# On the geometry of simply connected wandering domains

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April 19, 2021

We consider the dynamical system given by the iterates of an entire function $f : \mathbb{C} \to \mathbb{C}$ (not linear).

- **Fatou set** $\mathcal{F}_f = \{z \in \mathbb{C} \mid (f^n) \text{ is normal in a nbh of } z\}$
- **Fatou component** is a connected component of $\mathcal{F}_f$
- A Fatou component $\Omega$ is **pre-periodic** if there are non-negative integers $n \neq m$ such that $f^n(\Omega) \cap f^m(\Omega) \neq \emptyset$.
- A Fatou component which is not pre-periodic is called a wandering Fatou component or a **wandering domain**.

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# Classification in terms of internal dynamical behaviour

For entire functions we have a complete description of:

- Pre-periodic Fatou components (Fatou & examples of Siegel and Baker)
- Multiply connected wandering domains (Bergweiler-Rippon-Stallard '13)
- Simply connected wandering domains in terms of the hyperbolic distance between orbits of points and in terms of convergence to the boundary
  - (Benini-Evdoridou-Fagella-Rippon-Stallard '19) *All nine types can be realized by escaping wandering domains.*
  - (Evdoridou-Rippon-Stallard '20) *only six of these types can be realized by oscillating wandering domains.*

# External point of view

- **Escaping** wandering domain - the orbit leaves every compact,
- **Oscillating** wandering domain - there is a subsequence of the orbit that leaves every compact, and another subsequence that is bounded,
- **Dynamically bounded** wandering domains - the orbit is bounded.

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# Geometry of bounded simply connected wandering domains

**Theorem (B.T. 2021)**

*Let $\Omega \subset \mathbb{C}$ be a bounded connected regular open set whose closure has a connected complement. There exists an entire function $f$ for which $\Omega$ is an escaping (oscillating) wandering domain and the iterates $f^n|_\Omega$ are univalent.*

Recall that an open set $U$ is called **regular** if and only if $U = \mathrm{Int}(\overline{U})$ and notice that the conditions of the theorem imply that $\Omega$ is simply connected.

**Corollary**

*Every simply connected Jordan domain is a wandering domain of some entire function.*

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# Necessity of conditions

The condition that $\Omega$ is a regular open set is not only sufficient but also necessary.

- $\Omega$ wandering domain $\Rightarrow \mathcal{F}_f$ is disconnected.
- $f$ is continuous and open map $\Rightarrow f^n(\mathrm{Int}(\overline{\Omega})) \subset \mathrm{Int}(\overline{f^n(\Omega)}) \subset \mathrm{Int}(\overline{U_n})$ for all $n \geq 0$. Here $U_n$ is a Fatou component.
- If $\Omega$ is not regular $\Rightarrow \mathrm{Int}(\overline{\Omega}) \cap \mathcal{J}_f \neq \emptyset \Rightarrow \cup_{n=0}^{\infty} f^n(\mathrm{int}(\overline{\Omega}))$ covers the whole plane with at most one exception.
- This is now a contradiction since

$$\bigcup_{n=0}^{\infty} f^n(\mathrm{int}(\overline{\Omega})) \subset \bigcup_{U \text{ Fat.Comp.}} \mathrm{int}(\overline{U}) \subsetneq \mathbb{C} \setminus \{point\}.$$

If $\Omega$ is regular then $\partial\Omega = \partial(\mathbb{C} \setminus \overline{\Omega})$ hence there exists a sequence of points $(x_n) \subset \mathbb{C} \setminus \overline{\Omega}$ that accumulates everywhere on $\partial\Omega$.

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# Necessity of conditions

The other two conditions in our theorem, namely that $\Omega$ is bounded and that $\mathbb{C}\backslash\overline{\Omega}$ is connected, are needed for the application of the following stronger version of the well-known Runge's Approximation Theorem.

**Theorem**

*Let $K_1, \ldots, K_n \subset \mathbb{C}$ be pairwise disjoint compact sets whose complements $\mathbb{C}\backslash K_j$ are connected. Let $L_k \subset K_k$ be a finite set of points and $h_k : K_k \to \mathbb{C}$ a holomorphic map for every $1 \leq k \leq n$. For every $\epsilon > 0$ there exists an entire function $f$ satisfying:*

- $\|h_k - f\|_{K_k} < \epsilon$
- $f(x) = h_k(x)$ *for all* $x \in L_k$
- $f'(x) = h_k'(x)$ *for all* $x \in L_k$

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This theorem is a combination of several results due to Benhke-Stein 49', Florack 48', Royden 67'. A very similar approximation result was presented by Eremenko-Lyubich 87' (Main Lemma).

