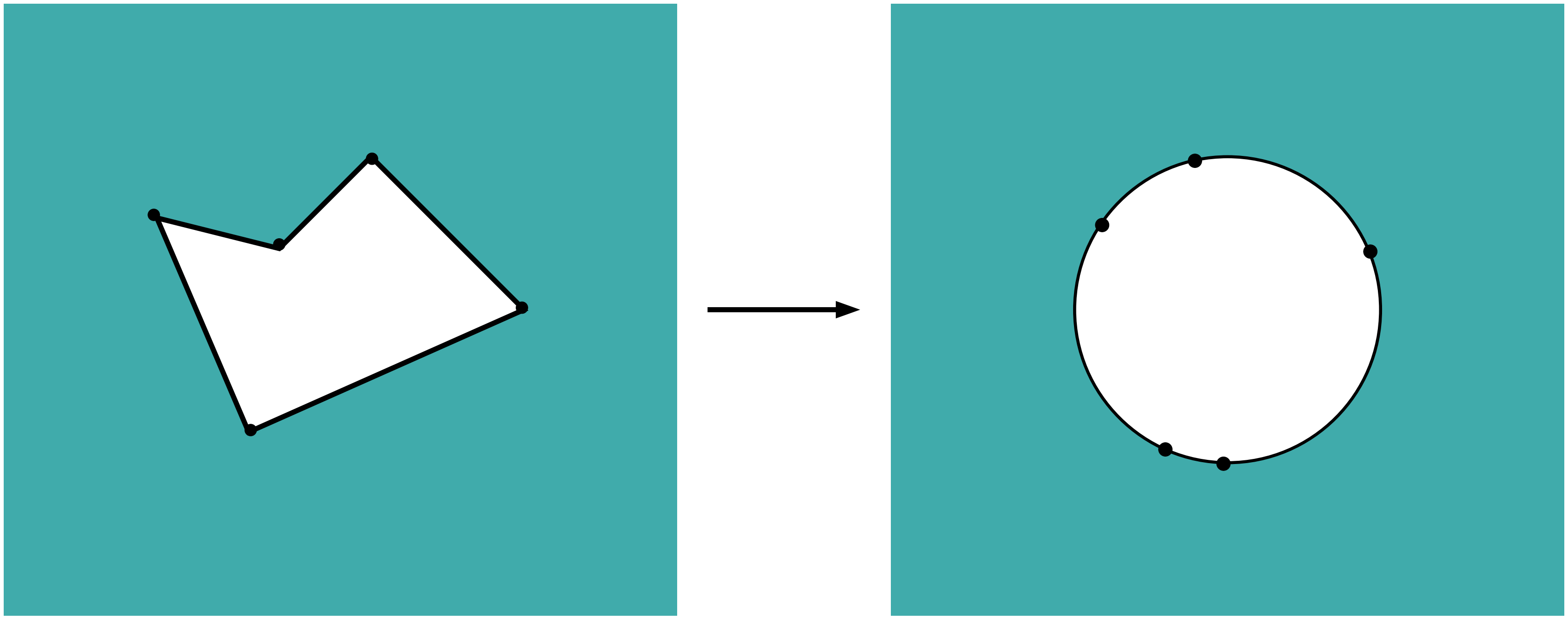
New edges are approximately balanced (universal constant).

Mapping theorem gives exactly balanced.

QC correction map is near identity if "spikes" are short.

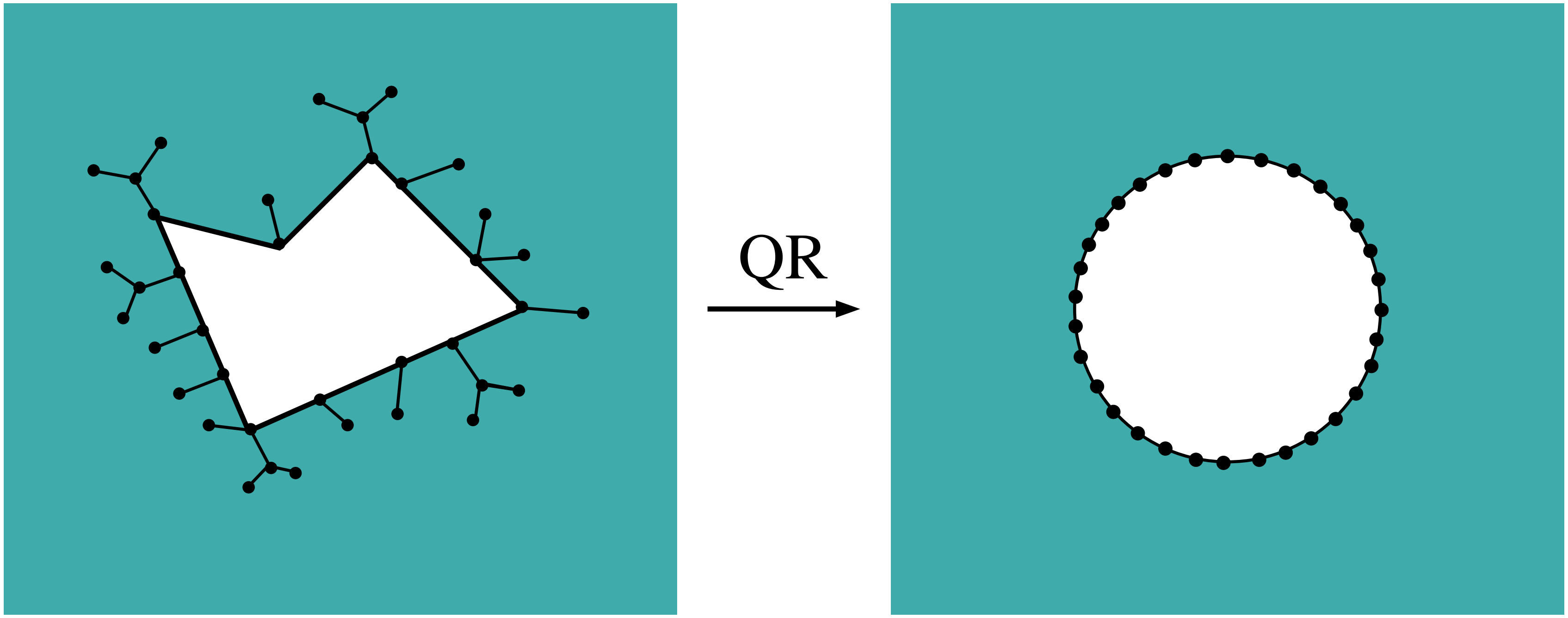
New tree approximates shape of old tree; different combinatorics.


YES



This works on other shapes besides trees.

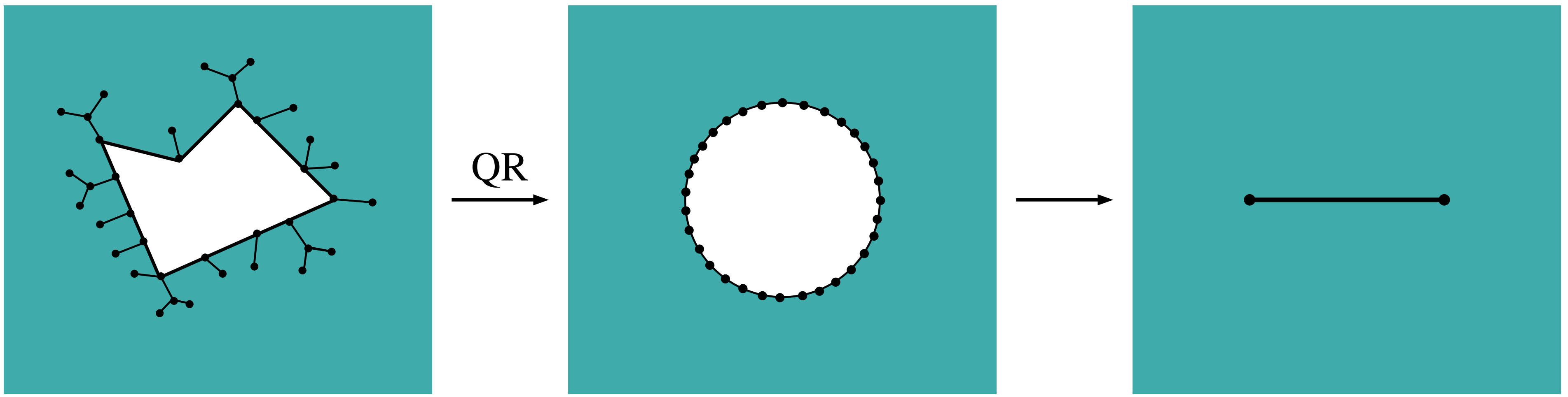
Suppose we are given “unbalanced polygon”.



We can add short branches to make it almost balanced.

Exterior is QR mapped to evenly divided circle.

