On the computational complexity of Julia sets in the exponential family

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— joint work with Artem Dudko (IMPAN) —

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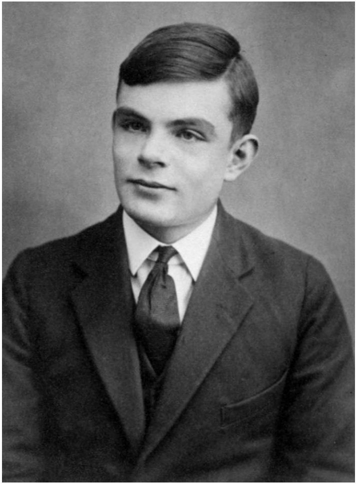


Alan M. Turing (1912 - 1954)

Roughly speaking, the **theory of computability** studies the existence of an algorithm which solves a problem in an effective manner.

A **Turing Machine** is a mathematical model of computation. It was introduced by Alan Turing in 1936 while studying the so-called Entscheidungsproblem (decision problem) posed in 1928 by Hilbert and Ackermann: *is there an algorithm that considers, as input, a statement and answers "Yes" or "No" according to whether the statement is universally valid?*

In this talk we use the terms

Turing Machine $\leftrightsquigarrow$ **algorithm**

interchangeably.

Computable numbers

Definition

A number $\alpha \in \mathbb{R}$ is called **computable** if there is a Turing Machine which given $n \in \mathbb{N}$ produces a number $\phi(n)$ such that

$$|\alpha - \phi(n)| < 2^{-n}.$$

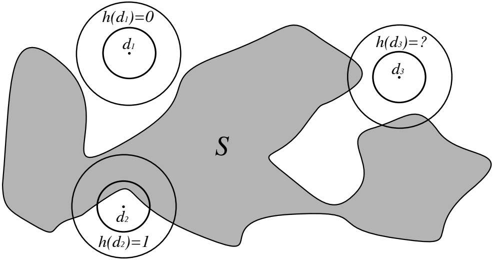
A complex number $z \in \mathbb{C}$ is computable if and only if $\operatorname{Re} z$ and $\operatorname{Im} z$ are computable.

Examples:

- algebraic numbers are computable;
- many famous transcendental numbers like e and π are computable;
- examples of non-computable numbers are usually obtained using *undecidable problems* such as the *halting problem*.

Note that there are only countably many Turing Machines, so the set of all computable numbers is countable.

Dyadic approximations



Consider the set of **dyadic numbers**

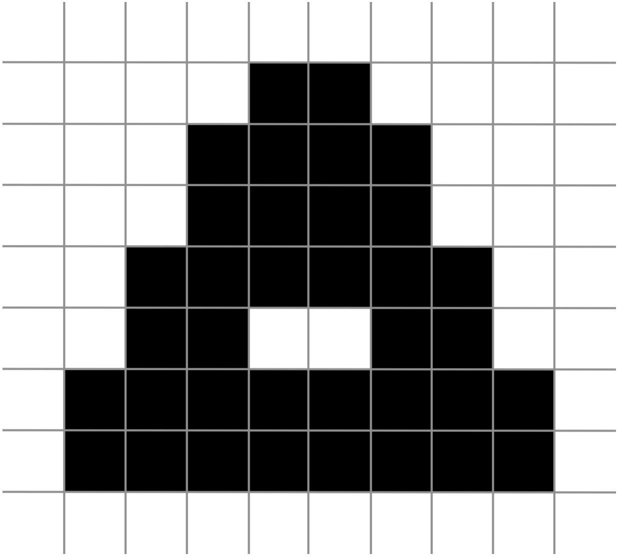
$$\mathbb{D} := \{z \in \mathbb{C} \colon z = k/2^n + i\,j/2^n \text{ for some } k, j \in \mathbb{Z} \text{ and } n \in \mathbb{N}\}.$$

A **dyadic approximation** of a set $S \subseteq \mathbb{C}$ can be described using a function

$$h_S(n,z) = \begin{cases} 1, & \text{if } \; d(z,S) \leqslant 2^{-n-1}, \\ 0, & \text{if } \; d(z,S) \geqslant 2 \cdot 2^{-n-1}, \\ 0 \text{ or } 1, & \text{otherwise}, \end{cases}$$

where $n \in \mathbb{N}$ and $z = k/2^{n+2} + i\,j/2^{n+2}$, for $k, j \in \mathbb{Z}$.