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We do not know whether the condition of $\mathbb{C}\backslash\overline{\Omega}$ being connected is also a necessary condition:

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Is the complement of the closure of a bounded simply connected Fatou component always connected?

If the answer is positive, then our result describes all possible geometries of bounded simply connected wandering domains.

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# Proof

Let $R > 0$ and let $(m_n)$ be a strictly increasing sequence of integers such that disks $\Delta(m_n, R)$ are pairwise disjoint and disjoint from $\Omega$.
Let $(x_n) \in \mathbb{C} \backslash \overline{\Omega}$ be a sequence of points which accumulates everywhere on $\partial\Omega$.

The idea is to inductively construct a sequence of entire functions $(f_n)$ that converges uniformly on compacts to the entire function $f$ with the following properties:

1. $f^n(\Omega) \subset \Delta(m_n, R)$, for all $n \geq 1$,
2. $f(\zeta) = \zeta \in \mathbb{C} \backslash \overline{\Omega}$ and $f'(\zeta) = \frac{1}{2}$
3. $f^n(x_n) = \zeta$ as for all $n \geq 1$,
4. $f^n|_{\Omega}$ is univalent for all $n \geq 1$

(1) implies that $\Omega$ is contained in the Fatou set and its orbit leaves every compact.
(2) and (3) imply that the pre-images of an attracting fixed point accumulate everywhere on $\partial\Omega$, hence $\Omega$ is a Fatou component.

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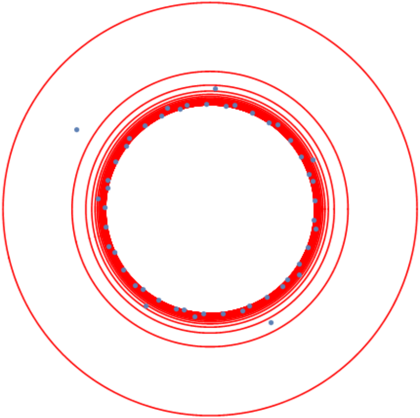
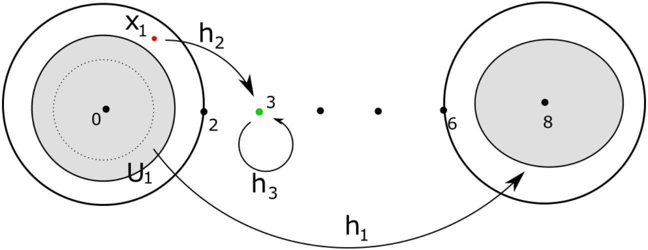
# How to obtain the sequence $(f_n)$ for the unit disk

We define $x_n := \frac{2n+1}{2n} e^{i\sqrt{2}n}$ and $U_n := \overline{\Delta}(0, \frac{2n+2}{2n+1})$.

- $U_{n+1} \subset \operatorname{int}(U_n) \subset \Delta(0,2)$ for all $n \geq 0$,
- $\overline{\Delta}(0,1) = \bigcap_{n\geq 0} U_n$.
- $x_n \in \operatorname{int}(U_{n-1}) \setminus U_n$ for all $n \geq 1$,
- $(x_n)$ accumulates everywhere on $\partial\Omega$.



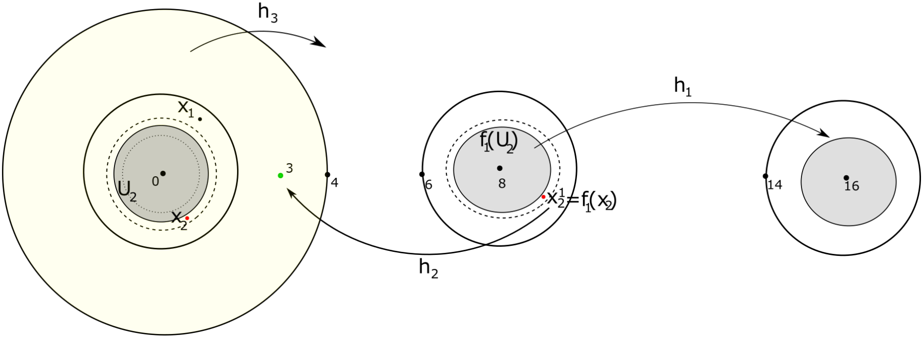
Let $K_1 = U_1$, $K_2 = \{x_1\}$, $K_3 = \{3\}$ and $h_1(z) = z + 8$, $h_2(z) = 3$, $h_3(z) = \frac{1}{2}(z-3)+3$.




