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Connectivity & unboundedness of Fatou components for elliptic functions

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Dynamics of elliptic functions

A few references

Seminal works by Jane Hawkins (UNC) & Lorelei Koss (Dickinson C.)

- Ergodic properties of Julia sets of Weierstrass elliptic functions, Mons. Math., 2002.
- Parametrized dynamics of the Weierstrass elliptic function, Conf. Geom. Dyn., 2004
- Connectivity properties of Julia sets of Weierstrass elliptic functions, Top. Appl. 2005

Several works by Janina Kotus (IMPAN) & Mariusz Urbański (UNT)

- Ergodic theorey, geometric measure theory, conformal measures and the dynamics of elliptic functions. arXiv:2007.13235 (math.DS).

Elliptic functions

Fix a lattice $\Lambda = \{n\lambda_1 + m\lambda_2 \ : \ n, m \in \mathbb{Z}, \operatorname{Im}(\lambda_2/\lambda_1) > 0\}$.

Elliptic function

An elliptic function is a function of a complex value, meromophic in $\mathbb{C}$ and doubly periodic w/r to Λ,

$$f_\Lambda : \mathbb{C} \to \widehat{\mathbb{C}}, \qquad f_\Lambda(z+\lambda_1) = f_\Lambda(z) = f_\Lambda(z+\lambda_2)$$

- $[z] := \{z + \lambda \ : \ \lambda \in \Lambda\}$ is the Λ-orbit or residue class of z.
- Natural projection $\operatorname{Proj} : \mathbb{C} \to \mathbb{C}/\Lambda$, with $z \mapsto [z]$.

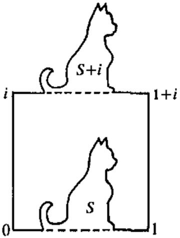
Double periodicity gives rise to interesting dynamical examples (but also to some headaches).

Fundamental domains

A closed and connected subset $\mathcal{Q} \subset \mathbb{C}$ is a fundamental domain if

- for each $z \in \mathbb{C}$, $\mathcal{Q}$ contains at least one point in its residue class, $[z] := \{z + \lambda \ : \ \lambda \in \Lambda\}$, and
- no two points in the interior of $\mathcal{Q}$ belong to the same residue class.

A fundamental parallelogram may have several shapes (square, rectangular, triangular) but the boundary of a fundamental domain may be more general (image from Jones & Singerman book).



Poles and zeros

Theorem

Any f_Λ is uniquely determined (up to a mult. constant) by its set of poles $\{\zeta_j\}$, its set of zeros $\{z_k\}$ and their multiplicities, satisfying

$$\sum_j \mu_j \zeta_j \equiv \sum \nu_k z_k \mod \Lambda.$$

- The field of elliptic functions over Λ is given by

$$\mathcal{E}(\Lambda) = \{R \circ \wp_\Lambda + \wp'_\Lambda (S \circ \wp_\Lambda) \ : \ R, S \in \text{Rat}\}.$$

Order and singular values

Fix Λ and let f_Λ be an elliptic function w/r to Λ.

- f_Λ has order $o_f \geq 2$, if for any given fundamental domain $\mathcal{Q}$ with no poles in its boundary, there are o_f poles (counting multiplicity) in the interior of $\mathcal{Q}$.
- Analogously, o_f is the number of solutions of $f_\Lambda(z) = a$ in $\mathcal{Q}$ for any $a \in \widehat{\mathbb{C}}$.

On each fundamental domain:

- f_Λ has $2o_f$ ramification points from which $o_f + r_f$ are critical points ($r_f =$ number of poles in the domain counted without multiplicity).
- f_Λ has no omitted values nor asymptotic values in $\mathbb{C}$.

Thus, $\#\mathrm{SV}(f_\Lambda) \leq o_f + r_f < \infty$.

Dynamics of elliptic functions

Theorem (Hawkins & Koss, 2002)

A periodic Fatou component of f_Λ is either a (super-)attracting/parabolic domain or a rotation domain. Every Fatou component is periodic or pre-periodic. Furthermore

$$\mathcal{J}(f_\Lambda) = \mathcal{J}(f_\Lambda) + \Lambda, \qquad \mathcal{F}(f_\Lambda) = \mathcal{F}(f_\Lambda) + \Lambda.$$

- $\mathcal{J}(f_\Lambda) = \overline{\bigcup_{k\geq 0} f_\Lambda^{-k}(\infty)}$ and $\mathcal{F}(f_\Lambda) = \widehat{\mathbb{C}} \setminus \mathcal{J}(f_\Lambda)$.
- $\Lambda-$invariance rules out existence of two completely invariant Fatou components.