Example



Computable sets

Definition

A set $S \subseteq \mathbb{C}$ is **computable** in time $t(n)$ if there is a Turing Machine which computes the function $h(n, z)$ for $z \in \mathbb{D}$ in time $t(n)$.
In other words, given $n \in \mathbb{N}$ and any dyadic point of the form

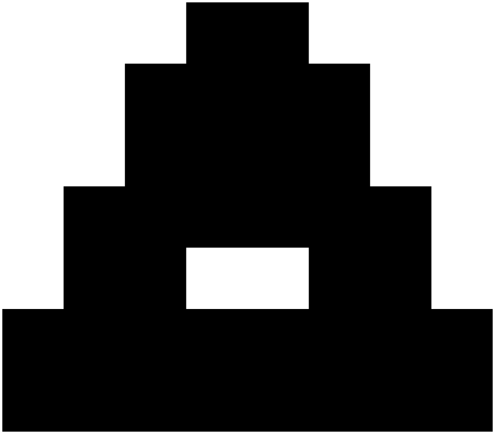
$$z = (k/2^{n+2}, j/2^{n+2}), \qquad \text{for } k, j \in \mathbb{Z},$$

this algorithm returns the value of $h(n, z)$ spending at most $t(n)$ time units.

Examples:

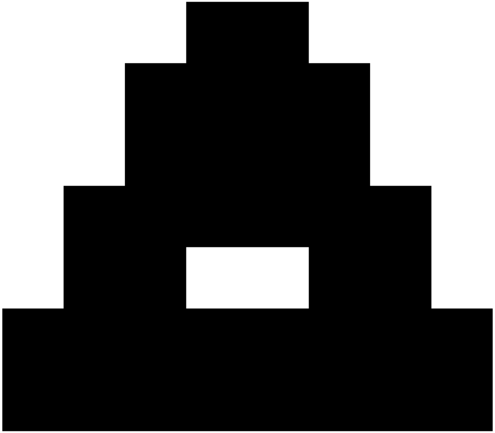
- a polygon (with or without interior) is computable if and only if all of its vertexes have computable coordinates;
- a disk is computable if and only if its center and its radius are computable.

Example



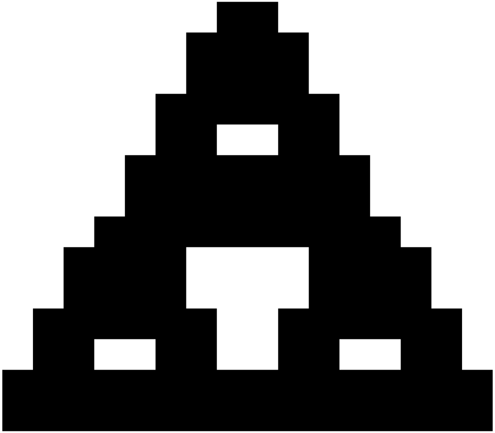
Perhaps... ?