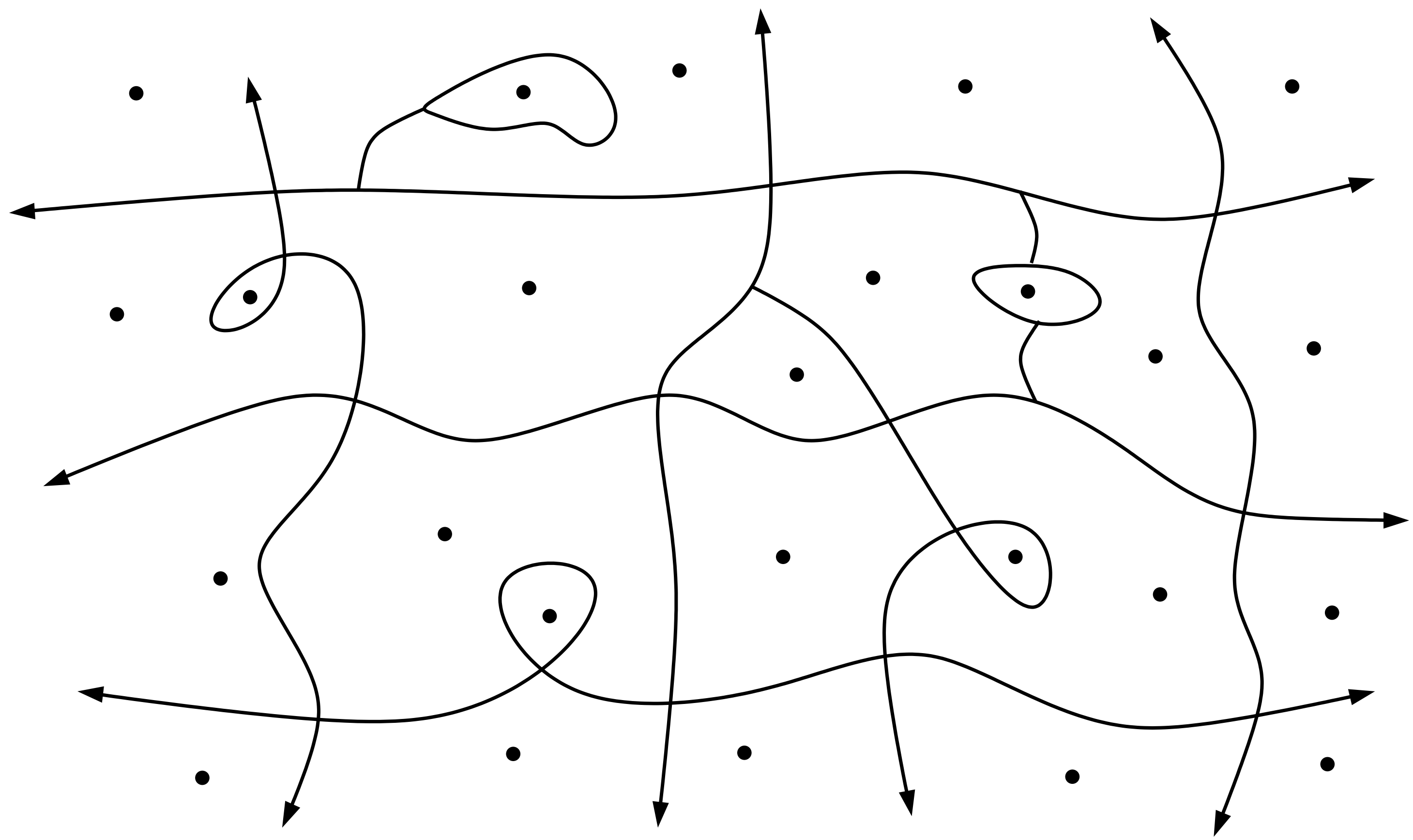
Again, QR distortion bounded, supported on tiny area.

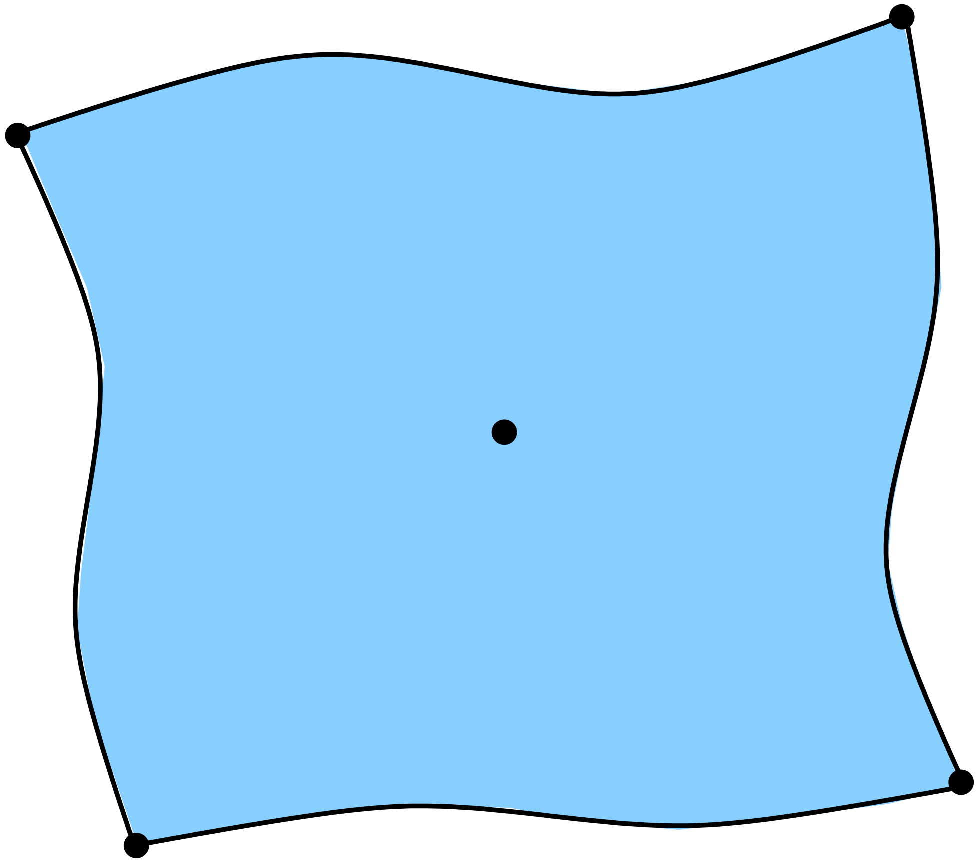


Important detail: QR map multiples arclength on each boundary segment.

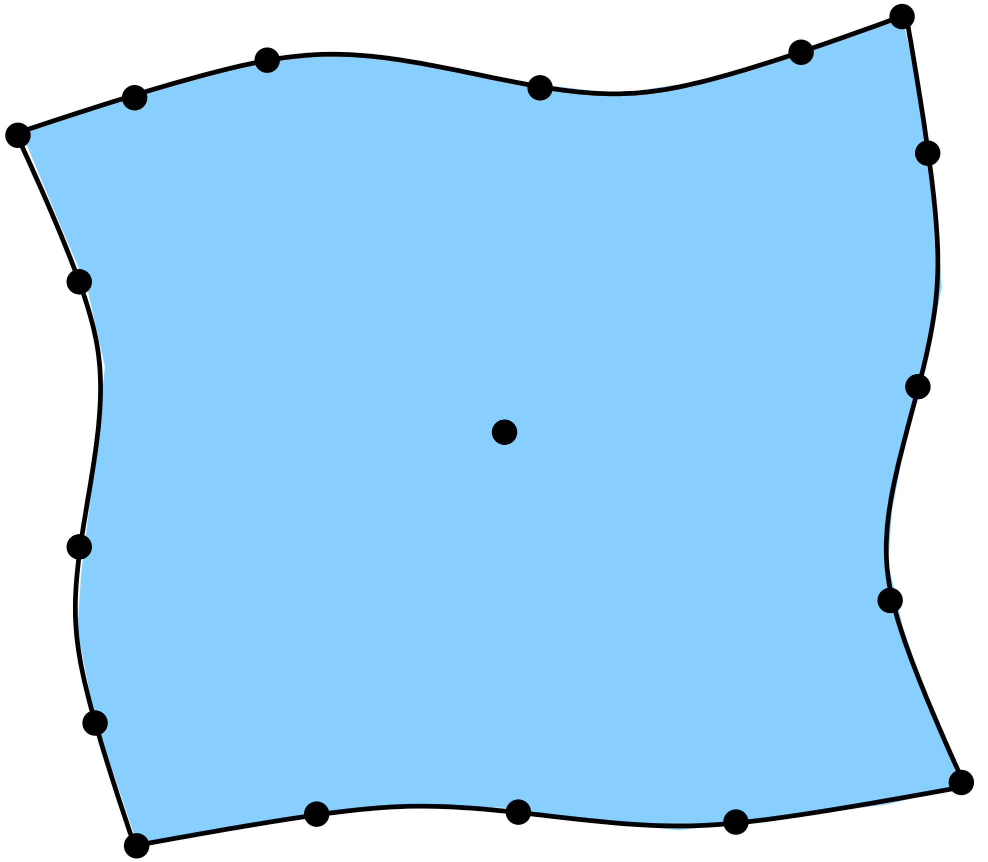
This allows us to “glue together” local maps to get a global QR map.

