Comparison of Newton and Ehrlich-Aberth rootfinding dynamical systems: practice and theory

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Introduction

Dynamical rootfinders

- *Finding polynomial roots by dynamical systems – A case study*, DCDS, 2020, Shemyakov et al.
- Natural emergence of iterative dynamical systems in rootfinders.
- We focus on Newton's method and Ehrlich-Aberth's method.
- The rootfinding goal: to find all roots of a polynomial.
- Newton: complex 1-dimensional

$$N_p(z) = z - \frac{p}{p'}(z).$$

- Ehrlich-Aberth: d-dimensional, updates in each coordinate depend on all other coordinates.

Theory and practice

- Theoretical results for Newton's: [Hubbard, Schleicher, Sutherland, 2001] and others. Not so many practical results.
- Theoretical results for Ehrlich-Aberth: local theory, recent ideas about general convergence [Reinke, 2020]. It's a non-trivial multi-dimensional dynamical system.
- Goal of the talk: experiments to compare the performance of Newton and Ehrlich-Aberth, try to explain results from practical and theoretical viewpoint.

Setup

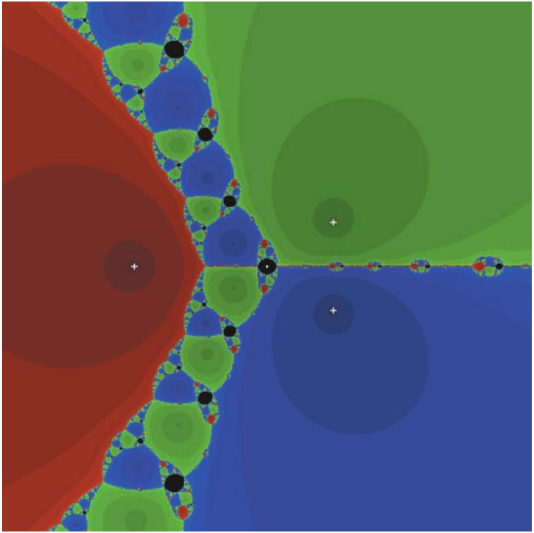
Newton's method: theory

- Newton update function:

$$N_p(z) = z - \frac{p}{p'}(z).$$

- Electrostatic interpretation: positive charges at the roots, negative charge at the iterate.
- Local behavior: quadratic convergence at simple roots (linear for higher order roots).
- The real challenge to find all roots: starting points.
- Attracting cycles.

Global dynamics and attracting cycles



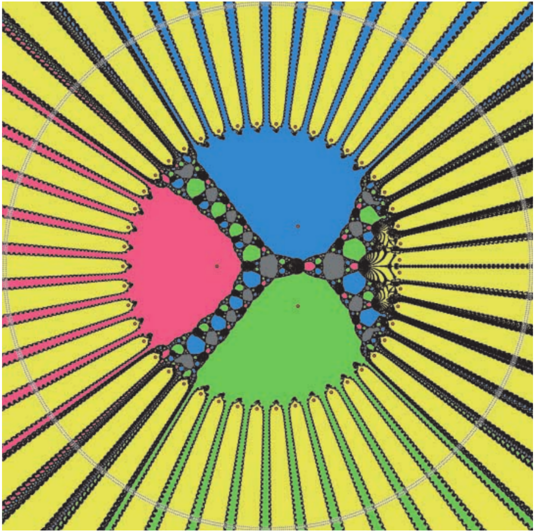
Global dynamics for Newton

- Infinity is a repelling point, zeros of p are (super-)attracting points.
- Poles at critical points of p.
- Basins of attraction and immideate basins. Fatou and Julia sets.
- Channels to infinity.

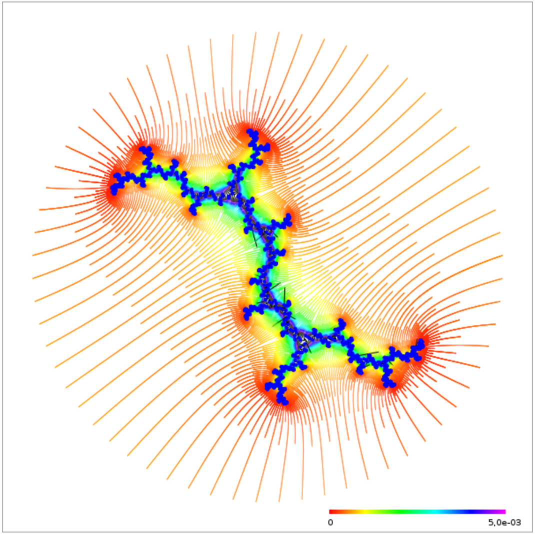
Newton's method: practice

- Channels help us to choose good starting points.
- Practical problem: we iterate too many points without interesting outcome.
- Idea: iterated refinement. Start with few points and add new orbits when something interesting happens.
- With this idea Robin Stoll efficently computed roots of polynomial of degree up to 1 million, Marvin Randig up to 1 billion (2017).