Example



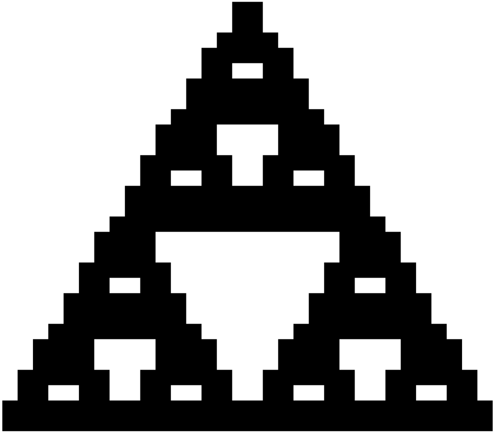
$n = 0$

Example



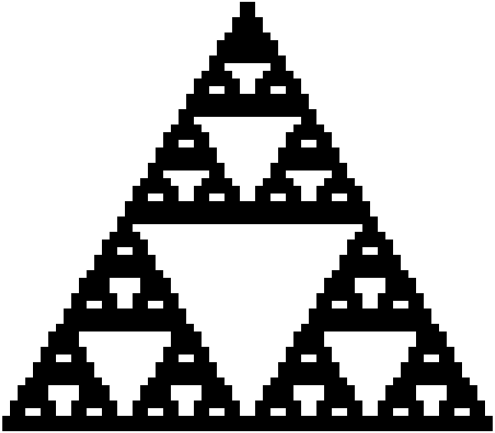
$n = 1$

Example



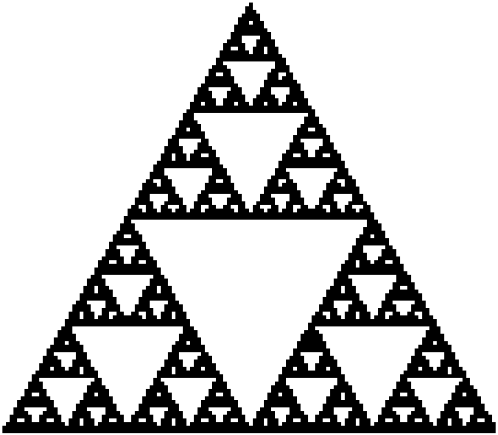
$n = 2$

Example



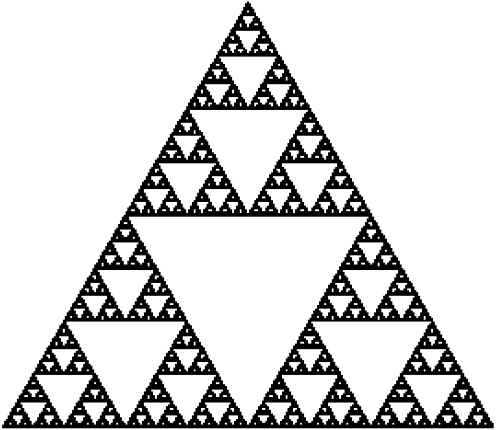
$n = 3$

Example



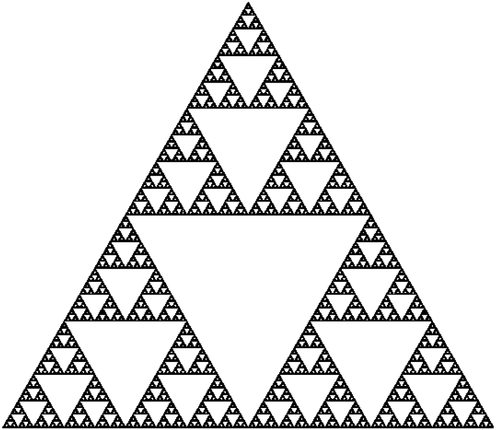
$n = 4$

Example



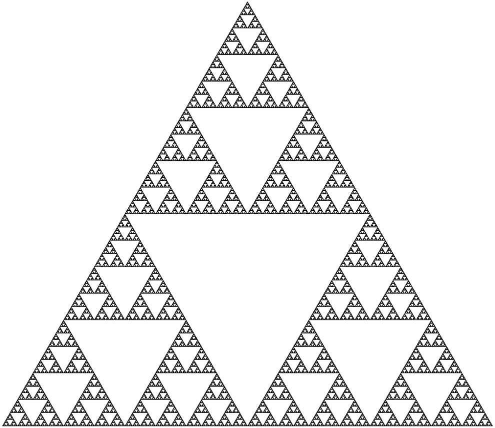
$n = 5$

Example



$n = 6$

Example



$n = 7$

Computability of Julia sets

In order to compute the Julia set of a rational or entire function f, we need to assume that a Turing Machine can obtain approximations of the values of $f(z)$.