Can replace trees by meshes. Can replace plane by other surfaces. Get rational or meromorphic functions.

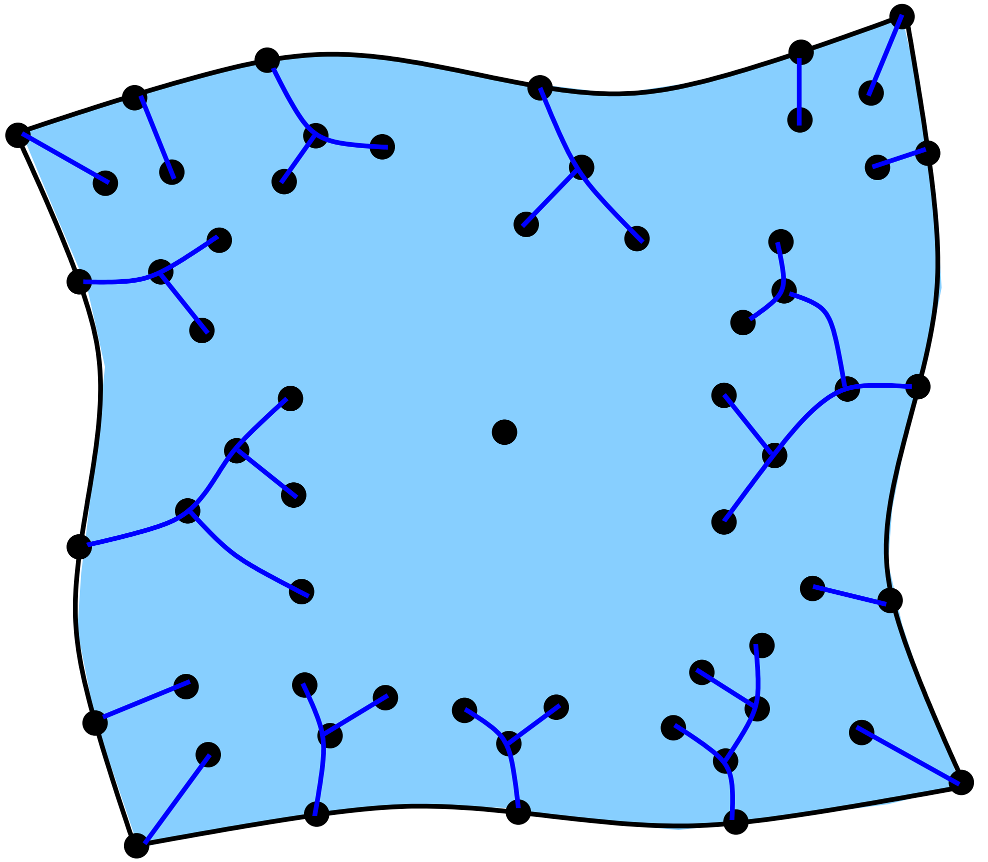




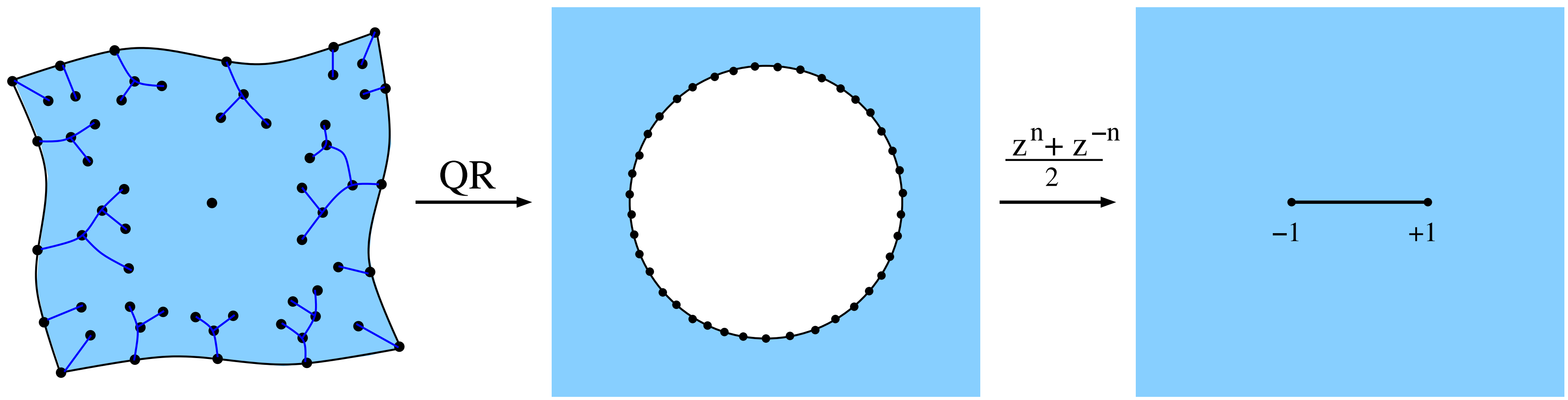
Using inversion we can work in a bounded domain.
Start with a piecewise smooth domain.



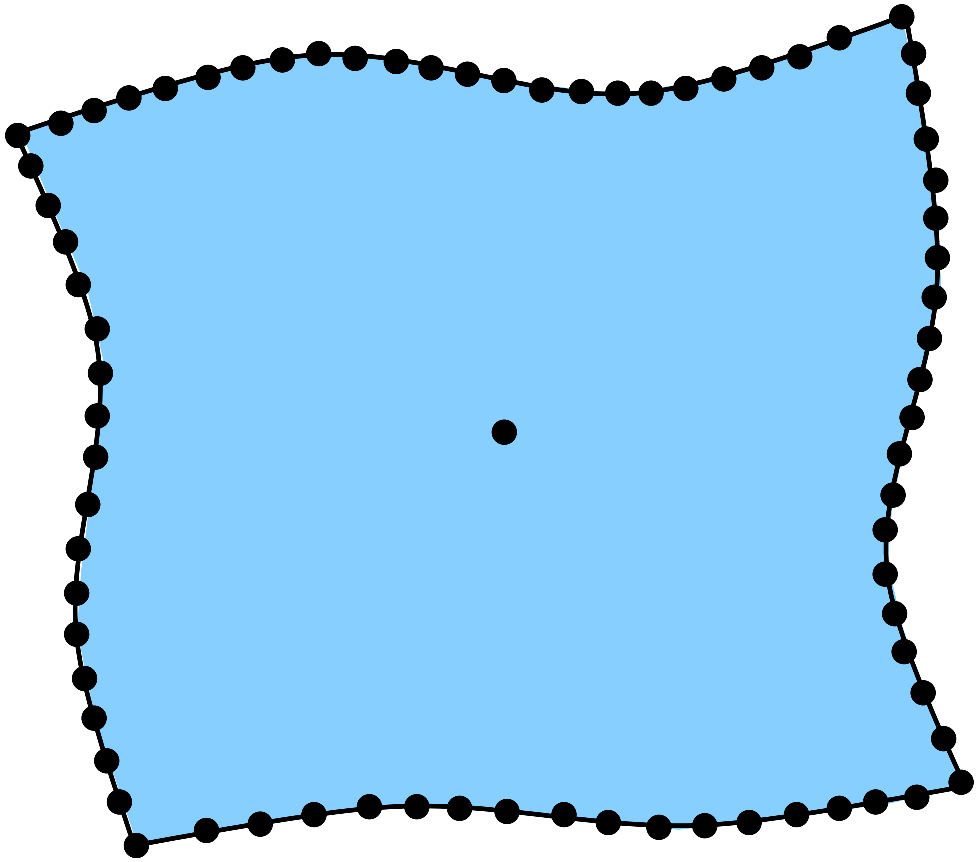
Add boundary points. Roughly evenly spaced.
Add interior point. Harmonic measures not all the same.