By the approximation theorem for every $\epsilon_1 > 0$ there is an entire function $f_1$, such that

- $\|f_1 - h_j\|_{K_j} < \epsilon_1$ for $j = 1, 2, 3$
- $f_1(x_1) = 3$
- $f_1(3) = 3$ and $f_1'(3) = \frac{1}{2}$.

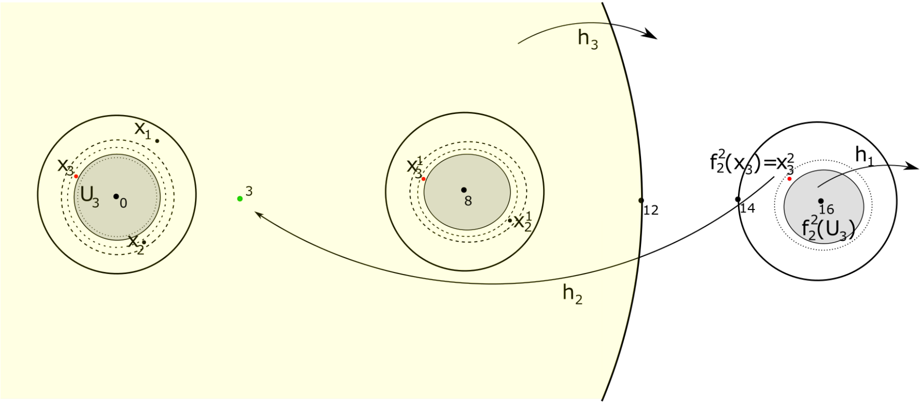
Let $K_1 = f_1(U_2)$, $K_2 = \{x_2^1\}$, $K_3 = \overline{\Delta}(0,4)$ and $h_1(z) = z + 8$, $h_2(z) = 3$, $h_3(z) = f_1(z)$.




Given $\epsilon_2 > 0$ there is an entire function $f_2$, such that

- $\|f_2 - h_j\|_{K_j} < \epsilon_2$ for $j = 1, 2, 3$
- $f_2(x_1) = f_1(x_1)$
- $f_2(x_2) = f_1(x_2) =: x_2^1$
- $f_2(3) = f_1(3)$ and $f_2'(3) = f_1'(3)$.
- $f_2(x_2^1) = 3$

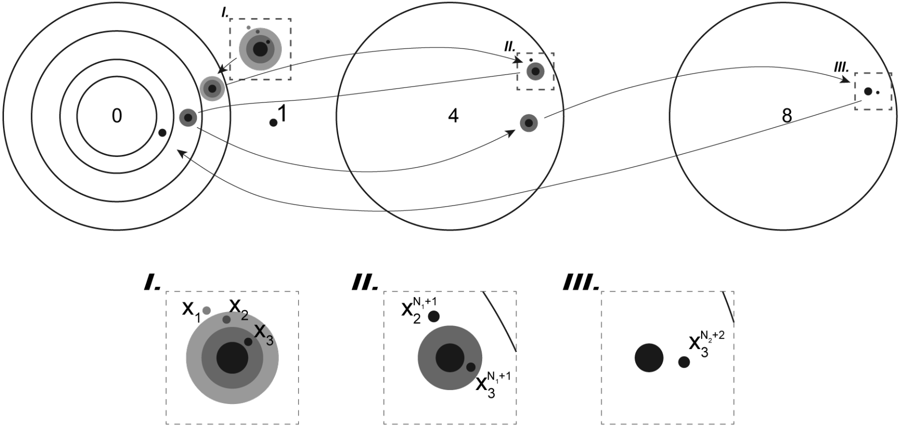
Let $K_1 = f_2^2(U_3)$, $K_2 = \{x_3^2\}$, $K_3 = \overline{\Delta}(0, 12)$ and $h_1(z) = z + 8$, $h_2(z) = 3$, $h_3(z) = f_2(z)$.