Definition

A function $\phi : \mathbb{N} \to \mathbb{D}$ is called an **oracle** for an element $z \in \mathbb{C}$, if $|\phi(n) - z| < 2^{-n}$ for all $n \in \mathbb{N}$.

An oracle works as a black box that is able to provide approximations of a number independently of whether it is computable or not.

Definition

Let f be an entire function. The Julia set $J(f)$ is called **computable** in time $t(n)$, if there is a Turing Machine with an oracle for the values of f, which computes the values of $h_S(n, z)$ for $S = J(f)$ in time $t(n)$. We say that $J(f)$ is **computable in polynomial time** if furthermore $t(n)$ can be bounded by a polynomial.

Example 1. Hyperbolic rational maps

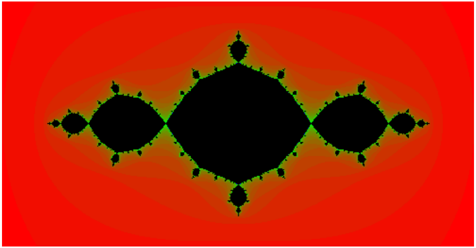
A periodic point z_0 of period $p \in \mathbb{N}$ is **attracting** if $|(f^p)'(z_0)| < 1$. Every attracting cycle has a **basin of attraction** given by

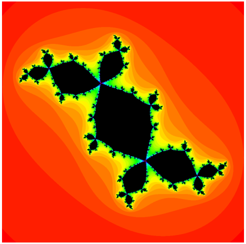
$$\mathcal{A}(z_0) := \{z \in \mathbb{C} \colon \exists\, 0 \leqslant j < p,\ f^{np+j}(z) \to z_0 \text{ as } n \to \infty\}.$$

A rational function f is **hyperbolic** if all the components of $F(f)$ are basins of attraction.

Theorem (Braverman '05, Rettinger '05)

For each $d \geqslant 2$ there exists a Turing Machine M with an oracle for the coefficients of a polynomial f of degree d such that if f is hyperbolic, then $J(f)$ is computable by M in polynomial time.





Lemma (Koebe distortion theorem)

Let $f : D(a,r) \to \mathbb{C}$ be a univalent function for $a \in \mathbb{C}$ and $r > 0$. Then, for any $z \in D(a,r)$,

$$\frac{|z-a||f'(a)|}{(1+|z-a|/r)^2} \leqslant |f(z)-f(a)| \leqslant \frac{|z-a||f'(a)|}{(1-|z-a|/r)^2},$$

and

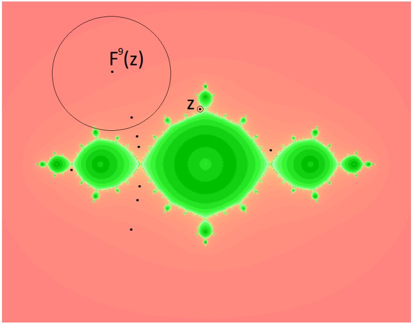
$$\frac{(1-|z-a|/r)|f'(a)|}{(1+|z-a|/r)^3} \leqslant |f'(z)| \leqslant \frac{(1+|z-a|/r)|f'(a)|}{(1-|z-a|/r)^3}.$$

In particular, if we put

$$r_1 := \frac{\alpha|f'(a)|r}{(1+\alpha)^2} \quad \text{and} \quad r_2 := \frac{\alpha|f'(a)|r}{(1-\alpha)^2},$$

then, for $0 < \alpha < 1$,

$$D(f(a), r_1) \subseteq f(D(a, \alpha r)) \subseteq D(f(a), r_2).$$



Sketch of the algorithm: There is a neighbourhood U of $J(f)$ and $C > 0$ such that

- given a point $z \in \mathbb{D}$ and $n \in \mathbb{N}$ compute approximate values of $z_k = f^k(z)$, for $1 \leqslant k \leqslant Cn$;
- if $z_k \in U$ for all $1 \leqslant k \leqslant Cn$ then $\mathrm{dist}(z, J(f)) < 2^{-n}$;
- if $z_k \notin U$ for some $1 \leqslant k \leqslant Cn$ then up to a constant factor

$$\mathrm{dist}(z, J(f)) \approx \frac{\mathrm{dist}(z_k, J(f))}{|(f^k)'(z)|} \approx \frac{1}{|(f^k)'(z)|}.$$