Starting points: how to hit all channels



Iterated refinement



Ehrlich-Aberth's method: theory

- Ehrlich-Aberth update function:

$$EA^{(i)}(\bar{z}) = z^{(i)} - \frac{\frac{p}{p'}(z^{(i)})}{1 - \frac{p}{p'}(z^{(i)}) \sum_{j \neq i} \frac{1}{z^{(i)} - z^{(j)}}}$$

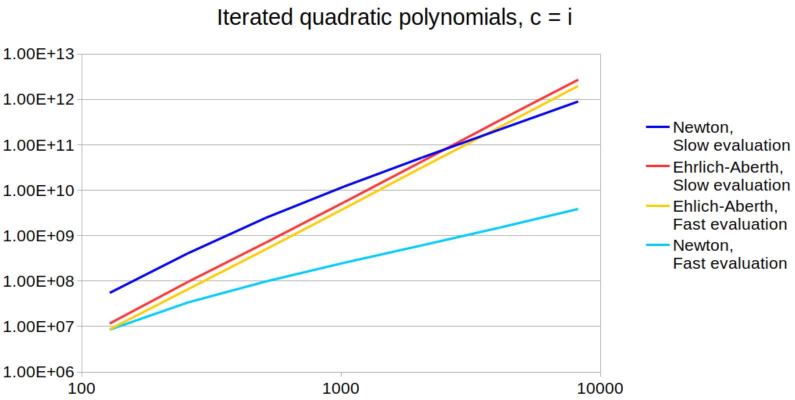
- Electrostatic interpretation: positive charges at the roots, negative at iterates, iterates see each other.
- Local convergence: superlinear for simple zeros and linear for multiple zeros.
- Attracting cycles for Ehrlich-Aberth?

The experiments

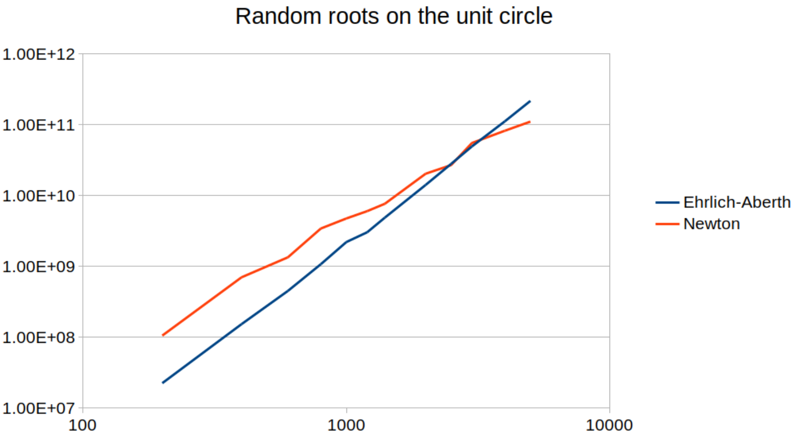
Experimental setup

- Polynomials considered: iterated quadratic (with roots on Julia set for $z^2 + i$), Chebyshev and Legendre (with real roots), with geometrical root configuration: on a circle, semicircle, on a perturbed circle, in a disk, on a grid, in clusters, on a segment.
- Fast and slow evaluation of polynomials, set of variable parameters. Evaluated total number of operations performed.
- Compare the total number of operations for Newton (iterated refinement) and Ehrlich-Aberth.