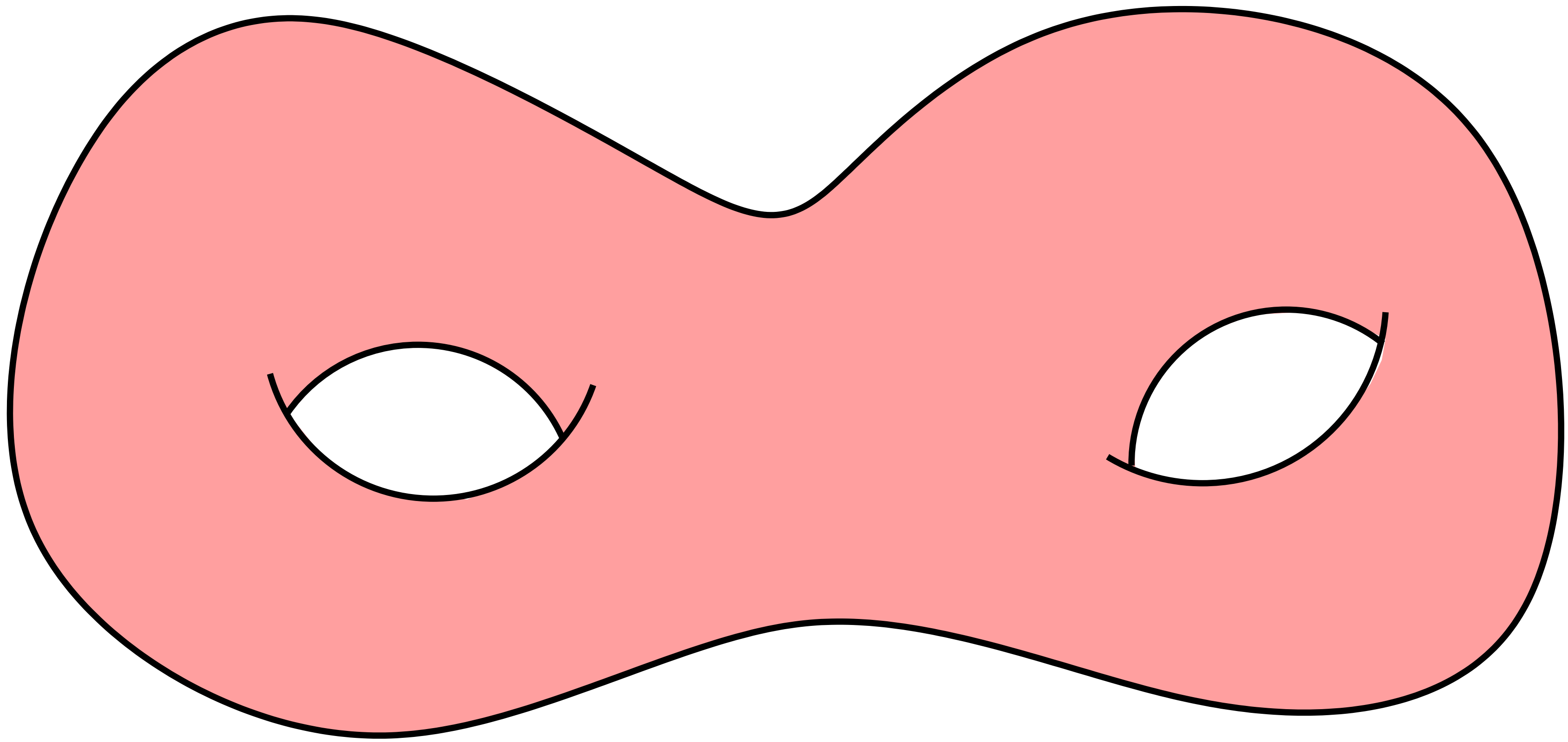
Add slits and trees so that all boundary arcs have roughly the same harmonic measure.



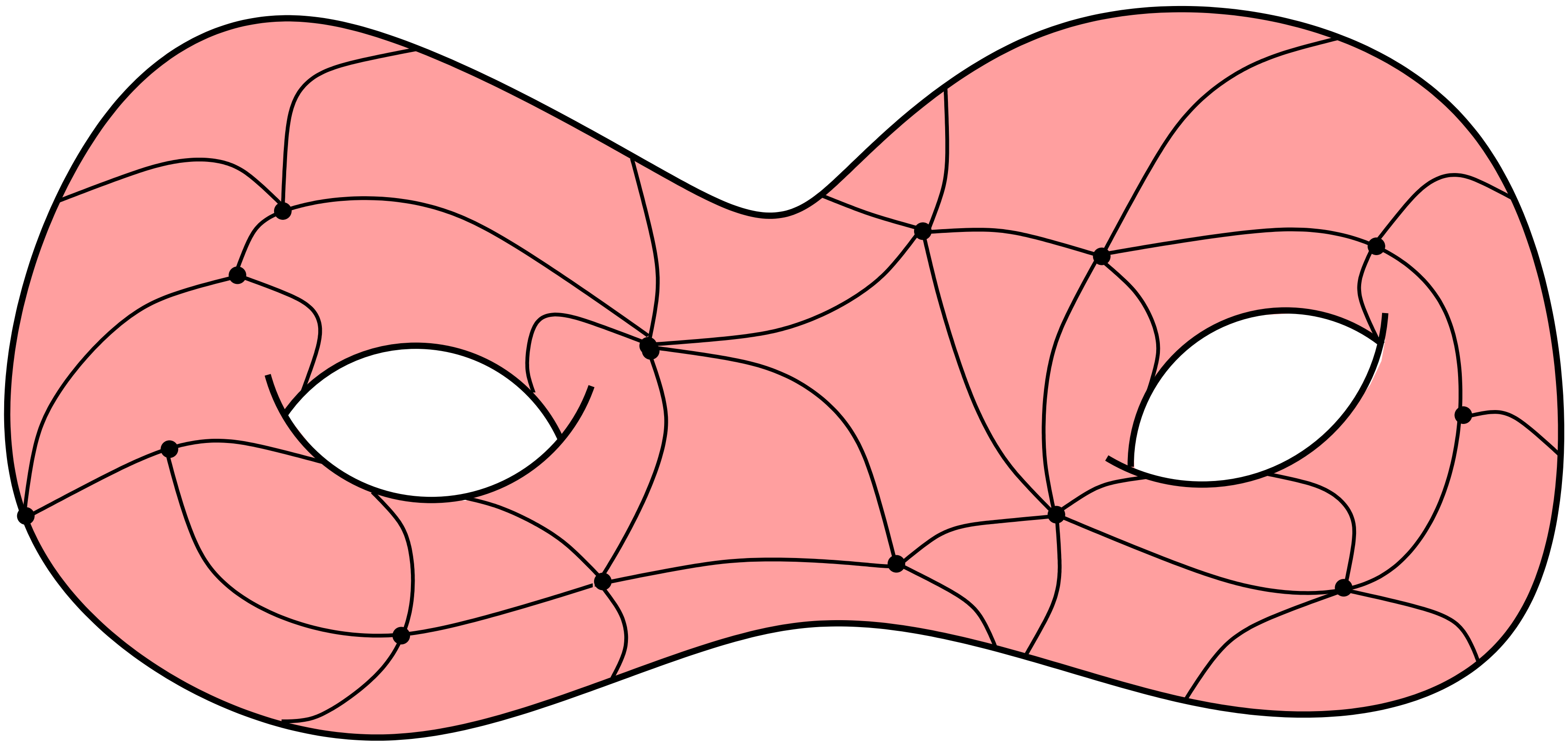
Find quasiregular map taking base point to ∞,
sending boundary arcs to equal intervals on circle.
Then each arc is mapped to $[-1, 1]$ via arclength.



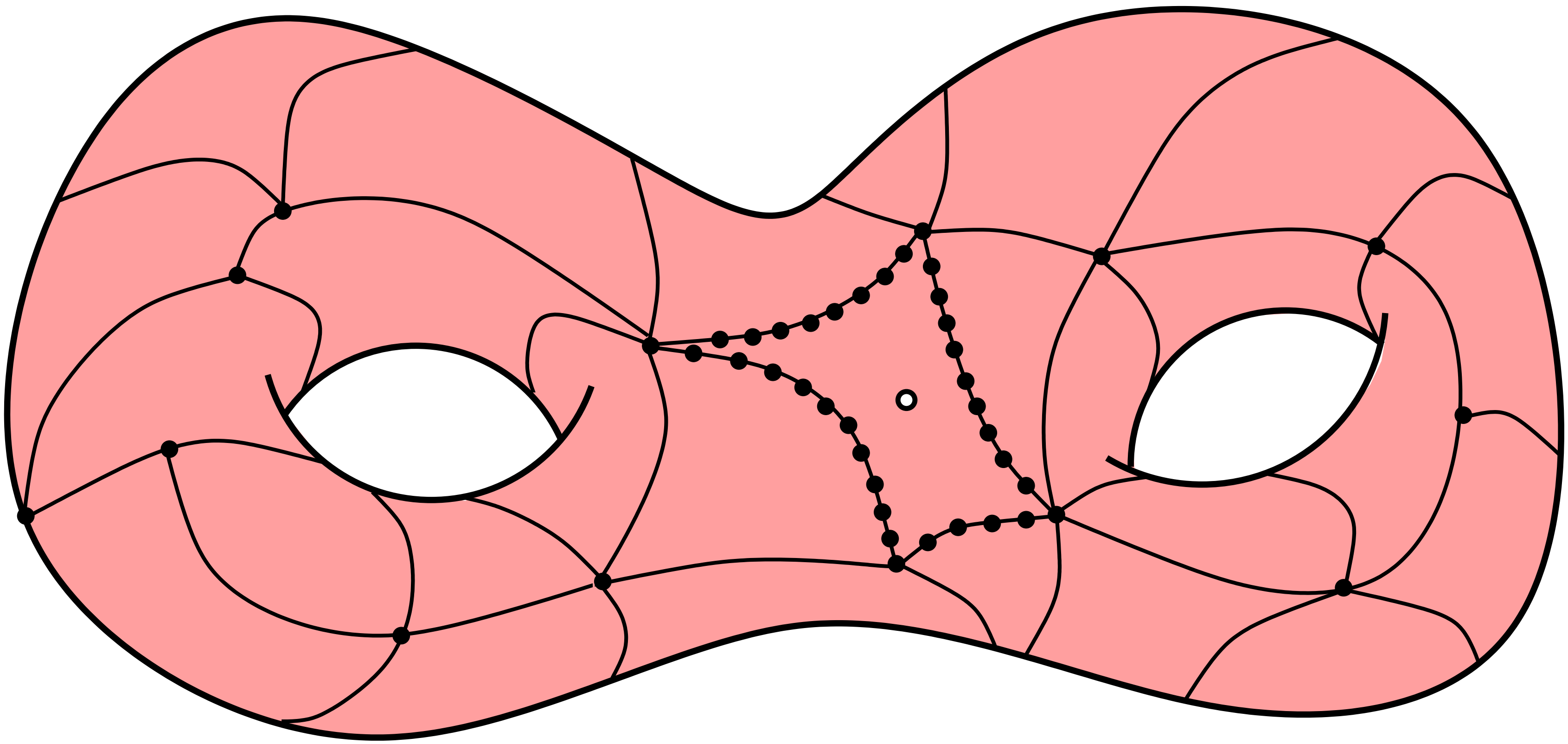
Repeat with more closely spaced points. Get QR map holomorphic except on thin neighborhood of boundary.



