Achievable connectivities of Fatou components for a family of rational maps

Dan Paraschiv

Universitat de Barcelona

April 2021

Joint work with Jordi Canela and Xavier Jarque

Table of Contents

Connectivity of Fatou components

Definition

The connectivity of a domain $D \subset \hat{\mathbb{C}}$ is the number of connected components of its boundary.

Periodic Fatou components can have connectivity 1, 2 or ∞. Strictly preperiodic Fatou components can have finite connectivity greater than 2. For any given $n \in \mathbb{N}^*$, there are examples of rational maps with Fatou components of connectivity n. However, the degree of n increases rapidly with n in these examples.

Theorem (Canela)

There exist rational maps whose dynamical plane contains Fatou components of arbitrarily large connectivity.

Examples of perturbation maps

Theorem (McMullen)

Let $F_\lambda = z^n + \frac{\lambda}{z^d}$, $n, d \in \mathbb{N}$ are such that $\frac{1}{n} + \frac{1}{d} < 1$. For $|\lambda|$ small enough, the Julia set is a Cantor set of quasicircles.

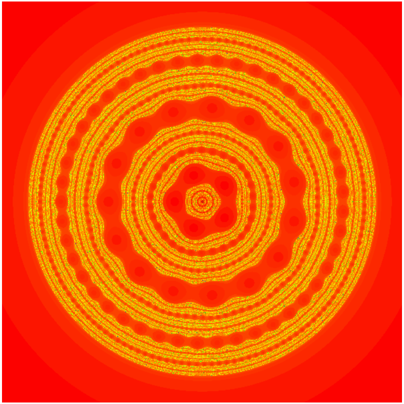
Theorem (Devaney, Look, Uminsky)

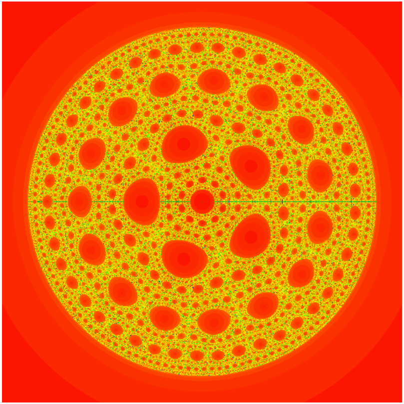
Let $F_\lambda = z^n + \frac{\lambda}{z^d}$. Suppose that the orbits of the free critical points of F_λ tend to ∞. Then, depending on the location of critical values, the Julia set is a Cantor set, a Cantor set of quasicircles, or a Sierpinski curve.

Theorem (Canela)

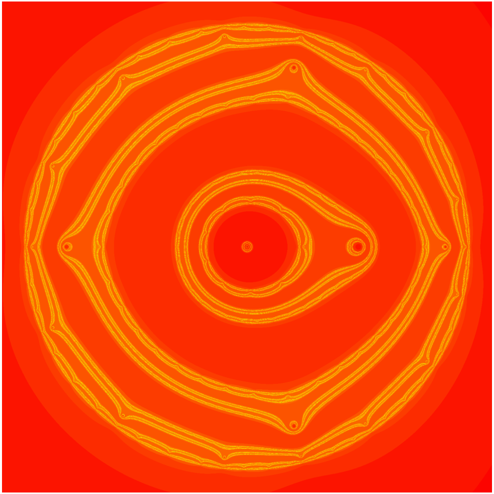
Let $B_{a,\lambda} = z^3 \frac{z-a}{1-\bar{a}z} + \frac{\lambda}{z^d}$, $n, d \in \mathbb{N}$ are such that $\frac{1}{n} + \frac{1}{d} < 1$. There exists λ such that the Julia set is a countable union of Cantor sets of quasicircles and uncountably many point components.

Dynamical plane of F_λ for $\lambda_1 = 10^{-4}$ and $\lambda_2 = 5 \cdot 10^{-3}$





Dynamical plane of $B_{0.5,10^{-7}}$



The unperturbed family

Let $S_{n,a,Q}$ be the rational map of degree $n+1$

$$S_{n,a,Q}(z) = \frac{z^n(z-a)}{Q(z)},$$

where $n \in \mathbb{N}$, $n > 1$, $a \in \mathbb{C}^*$, and Q is a polynomial, satisfying the following conditions:

(a) The polynomial Q has degree at most n, such that $Q(0) \neq 0$. Hence, $z = 0$ is a superattracting fixed point of degree n and ∞ is a fixed point.