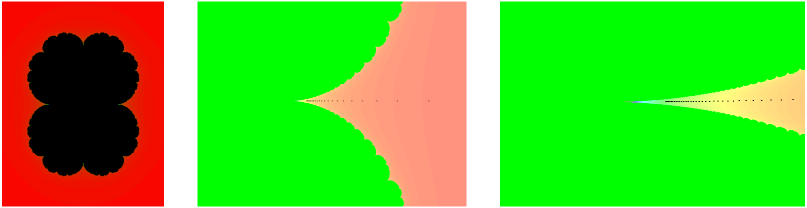
Example 2: Parabolic fixed points

A periodic point z_0 of period $p \in \mathbb{N}$ is **parabolic** if $(f^p)'(z_0) = e^{2\pi i \theta}$ with $\theta \in \mathbb{Q}$. Every parabolic cycle has a **parabolic basin of attraction** given by

$$\mathcal{A}(z_0) := \{z \in \mathbb{C} \colon \exists\, 0 \leqslant j < p,\ f^{np+j}(z) \to z_0 \text{ as } n \to \infty\}.$$

Theorem (Braverman '06)

For any $d > 2$ there exists a Turing Machine M with an oracle for the coefficients of a rational map f of degree d such that if every critical orbit of f converges either to an attracting or to a parabolic orbit, M computes $J(f)$ in polynomial time.



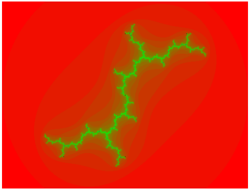
Problem: orbits escape exponentially slowly from a parabolic fixed point.

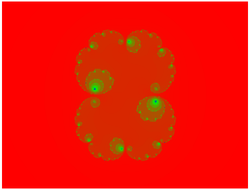
- Braverman showed directly that exponential iterates of f near the fixed point can be computed in polynomial time.
- Dudko-Sauzin '14 obtained efficient approximations of the *Fatou coordinates* which conjugate f to the translation $z \mapsto z + 1$ near the fixed point.

Example 3. Filled Julia sets with no interior

Theorem (Binder, Braverman & Yampolsky '07)

For each $d \geqslant 2$ there exists a Turing Machine M with an oracle for the coefficients of a polynomial f of degree d such that if the filled-in Julia set $K(f)$ has no interior, then $J(f)$ is computable by M.

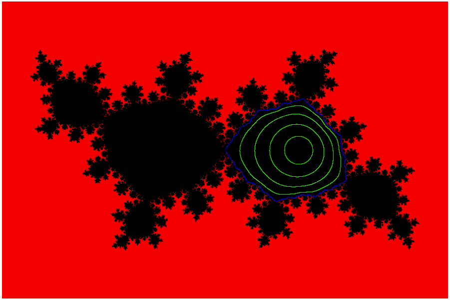




Sketch of the algorithm: Using that repelling periodic points are dense in $J(f)$, we can compute enough approximations of the periodic points of f so that a given neighborhood of $K(f)$ is well approximated by balls centred at these points. On the other hand, we can compute the preimages of a large disc. The algorithm stops when the second approximation is contained in the first.

Example 4. Quadratic Siegel discs

A periodic point z_0 of period $p \in \mathbb{N}$ is of **Siegel type** if $(f^p)'(z_0) = e^{2\pi i \theta}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and f is linearizable in a neighbourhood of z_0. Every Siegel periodic point belongs to a Fatou component known as a **Siegel disc** where f is conjugated to an irrational rotation.



Theorem (Braverman & Yampolsky '06, '09)

There exists a quadratic polynomial f_θ with a Siegel fixed point at the origin of rotation number $\theta \in \mathbb{R} \setminus \mathbb{Q}$ such that $J(f_\theta)$ is not computable. Moreover, θ can be chosen computable and such that $J(f_\theta)$ is locally connected.