Standing hypothesis

Throughout this talk

Λ will be fixed and we will consider the dynamics of a nonconstant elliptic function f_Λ.

Remark (Parametrized dynamics)

- $f_\Lambda + b$ for $b \in \mathbb{C}$.
- $f_{k\Lambda}$ with $k \in \mathbb{C} \setminus \{0\}$ (homogeneity properties of $\wp_\Lambda, \wp'_\Lambda$).
- The invariants of $\Lambda = \Lambda(g_2, g_3)$ satisfy $\triangle := g_2^3 - 27g_3^2 \neq 0$, then $f_{\Lambda(g_2,g_3)}$ as (g_2, g_3) varies in $\mathbb{C}^2 \setminus \{\triangle = 0\}$.

Fatou components :: Connectivity

Existence of Herman rings

- A periodic Fatou component of f_Λ has connectivity 1, 2 or ∞.

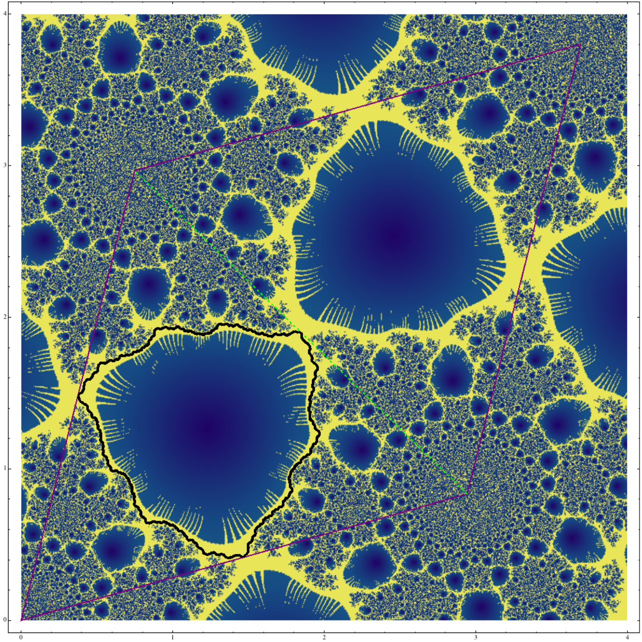
Theorem (MR 2020)

There exists an elliptic function of order at least 3 *with a forward invariant Herman ring.*

- Existence shown using Shishikura's quasiconformal surgery techniques.
- Same procedure can be extended to show existence of elliptic functions with cycles of Herman rings.

Still missing...

An explicit example of an elliptic function with a Herman ring.



By J. Hawkins

Overview of the construction

- Elliptic function f_Λ of order $o_f \geq 2$ and a rational function W of degree $d \geq 2$, both with invariant Siegel disks.
- Obtain quasiregular function $g : \mathbb{C} \to \widehat{\mathbb{C}}$,
  - doubly periodic w/r Λ,
  - $o_f + d - 1$ poles on each fundamental domain.
  - g-invariant annular domain A.

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  - doubly periodic w/r Λ,
  - $o_f + d - 1$ poles on each fundamental domain.
  - g-invariant annular domain A.
- A Beltrami differential μ defined a.e. over $\mathbb{C}$, of bounded dilation, g-invariant and satisfying $\mu(z+\lambda) = \mu(z)$ for all $\lambda \in \Lambda$.

Theorem

There exists a qc-map $\psi : \mathbb{C} \to \widehat{\mathbb{C}}$ that integrates μ and

$$G = \psi \circ g \circ \psi^{-1} : \mathbb{C} \to \widehat{\mathbb{C}}$$

is doubly periodic w/r $\psi(\Lambda)$, of order $o_f + d - 1 \geq 3$ and has an invariant Herman ring $\psi(A)$.

Herman rings

• Over any lattice Λ, the Weierstrass $\wp$ function has no cycles of Herman rings (Hawkins & Koss, 2002)

In analogy to Shishikura's result for quadratic rational functions, one has

Theorem (MR 2020)

An elliptic function of order 2 cannot have Herman rings.