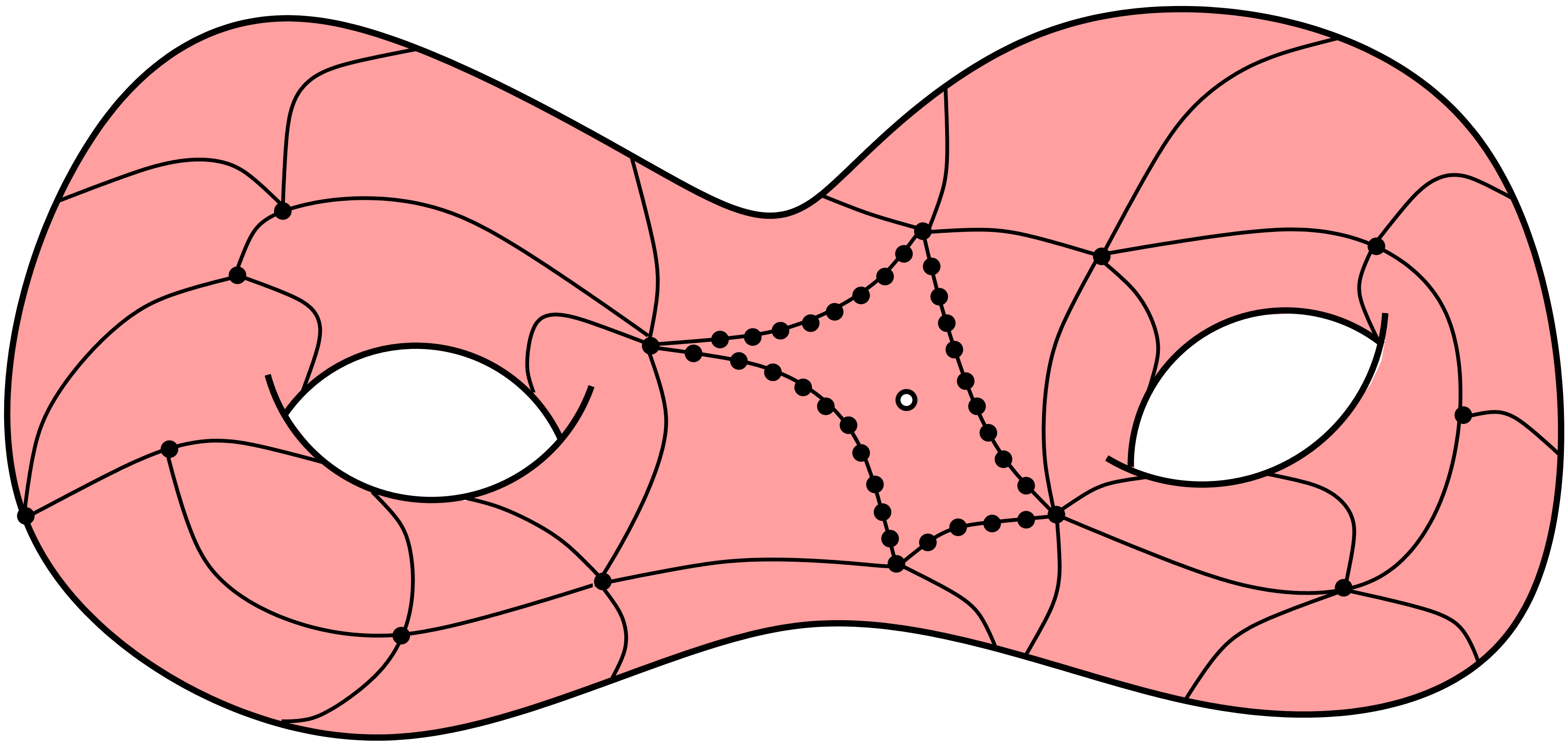
Now do this on a Riemann surface.



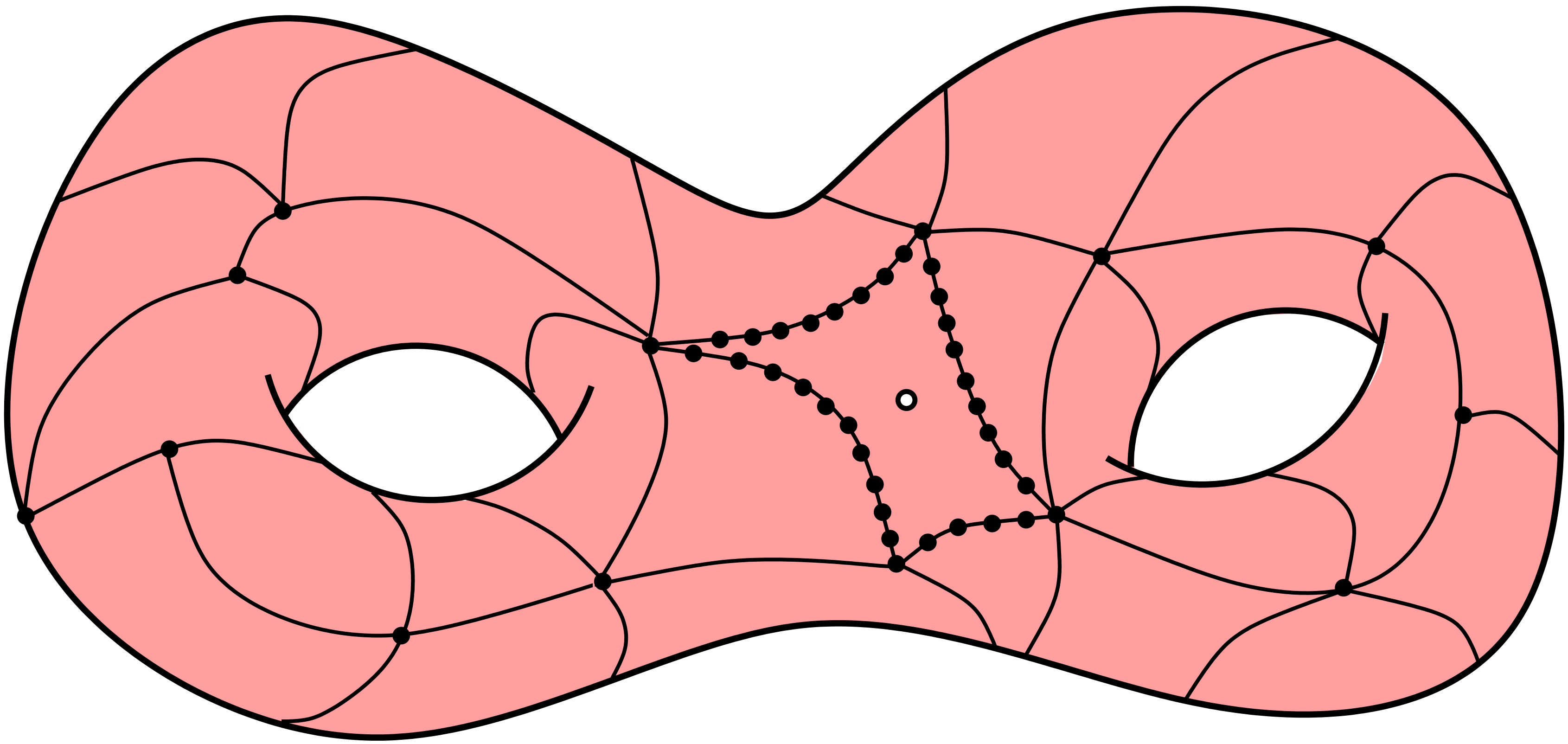
Cut the surface into nice pieces.