Theorem (Binder, Braverman & Yampolsky '06)

There exist quadratic polynomials f_θ with a Siegel disc such that $J(f_\theta)$ has arbitrarily large computational complexity.

Theorem (Binder, Braverman & Yampolsky '06)

Let $\Delta(\theta)$ be the Siegel disc of the quadratic polynomial f_θ. TFAE:

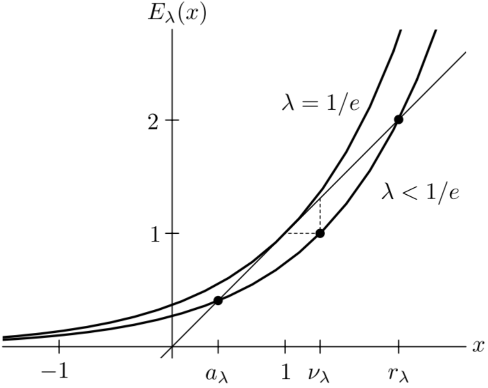
- *the Julia set $J(f_\theta)$ is computable;*
- *the inner radius $\rho(\theta) := \inf_{z \in \partial\Delta(\theta)} |z|$ is computable;*
- *the conformal radius $r(\theta)$ of $\Delta(\theta)$ is computable.*

The exponential family

Theorem (Dudko & MP)

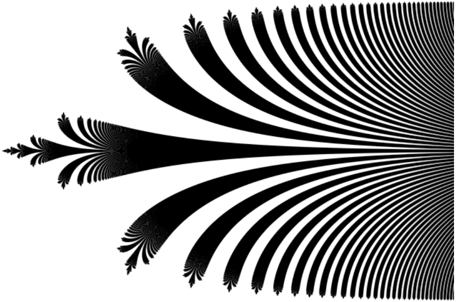
Let $E_\lambda(z) = \lambda e^z$ with $\lambda \in (0, 1/e)$. There exists a Turing Machine M with an oracle for the value of λ such that $J(E_\lambda)$ is computable by M in polynomial time.

For such parameters, the map E_λ has two real fixed points $a_\lambda < r_\lambda$.



Cantor bouquets

For such maps, the basin of attraction of a_λ is connected, and its complement $J(E_\lambda)$ is an uncountable collection of disjoint curves each joining a finite point to ∞ which is known as a **Cantor bouquet**.



Picture by Arnaud Chéritat.

$J(E_\lambda)$ has many **interesting properties**! For example, if End_λ denotes the set of the endpoints of the curves in $J(E_\lambda)$, the set $\mathrm{End}_\lambda \cup \{\infty\}$ is connected, but End_λ is totally disconnected. Moreover, $\dim_H(J(E_\lambda)) = 2$ but $\dim_H(J(E_\lambda) \setminus \mathrm{End}_\lambda) = 1$.

A first approximation

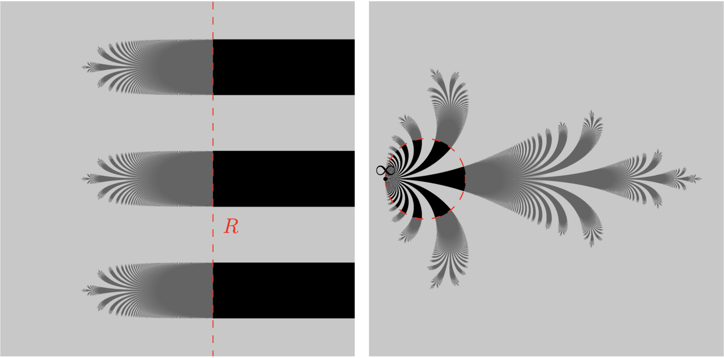
For $R > 0$ define

$$B_R := \{z \in \mathbb{C} \colon \operatorname{Re} z > R,\ |\operatorname{Im} z - 2k\pi| < \tfrac{\pi}{2} \text{ for some } k \in \mathbb{Z}\}.$$