- If f_Λ has a double pole, then same ideas as for $\wp_\Lambda$.
- If f_Λ has two simple poles (a Jacobi elliptic function) then a new approach is necessary.

Herman rings

Two other consequences from quasiconformal surgery are.

Theorem

Any elliptic function has at most $o_f - 2$ forward invariant Herman rings.

• If $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is meromorphic with $N < \infty$ poles, then f has at most $N - 1$ invariant Herman rings (Domínguez & Fagella; Fagella & Peters).

Corollary

An elliptic function with a unique pole of multiplicity o_f cannot have Herman rings.

Connectivity and critical values

- If every Fatou component has at most one critical value, then $\mathcal{J}$ is connected and every Fatou component is simply connected.
- If f_Λ is hyperbolic[1] and there exists a Fatou component U that contains all critical values, then $\mathcal{J}(f_\Lambda)$ is a Cantor set and U has ∞-connectivity (H&K 2005).

Remark

A Fatou component U containing more than one critical value, may give rise to unbounded Fatou components.

[1] Rippon & Stallard: for each $n \in \mathbb{N}$ and $z \in \mathcal{J}(f_\Lambda)$ which is not a prepole, there exist $C > 0$ and $K > 1$ so that $|(f_\Lambda^n)'(z)| \geq CK^n$.

Fatou components :: Unboundedness

Bounded vs. unbounded components

Unbounded Fatou components

Any Fatou component not contained in a fundamental domain is unbounded.

There are examples of unbounded Fatou components coming from

1. forward invariant basins (or their preimages) of attracting, superattracting or parabolic cycles (Hawkins & Koss; MR & Pérez Lucas; Hawkins & MR).
2. A preimage of a fixed Siegel disk (recently announced by Hawkins & Koss).

Bounded vs. unbounded components

Bounded components

Every periodic rotation domain is always bounded.

Proposition (Hawkins & Koss, 2002)

If $U_1, \ldots, U_n$ is an n-cycle of Siegel disks for an elliptic function f_Λ, then each U_j lies in the interior of a fundamental domain.

- This is a consequence of f_Λ^n acting injectively on each U_j.
- Same result holds for cycles of Herman rings.

Single or double?

Definition (Toral band Fatou components)

A component $U \subset \mathcal{F}(f_\Lambda)$ is called a

1. **Single toral band** if $\mathrm{Proj}(U)$ is a topological band in $\mathbb{C}/\Lambda$ that contains a homotopically non-trivial curve.

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• If $\mathcal{F}(f_\Lambda)$ has a double toral band component, then $\mathcal{J}(f_\Lambda)$ is disconnected.

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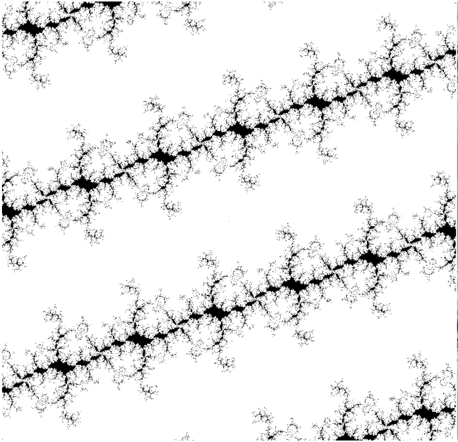
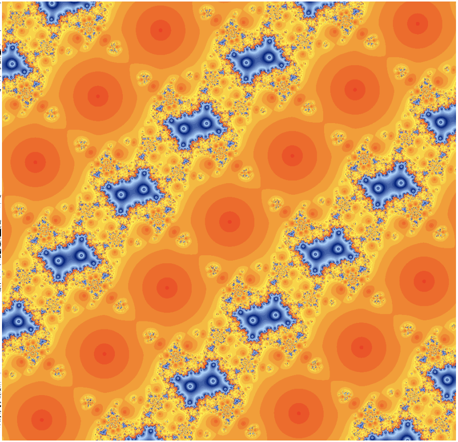
2. **Double toral band** if U contains the boundary of a fundamental domain of Λ.

• If $\mathcal{F}(f_\Lambda)$ has a double toral band component, then $\mathcal{J}(f_\Lambda)$ is disconnected.