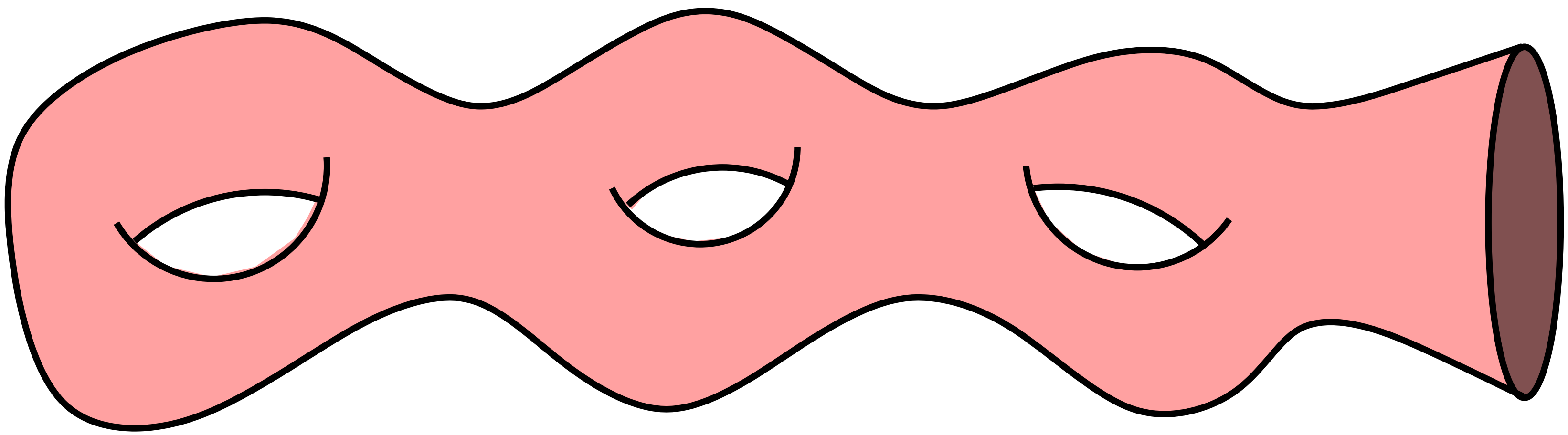
Subdivide edges into many small arcs.
Our construction gives QR maps on each piece.
Continuous across boundaries. QR on whole surface.



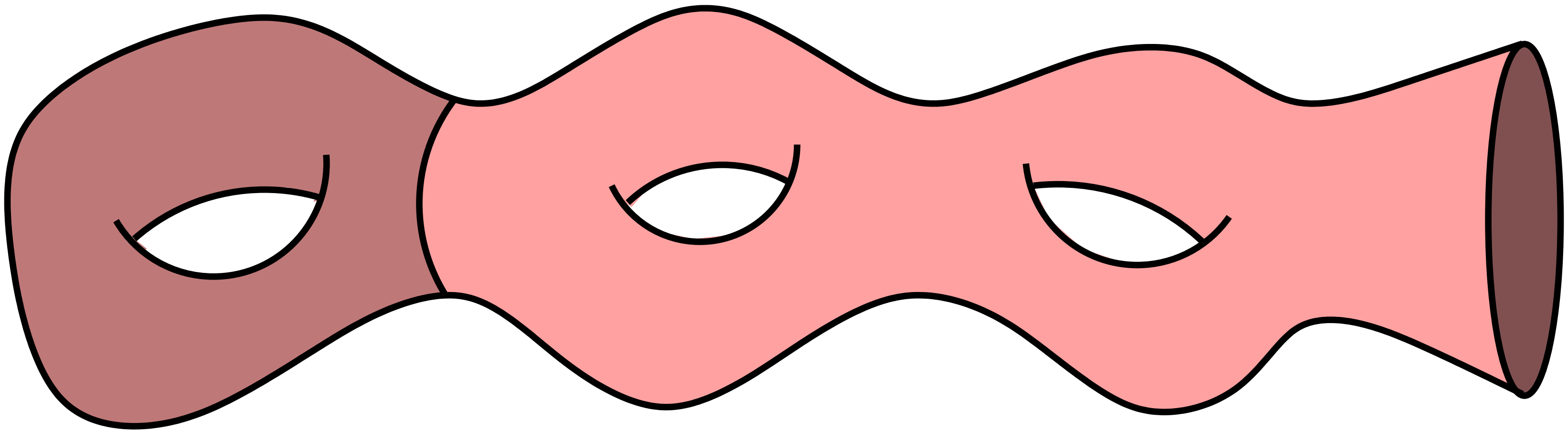
Change conformal structure to make QR map holomorphic.
Closely spaced dots $\Rightarrow$ small conformal change.
$\Rightarrow$ Belyi surfaces are dense in all surfaces.



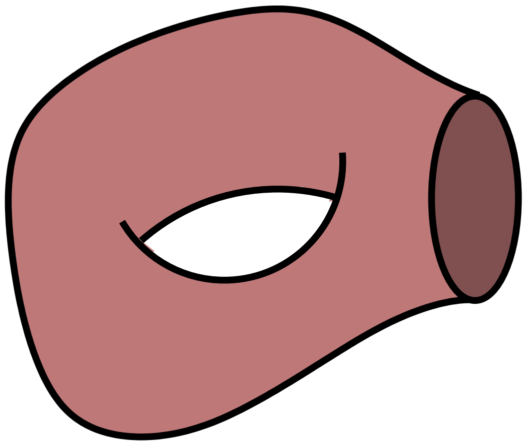
Every Riemann surface has a "QR Belyi" function.
Solving Beltrami gives countably, dense set of compact surfaces.
What about non-compact surfaces? .



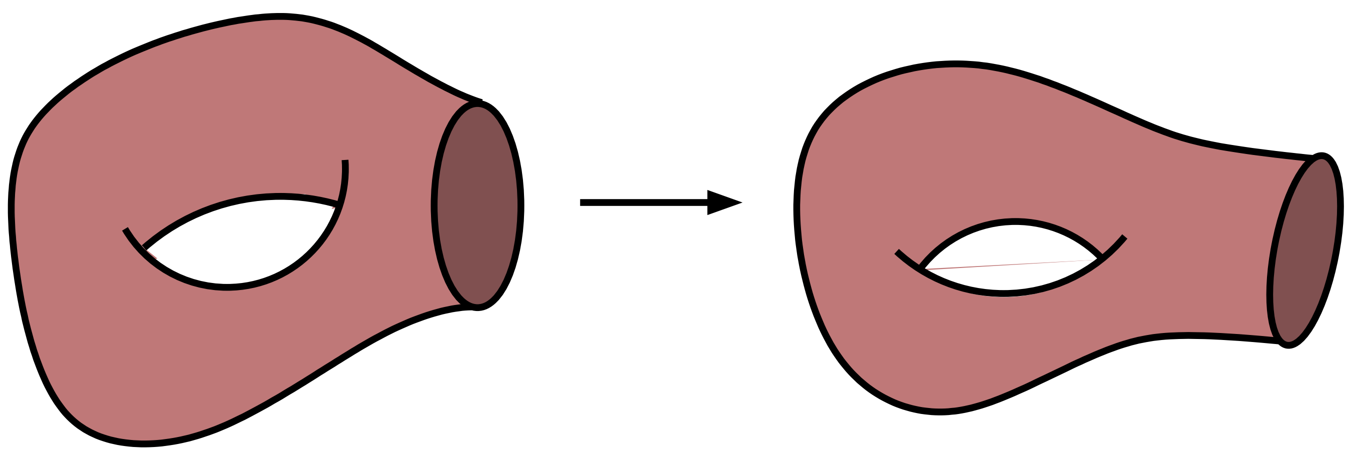
Consider a non-compact surface R.



