Applications of harmonic functions in transcendental dynamics

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Eremenko's conjecture (1989):
All the components of $I(f)$ are unbounded.

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- $J(f) = \partial A(f)$
- All the components of $A(f)$ are unbounded.

Useful properties of the maximum modulus

Bergweiler + R + S, 2013

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Lemma (Convexity)

$$\log M(r^c, f^n) \geq c \log M(r, f^n)$$

for all $c > 1$, $n \in \mathbb{N}$, $r \geq R = R(f)$.

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The second follows from a refinement of Eremenko's proof that $I(f) \neq \emptyset$, based on Wiman-Valiron theory.

Lemma (Eremenko points)

Given $\epsilon > 0$,

$$A_r(f) \cap \{z : r \leq |z| \leq r(1+\epsilon)\} \neq \emptyset,$$

for all $r \geq R = R(f, \epsilon)$.

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$$m(\rho, f) \geq M(\rho, f)^{1-\delta}, \text{ for } \rho \in A(r^{a+2\pi\delta}, r^{b-2\pi\delta}).$$

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(b) *f behaves like a monomial in $A(r^{a+2\pi\delta}, r^{b-2\pi\delta})$.*

Proof of $m(\rho, f) \geq M(\rho, f)^{1-\delta}$, for $\rho \in A(r^{a+2\pi\delta}, r^{b-2\pi\delta})$

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and $|\Im t_2 - \Im t_1| \leq \pi$,
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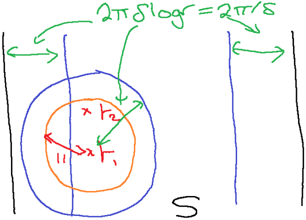

Applications to wandering domains

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Definition

A component U of $F(f)$ is a **wandering domain** if

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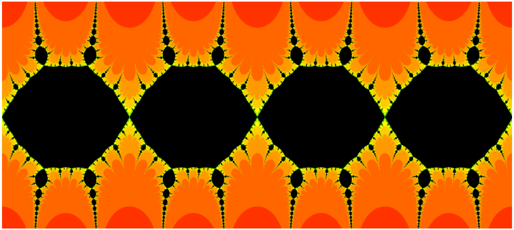

$f(z) = z + \sin z + 2\pi$

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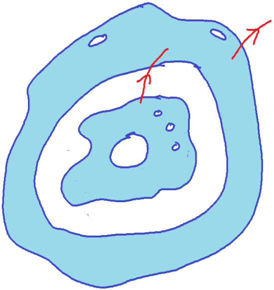

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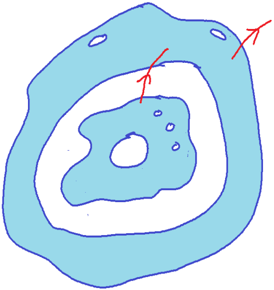

Theorem (Zheng, 2006)

If U is a multiply connected wandering domain then there exist sequences (r_n) and (R_n) such that, for large n,

$$U_n \supset \{z : r_n \leq |z| \leq R_n\}, \text{ where } R_n/r_n \to \infty \quad \text{as } n \to \infty.$$

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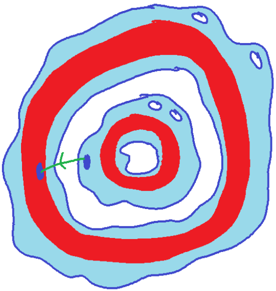

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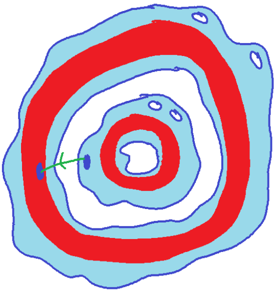

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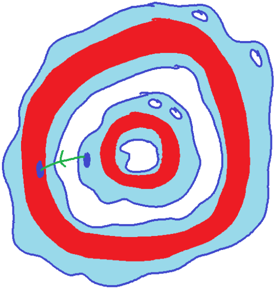

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Proof of Bergweiler + R + S result

Let U be a multiply connected wandering domain, and $z_0 \in U$.

- The functions
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If U is a component of $F(f)$ and $\partial U \cap I(f)$ has positive harmonic measure in U, then $U \subset I(f)$.

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Claim implies

$$\omega(z_0, \bigcup_{n \geq m} A_n, U) \leq \sum_{n \geq m} \omega(z_0, A_n, U) \to 0 \text{ as } m \to \infty$$

and hence $\omega(z_0, \bigcap_{m=1}^{\infty} \bigcup_{n \geq m} A_n, U) = 0$.

Proof of claim that $\sum_{n \geq N} \omega(z_0, A_n, U) < \infty$

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Step 1 By Löwner's Lemma,

$$\omega(z_0, A_n, U) \leq \omega(z_n = f^n(z_0), E_n = f^n(A_n), U_n).$$

Proof of claim that $\sum_{n\geq N}\omega(z_0, A_n, U) < \infty$

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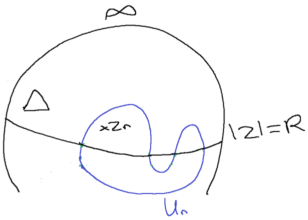
$$\omega(z_0, A_n, U) \leq \omega(z_n = f^n(z_0), E_n = f^n(A_n), U_n).$$

Step 2 Consider the sets

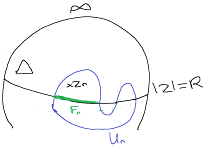
- $\Delta = \{z : |z| > R\} \cup \{\infty\}$
- $E_n = \partial U_n \cap \{z : |z| \leq R\}$
- V_n is component of $U_n \cap \Delta$ containing z_n
- $F_n = \partial V_n \cap \{z : |z| = R\}$



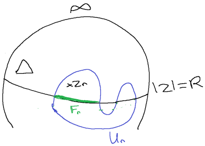
By the extended maximum principle

$$\omega(z, E_n, U_n) \leq \omega(z, F_n, \Delta), \text{ for } z \in V_n.$$

Proof that $\sum_{n \geq N} \omega(z_n, F_n, \Delta) < \infty$

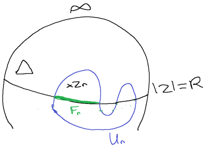


Proof that $\sum_{n \geq N} \omega(z_n, F_n, \Delta) < \infty$



Step 3 Choose N so that $|z_n| = |f^n(z_0)| > 2R$ for $n \geq N$.

Proof that $\sum_{n\geq N} \omega(z_n, F_n, \Delta) < \infty$



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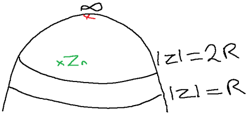
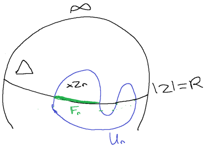
Step 4 By Harnack's Inequality, $\omega(z_n, F_n, \Delta) \leq 3\omega(\infty, F_n, \Delta)$.



Proof that $\sum_{n\geq N} \omega(z_n, F_n, \Delta) < \infty$



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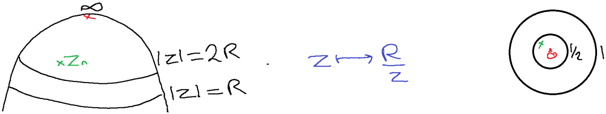

Step 5 Since $F_n \subset U_n$, the sets F_n are distinct and so

$$\sum_{n\geq N} \omega(z_n, F_n, \Delta) \leq 3 \sum_{n\geq N} \omega(\infty, F_n, \Delta) \leq 3.$$