Lemma

Let $\lambda \in (0, 1/e)$. There exists a linear function $R = R(n) > 0$ (which depends on λ) such that for $n \in \mathbb{N}$,

$$d(z, J(E_\lambda)) < 2^{-n}, \qquad \text{for all } z \in B_R.$$



Proposition

Let $\lambda \in (0, 1/e)$ *and suppose that* $M \in (a_\lambda, r_\lambda)$. *There exist* $0 < C_1 < C_2$ *such that the following is true. Assume that* $z \in \mathbb{C}$ *and* $k \in \mathbb{N}$ *are such that* $\operatorname{Re}(E_\lambda^k(z)) < M$ *and* $\operatorname{Re}(E_\lambda^j(z)) \geqslant M$ *for all* $0 \leqslant j < k$. *Then*

$$\frac{C_1 d(E_\lambda^k(z), J(E_\lambda))}{|(E_\lambda^k)'(z)|} \leq d(z, J(E_\lambda)) \leq \frac{C_2 d(E_\lambda^k(z), J(E_\lambda))}{|(E_\lambda^k)'(z)|}. \tag{1}$$

Proposition

Let $\lambda \in (0, 1/e)$. *If* $z \in \mathbb{C}$ *satisfies* $\operatorname{Re}(E_\lambda^k(z)) > 2\pi + r_\lambda$ *for some* $k \in \mathbb{N}$, *then*

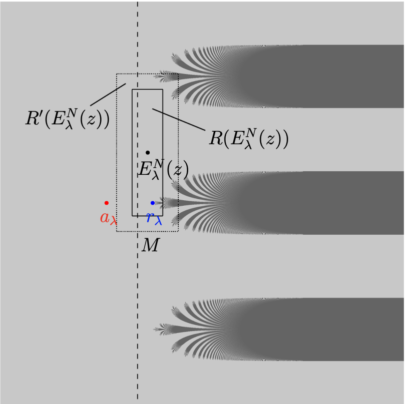
$$\frac{d(E_\lambda^k(z), J(E_\lambda))}{4|(E_\lambda^k)'(z)|} \leq d(z, J(E_\lambda)) \leq \frac{4d(E_\lambda^k(z), J(E_\lambda))}{|(E_\lambda^k)'(z)|}. \tag{2}$$

Proposition

Let $\lambda \in (0, 1/e)$ and $M_1 \in (1, r_\lambda)$. There exists a linear function $N : \mathbb{N} \to \mathbb{R}$ such that for every $n \in \mathbb{N}$, if

$$M_1 \leqslant \operatorname{Re}(E_\lambda^j(z)), \qquad \text{for all } 0 \leqslant j \leqslant N(n),$$

then $d(z, J(E_\lambda)) < 2^{-n}$.



The algorithm

Input: integer n (precision), real λ (parameter), complex $z \in \mathbb{D}(n+2)$ (dyadic point)

$1 < M_1 < M_2 < r_\lambda$ dyadic constants for using Propositions 3.3 and 3.6

$\epsilon \leftarrow$ dyadic constant from the proof of Proposition 3.3; auxiliary constant

$C \leftarrow \mathrm{Up}(\sqrt{\lambda^{-2} + \pi^2})$; auxiliary constant

$C_1 \leftarrow$ dyadic constant from Proposition 3.3; auxiliary constant

$K \leftarrow$ dyadic distortion constant from Proposition 3.6; Koebe constant

$R \leftarrow \mathrm{Up}((n+10)\ln 2 + \ln(C) - \ln(\lambda))$; upper bound for $\mathrm{Re}\, z$

$N \leftarrow \mathrm{Up}(((n+1)\ln 2 + \ln(KC))/\ln M_1)$; maximum number of iterations

$\tilde{\pi} \leftarrow 2^{-n-10}$ – approximation of π;

$k \leftarrow 0$;

while $k < N$ **do**

  compute dyadic approximations

   $z_k \approx E_\lambda^k(z) \pmod{2\pi i} = \lambda \exp(E_\lambda^{k-1}(z)) \pmod{2\pi i}$;