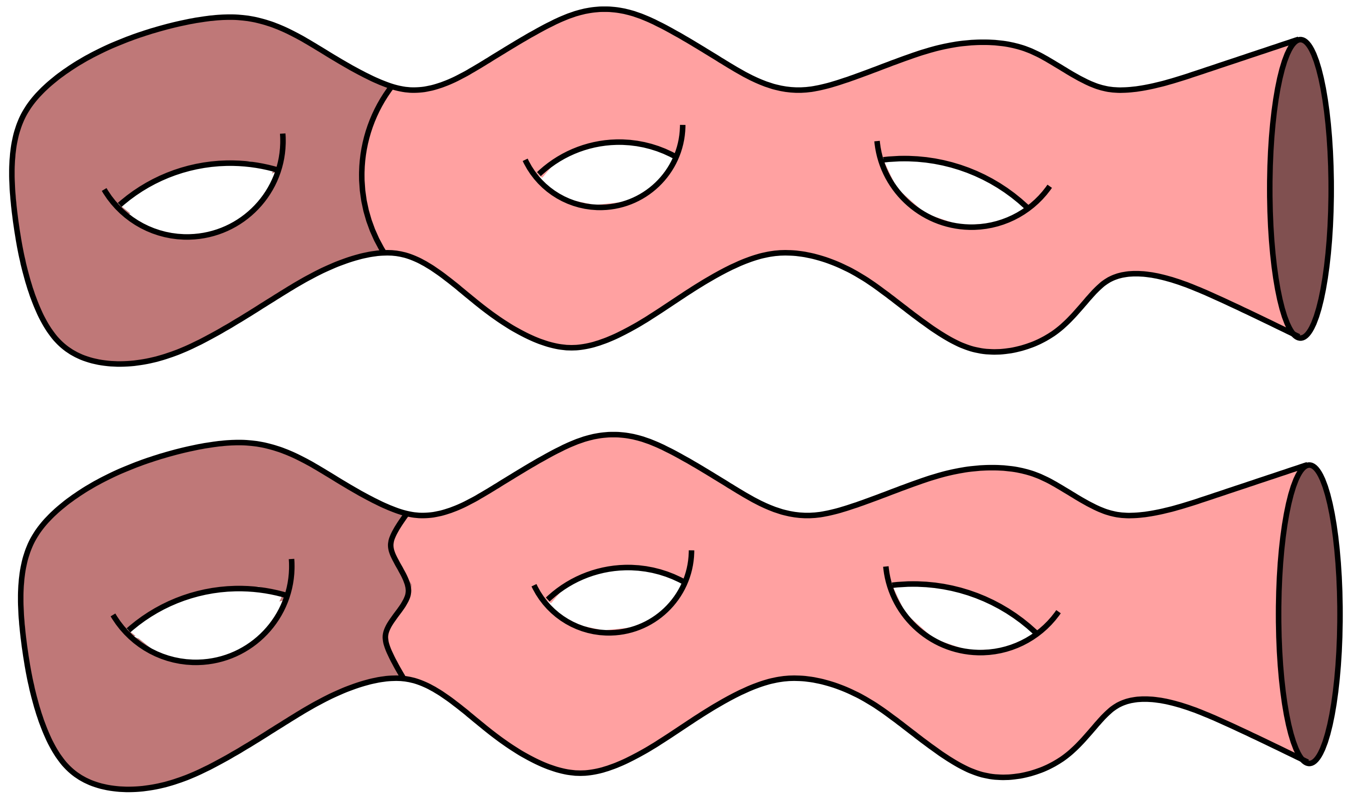
Take a compact, smoothly bordered piece Y_1.



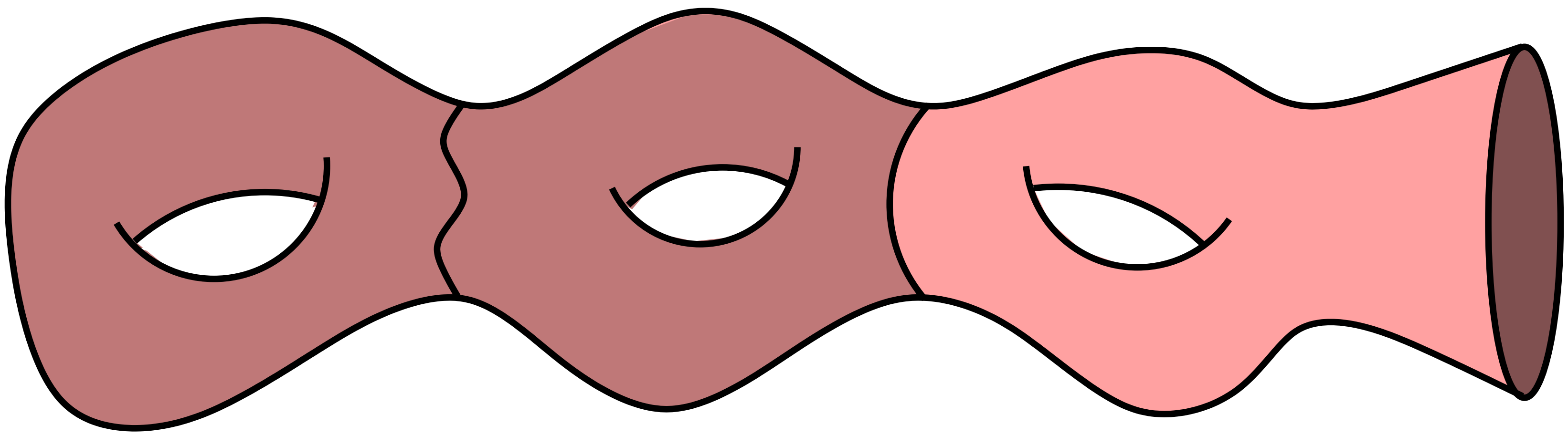
Take a compact, smoothly bordered piece Y_1.
Build a QR covering map on this piece as before.



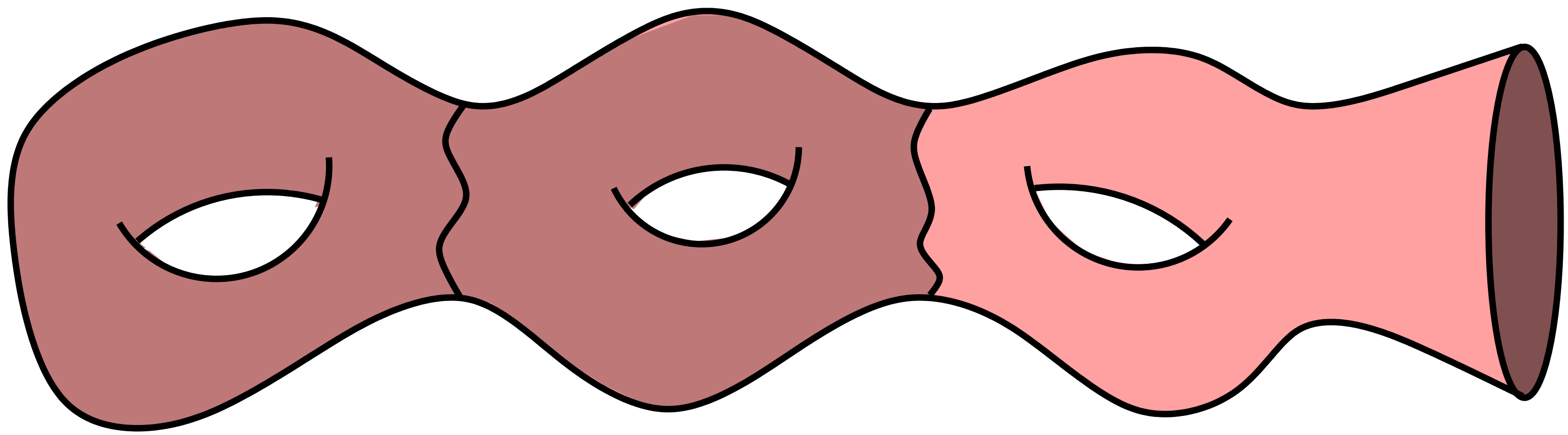
Take a compact, smoothly bordered piece Y_1.
Build a QR covering map on this piece as before.
Solve Beltrami on the compact piece. Get new surface $\widetilde{Y}_1 \neq Y_1$.



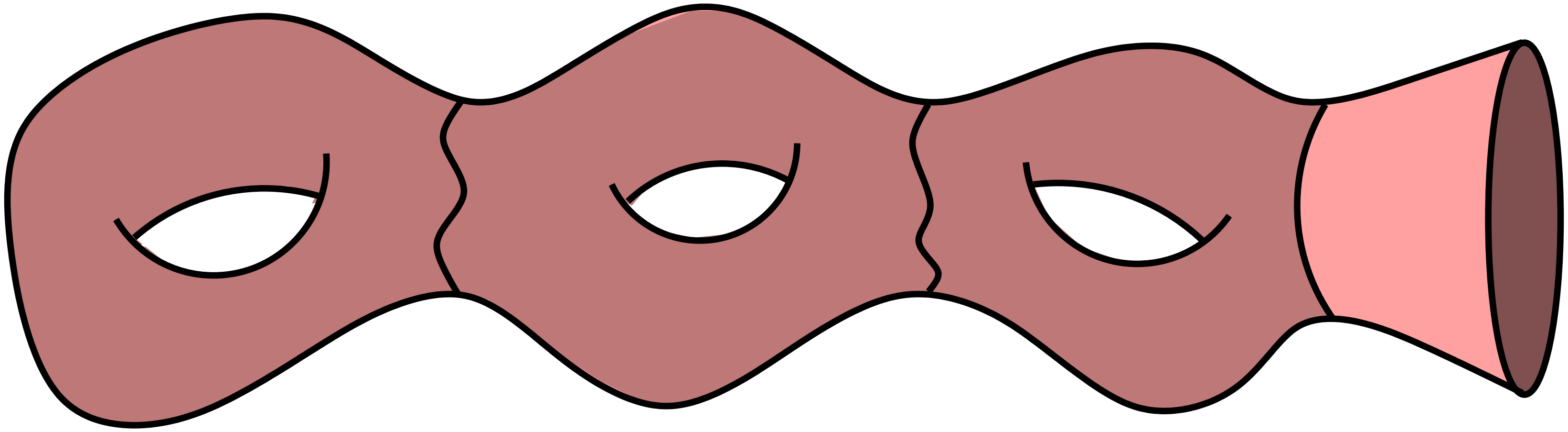
Key idea: small dilatation implies $\widetilde{Y}_1$ embeds in R.