(b) The fixed point $z = \infty$ is attracting.

(c) There are exactly two Fatou components, the immediate basins of attraction of $z = 0$ and ∞, each containing exactly n critical points, counting multiplicity.

The perturbed family

We now consider the perturbed map

$$\mathcal{S}_{n,d,\lambda}(z) = \frac{z^n(z-a)}{Q(z)} + \frac{\lambda}{z^d},$$

where $\lambda \in \mathbb{C}^*$.
We say that $\mathcal{S}_{n,d,\lambda}$ satisfies (a), (b), and (c) if $S_{n,a,Q}$ satisfies (a), (b), and (c). We also assume that $\mathcal{S}_{n,d,\lambda}$ satisfies

(d) The numbers $n, d \in \mathbb{N}$ are such that $\frac{1}{n} + \frac{1}{d} < 1$.

Dynamical planes of $S_{n,a,Q}$ and $\mathcal{S}_{n,d,\lambda}$, where $\mathcal{S}_{n,d,\lambda} = z^2(z-a) + \frac{\lambda}{z^3}$, for $a = (0.9 + 0.6i)$ and $\lambda = 10^{-7}$

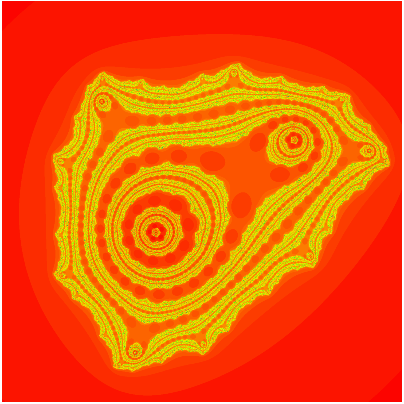


Table of Contents

Skeleton of the dynamical plane

Proposition

Let $|\lambda|$ small enough. Then the following happen:

(i) *The Fatou component $\mathcal{A}_\lambda^*(\infty)$ is mapped onto itself with degree $n+1$. The boundary of $\mathcal{A}_\lambda^*(\infty)$ is a quasicircle that moves continously with respect to λ. The set $\mathcal{A}_\lambda^*(\infty)$ contains exactly n critical points counting multiplicity.*

(ii) *The Fatou component T_λ, which contains $z=0$, is simply connected, and it is mapped with degree d onto $\mathcal{A}_\lambda^*(\infty)$. There are no other preimages of $\mathcal{A}_\lambda^*(\infty)$.*

(iii) *There exists a Fatou component A_λ which is doubly connected and contains exactly $n+d$ simple critical points, given by $c_{\lambda,\xi}$, and $n+d$ zeros, given by $z_{\lambda,\xi}$. Moreover, A_λ is mapped with degree $n+d$ onto T_λ and surrounds the origin.*

Skeleton of the dynamical plane

Proposition

Let $\mathcal{S}_{n,d,\lambda}$ satisfying conditions (a), (b), (c), and (d). Then, there exists a constant $\mathcal{C} = \mathcal{C}(a, Q, n, d)$ such that if $\lambda \neq 0$ and $|\lambda| < \mathcal{C}$ the following statements are satisfied:

- *(i) Let A_{out} be the annulus bounded by $\overline{A_\lambda}$ and $\partial\mathcal{A}^*_\lambda(\infty)$. There exists a Fatou component $D_\lambda \subset A_{out}$ which is simply connected, is mapped with degree 1 onto T_λ, and contains w_λ.*
- *(ii) The critical point ν_λ lies in $A_{out} \setminus D_\lambda$.*
- *(iii) There are no preimages of T_λ other than D_λ and A_λ.*

The free critical point

Theorem

(Riemann-Hurwitz formula) Let $U, V \subset \hat{\mathbb{C}}$ be two connected domains of connectivity $m_U, m_V \in \mathbb{N}^*$ and let $f : U \to V$ be a degree d proper map branched over r critical points, counted with multiplicity. Then,

$$m_U - 2 = d(m_V - 2) + r.$$

Proposition

Assume that $\nu_\lambda \in \mathcal{U}_\nu$, where $\mathcal{U}_\nu$ is a Fatou component which is eventually mapped onto A_λ and surrounds $z = 0$. Then, $\mathcal{U}_\nu$ is triply connected and $\mathcal{U}_\nu \subset A_{out}$.