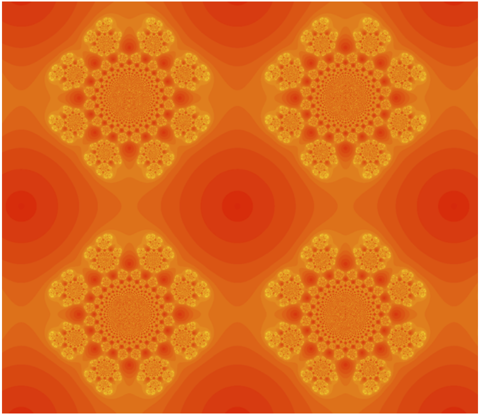
• Koss (2015) gave first example of double toral band with disconnected (but not Cantor) Julia set.

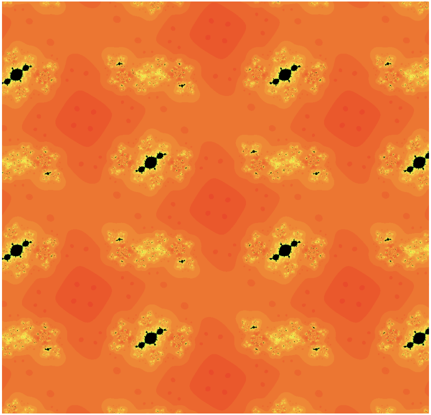
Single toral bands





Double toral bands





Unboundedness and critical values

Theorem

Let f_Λ be an elliptic function of order 2 over a lattice Λ. If U is a Fatou component that contains n critical values. Then

1. *if $n \geq 2$, $\mathcal{F}(f_\Lambda)$ contains a toral band. Furthermore*
2. *if $n \geq 3$, $\mathcal{F}(f_\Lambda)$ contains a double toral band.*

Since $o_f = 2$ we have

- $f_\Lambda = M \circ \wp_\Lambda$, with M a Möbius transformation.
- $\#SV(f_\Lambda)$ is either 3 (if f_Λ has double pole) or 4 (two simple poles)
- Critical symmetry: if $c \in \mathrm{Crit}(f_\Lambda)$ and $c \in U \subset \mathcal{F}(f_\Lambda)$,

$$c + z \in U \qquad \text{iff} \qquad c - z \in U.$$

• H&K have shown (1) using critical symmetry for $\wp_\Lambda$.

Sketch of the proof

An order 2 elliptic function takes the form $f_\Lambda = M \circ \wp_\Lambda$ where either

1. $M(z) = A\left(\frac{z-k}{z-h}\right)$, for $A, h, k \in \mathbb{C}$, with $A \neq 0$. In this case, f_Λ has two simple poles and four critical points,

$$\text{Crit}(f_\Lambda) = \left\{\frac{\lambda_1}{2}, \frac{\lambda_2}{2}, \frac{\lambda_3}{2}, 0\right\} + \Lambda, \qquad \text{Poles}(f_\Lambda) = \{\zeta_0, \zeta_1\} + \Lambda$$

   where $\wp_\Lambda(\zeta_i) = h$.

2. $M(z) = A(z-k)$, for $A, k \in \mathbb{C}$, $A \neq 0$. In this case, f_Λ has one double pole and three critical points,

$$\text{Crit}(f_\Lambda) = \text{Crit}(\wp_\Lambda), \qquad \text{Poles}(f_\Lambda) = \text{Poles}(\wp_\Lambda) = \Lambda.$$

Sketch of the proof

- Assume f_Λ has double pole and $U \subset \mathcal{F}(f_\Lambda)$ contains two critical values, $v_1, v_2 \in U \cap SV(f_\Lambda)$, where $f_\Lambda(\lambda_i/2) = v_i$.
- W.l.o.g. let $V \subset f_\Lambda^{-1}(U)$ with $\frac{\lambda_1}{2}, \frac{\lambda_2}{2} \in V$.

Questions

Unboundedness and critical values

Claim

If f_Λ has order 2 and U contains exactly two critical values, then $\mathcal{F}(f_\Lambda)$ contains a single toral band.

Question

For any f_Λ of order $o_f \geq 3$, what is the minimal number of critical values necessary inside a Fatou component so its preimage is a single toral band?

Connectivity of unbounded components

- Double toral bands are by definition ∞-connected.
- Example of high order elliptic function with single toral bands of ∞-connectivity.

Question

Are single toral bands of ∞-connectivity possible for lower orders?

Força Barça!