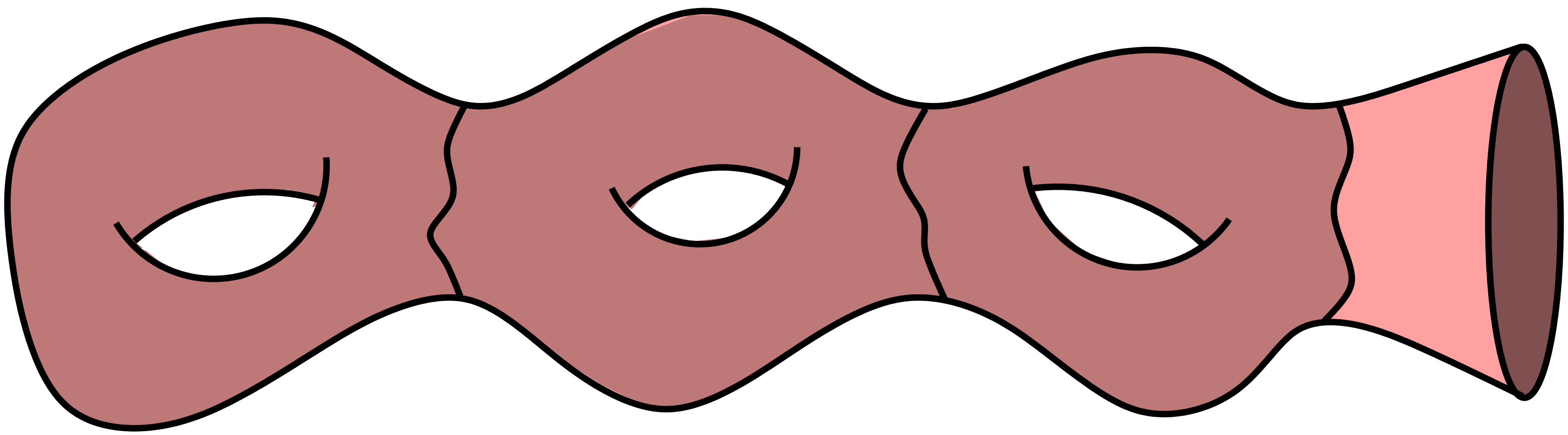
Take a larger compact, bordered piece Y_2.



Construct QR, solve Beltrami, embed $\widetilde{Y}_2$ in R.



Induction: Construct, Beltrami, re-embed in R,...



Induction: Construct, Beltrami, re-embed in R,...
Eventually build Belyi function on R.

Thm (B-Rempe): Every non-compact surface has a Belyi function.

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Corollary: Every non-compact surface can be obtained by gluing together equilateral triangles.

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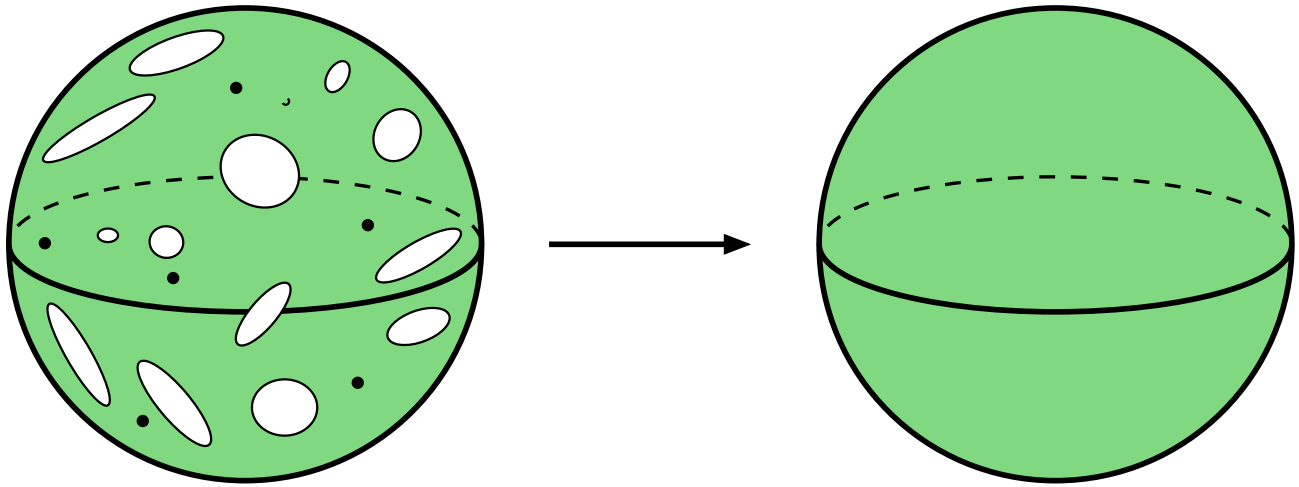
Corollary: Every Riemann surface is a branched cover of the sphere, branched over finitely many points.

For compact surfaces, this is Riemann-Roch.

Compact, genus g sometimes needs $3g$ branch points.

3 branch points suffice for all non-compact surfaces.

Corollary: Any open $U \subset \mathbb{C}$ is 3-branched cover of the sphere.



A. Epstein: a holomorphic map $X \to Y$ is finite type if

- Y is compact,
- f is open,
- f has no isolated removable singularities,
- the set of singular values is finite.

Gives many new dynamical systems of finite type.

E.g., escaping points have more than one point they can escape to.

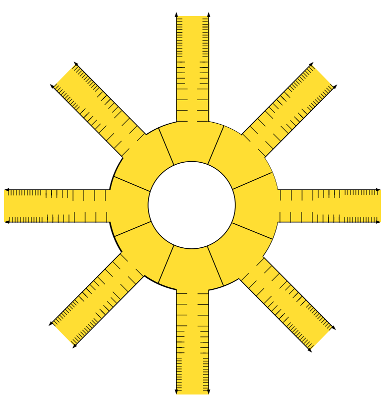
THANKS

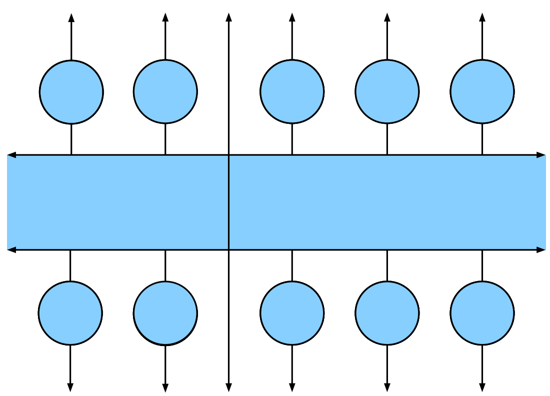
Connection between finite planar trees and polynomials can be extended to infinite trees and entire functions.

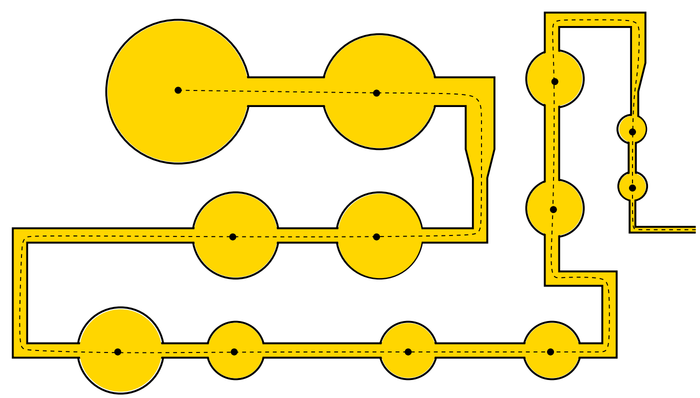
Folding theorem: given an infinite planar tree T with mild geometric assumptions, there is an entire function f with two singular values so that $f^{-1}([-1,1])$ approximates T.

This has various applications, e.g.,

- existence of certain "small" Julia sets,
- new wandering domains for certain entire functions,
- meromorphic functions with prescribed postcritical dynamics,







THANKS