Given $\epsilon_3 > 0$ there is an entire function $f_3$, such that

- $\|f_3 - h_j\|_{K_j} < \epsilon_3$ for $j = 1, 2, 3$
- $f_3(x_1) = f_2(x_1)$
- $f_3^j(x_2) = f_2^j(x_2)$ for $j = 1, 2$
- $f_3^j(x_3) = f_2^j(x_3) =: x_3^j$ for $j = 1, 2$
- $f_3(3) = f_2(3)$ and $f_3'(3) = f_2'(3)$.
- $f_3(x_3^2) = 3$

# Oscillating



# Automorphisms of $\mathbb{C}^{m\geq 2}$

The *polynomially-convex hull* of a compact set $K \subset \mathbb{C}^m$ is defined as

$\widehat{K} = \{z \in \mathbb{C}^m : |p(z)| \leq \sup_K |p| \ \text{ for all holomorphic polynomials } p\}$

We say that $K$ is **polynomially convex** if $\widehat{K} = K$.

A compact set $K \subset \mathbb{C}$ is polynomially convex if and only if $\mathbb{C}\backslash K$ is connected.

**Theorem (B.T. 2020)**

*Let $\Omega \subset \mathbb{C}^{m\geq 2}$ be a bounded regular open set whose closure is polynomially convex. There exists an automorphism of $\mathbb{C}^m$ with an escaping (oscillating) wandering domain equal to $\Omega$.*

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# Examples of such domains

- Any bounded convex domain in $\mathbb{C}^n$.
- Topologically non-trivial examples: It is known that any totally real compact manifold $M \subset \mathbb{C}^n$ of dimension $k < n$ can be smoothly perturbed so that its perturbation $M'$ is totally real compact manifold which is polynomially convex, in particular $M'$ has the same topology as $M$. By taking an appropriate tubular neighbourhood of $M'$ we obtain an open set with desired properties.

This shows that there is rich variety of wandering domains which are topologically non-equivalent.

## About the proof

- The idea of the proof is essentially the same, but its complexity increases due to the restrictive nature of automorphisms.
- Trouble: Uniformly convergent sequence of automorphisms may not converge to an automorphism (surjectivity can be lost, e.g Fatou-Bieberbach domains).
- The key ingredient: Approximation result of the Andersén–Lempert theory

**Theorem**

*Let $K_1, K_2, \ldots, K_n$ be pairwise disjoint compact sets in $\mathbb{C}^m$ such that all but one are starshape. Let $F_j \in Aut(\mathbb{C}^m)$ ($j = 1, \ldots, n$) be such that the images $K_j' = F_j(K_j)$ are pairwise disjoint.*
*If the sets $A = \cup_{j=1}^n K_j$ and $B = \cup_{j=1}^n K_j'$ are polynomially convex, then for every $\epsilon > 0$ there exists $G \in Aut(\mathbb{C}^m)$ such that $\|G - F_j\|_{K_j} < \epsilon$ for all $j = 1, \ldots, m$.*
*In particular the automorphism $G$ can be chosen so that its finite order jets agree with the corresponding jets of $F_j$ at any given finite set of points in $K_j$, for $1 \leq j \leq m$.*

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- The key ingredient: Approximation result of the Andersén–Lempert theory

## Theorem

*Let $K_1, K_2, \dots, K_n$ be pairwise disjoint compact sets in $\mathbb{C}^m$ such that all but one are starshape. Let $F_j \in Aut(\mathbb{C}^m)$ ($j = 1, \dots, n$) be such that the images $K_j' = F_j(K_j)$ are pairwise disjoint.*
*If the sets $A = \cup_{j=1}^n K_j$ and $B = \cup_{j=1}^n K_j'$ are polynomially convex, then for every $\epsilon > 0$ there exists $G \in Aut(\mathbb{C}^m)$ such that $\|G - F_j\|_{K_j} < \epsilon$ for all $j = 1, \dots, m$.*
*In particular the automorphism $G$ can be chosen so that its finite order jets agree with the corresponding jets of $F_j$ at any given finite set of points in $K_j$, for $1 \le j \le m$.*

THANK YOU