Conjugacy to Blaschke product

Proposition

Let $\mathcal{S}_{n,d,\lambda}$ satisfying (a), (b), (c), and (d). Let $\lambda \neq 0$, $|\lambda| < \mathcal{C}$. Then, there exist an analytic Jordan curve $\Gamma \subset A_\lambda$ which surrounds $z = 0$, $b \in \mathbb{D}^*$, $\theta \in [0, 1)$, and a quasiconformal map $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\varphi \circ R_{b,\theta} = \varphi \circ \mathcal{S}_{n,d,\lambda}$ on $A(\Gamma, \partial\mathcal{A}^*_\lambda(\infty))$, where

$$R_{b,\theta} = e^{2\pi i\theta} z^n \frac{z - b}{1 - \overline{b}z}$$

is a Blaschke product.

Table of Contents

Theorem A

Theorem A

Let $\mathcal{S}_{n,d,\lambda}$ satisfying (a), (b), (c), and (d) and let $\lambda \neq 0$, $|\lambda| < \mathcal{C}$. Assume also that $\nu_\lambda \in \mathcal{U}_\nu$, where $\mathcal{U}_\nu$ is an iterated preimage of A_λ which surrounds $z = 0$. Let k be the minimal number of iterations needed by the free critical point ν_λ to be mapped into $Bdd(A_\lambda)$. Let U be a Fatou component of connectivity $\kappa > 2$. Then, there exist $i, j, \ell \in \mathbb{N}$ such that $\kappa = (n+1)^i d^j n^\ell + 2$ and $\ell \leq jk$.

Sketch of the proof of Theorem A

Proof.

(i) Iterated preimages of D_λ have connectivity 1. Iterated preimages of A_λ, except for eventual preimages of $\mathcal{U}_\nu$ have connectivity 2.

(ii) Preimages, of a Fatou component U which does not surround $z = 0$, map with degree 1 onto U and do not surround $z = 0$.

(iii) An eventual preimage of $\mathcal{U}_\nu$ which surrounds $z = 0$ may map forward with degree d, n, or $n + 1$.

(iv) A Fatou component can only iterate backwards at most k times with degree n before having to iterate with degree d.

(v) We can iterate backwards with degree d and $n + 1$ as many times as needed.

(vi) Apply Riemann-Hurwitz to obtain the possible connectivities.

□

Theorem B

Theorem B

Let $\mathcal{S}_{n,d,\lambda}$ satisfying (a), (b), (c), and (d) and let $\lambda \neq 0$, $|\lambda| < \mathcal{C}$. Assume also that $\nu_\lambda \in \mathcal{U}_\nu$, where $\mathcal{U}_\nu$ is an iterated preimage of A_λ which surrounds $z = 0$. Let $k \geq 1$ be the minimal number of iterations needed by the free critical point ν_λ to be mapped into $Bdd(A_\lambda)$. For any given $i, j, \ell \in \mathbb{N}$ such that $\ell \leq j(k-1)$, there exists a Fatou component U of connectivity $\kappa = (n+1)^i d^j n^\ell + 2$.

Theorem C

Theorem C

Let $\mathcal{S}_{n,d,\lambda}$ satisfying (a), (b), (c), and (d) and let $\lambda \neq 0$, $|\lambda| < \mathcal{C}$. For any given $i, l \geq 0$ and $j > 0$, there exists a parameter λ such that $\mathcal{S}_{n,d,\lambda}(z)$ has a Fatou component of connectivity $\kappa = (n+1)^i d^j n^\ell + 2$, and a Fatou component of connectivity $\kappa = (n+1)^i + 2$.

Idea of the proof:
As λ approaches $z = 0$, the number of iterations k required by ν_λ to reach $\mathrm{Bdd}(A_\lambda)$ increases. We prove that for all large enough k there exists λ such that $\mathcal{U}_\nu$ is properly defined, $\mathcal{S}^k_{n,d,\lambda}(\mathcal{U}_\nu) \subset \mathrm{Bdd}(A_\lambda)$ and $\mathcal{S}^q_{n,d,\lambda}(\mathcal{U}_\nu) \not\subset \mathrm{Bdd}(A_\lambda)$, for $q < k$. The conclusion follows from Theorem B.

Thank you for your attention!