   $d_k \approx |DE_\lambda^k(z)| \pmod{2\pi i} = |E_\lambda^k(z) \cdot DE_\lambda^{k-1}(z)| \pmod{2\pi i}$;

  with precision $\min(2^{-n-10}, \epsilon/10)$;

  if $\mathrm{Re}\, z_k < (M_1 + M_2)/2$ **then**

   Compute a dyadic approximation T

   of $\min\{d(z_k, [0, a_\lambda]), d(z_k, B_{r_\lambda})\}$

   with precision $\epsilon/10$ Proposition 3.3

   if $T \leqslant 3 \cdot 2^{-n} d_k$ **then**

    return 1

   else

    return 0

   end if

  else

The algorithm

of $\min\{d(z_k,[0,a_\lambda]), d(z_k, B_{r_\lambda})\}$
with precision $\epsilon/10$ — Proposition 3.3
if $T \leqslant 3 \cdot 2^{-n} d_k$ **then**
  return 1
else
  return 0
end if
else
if $\mathrm{Re}\, z_k > R+1$ **then**
  if $|\mathrm{Im}\, z_k| < \tilde{\pi}/2 + 2^{-n-3}$ **then** — Proposition 4
    return 1;
  else — Proposition 3
    if $\frac{|\mathrm{Im}\, z_k| - \tilde{\pi}/2}{4 d_k} < 2^{-n}$ **then**
      return 1;
    else
      return 0;
    end if
  end if
end if
end if
$k \leftarrow k+1$;
end while
if $k = N$ **then** — Proposition 2
  return 1;
end if

Output: the program returns 1 if the point $z \in \mathbb{D}_n$ is at a distance smaller than 2^{-n} to $J(E_\lambda)$ and 0 otherwise

Hyperbolic maps in the exponential family

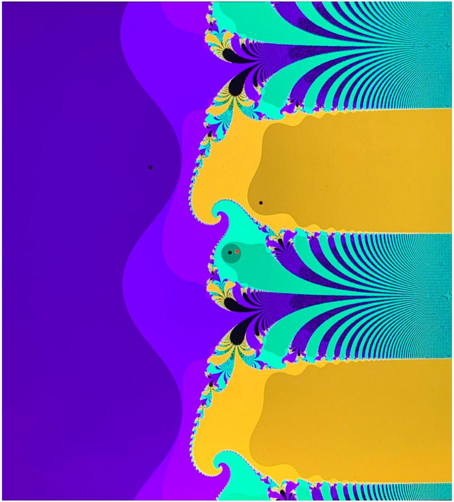
Let $\lambda \in \mathbb{C}^*$ be such that $E_\lambda(z) = \lambda e^z$ has an attracting periodic cycle of period $p \in \mathbb{N}$.

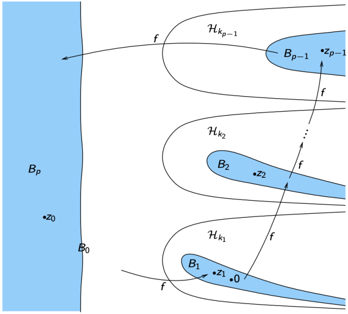
- $p = 1$: $F(E_\lambda)$ is connected and $J(E_\lambda)$ is a **Cantor bouquet**;
- $p > 1$: $F(E_\lambda)$ has ∞ components and $J(E_\lambda)$ is a **pinched Cantor bouquet**.

Theorem (Dudko & MP)

Let $\lambda \in \mathbb{C}^*$ be such that E_λ has an attracting periodic cycle. There exists a Turing Machine M with an oracle for the value of λ such that $J(E_\lambda)$ is computable by M in polynomial time.

Case $p > 1$





Sketch by Krzysztof Barański, Bogusława Karpińska and Anna Zdunik.

Combinatorics of hyperbolic exponential maps with a cycle of period $p > 1$ were studied by Bhattacharjee and Devaney.

Moltes gràcies per la vostra atenció!

Thank you for your attention!