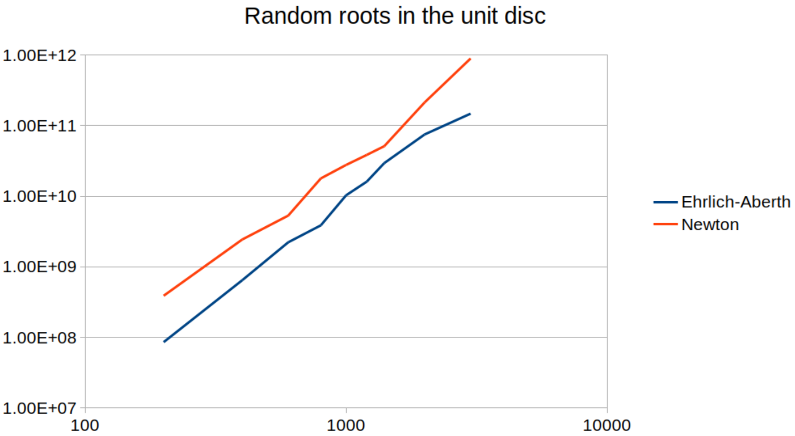
Results for iterated quadratic



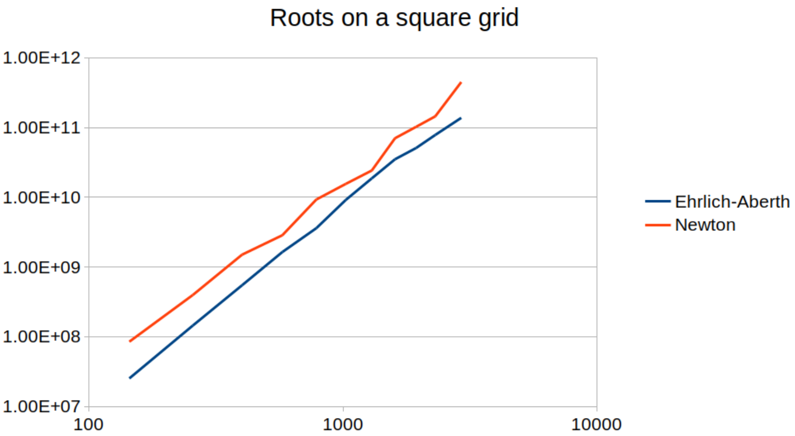
Results for roots on the circle



Results for roots in the disk



Results for roots on a grid

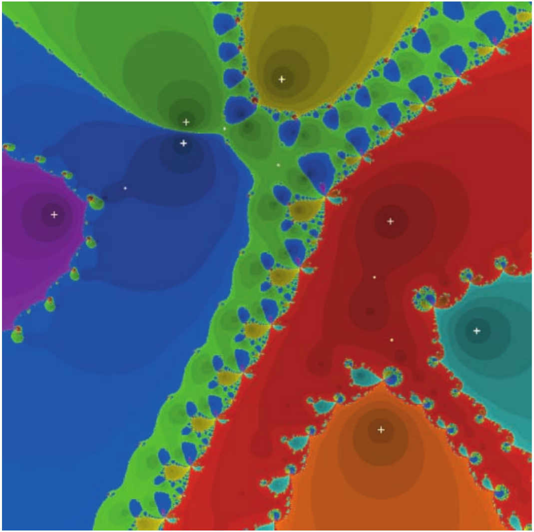


Practical observations

- Logarithmic evaluation: Newton wins.
- Conjecture is that Newton works better if roots are well-exposed to the infinity. Otherwise Ehrlich-Aberth seems to be more efficient.
- Newton searches roots from the outside, iterations move slower when inside the convex hull of roots.
- Ehrlich-Aberth searches all the roots simultaniously, there are less orbits but they require more time to compute.

Theoretical viewpoint

Immideate basisn and channels



Channels and efficiency for Newton

Definition (Width of a channel)

Let W be a channel, far enough from the origin we factorize W by the dynamics. We get a Riemann surface homeomorphic to an annulus, the modulus of this annulus is the width of W.

Proposition (Estimate for width from below)

Each root α has a channel with width at least $\pi/\log m$, where m is the number of critical points of N_p in the immediate basin of α.

- Wider channels are easier to hit with starting points.

The case of real roots

Proposition (Critical points control channels)

If some immediate basin has m critical points of N_p, then it has m different channels.

Proposition (Channels for real roots)

Let p be a polynomial with simple real roots. Then each root has a channel with width at least $\pi/\log 3$.

- Observe that N_p has $2d-2$ critical points.
- Each root except the smallest and the largest has at least two channels due to symmetry.
- There are only two channels left to the smalles and the largest roots.
- Every root has a channel with width at least $\pi/\log 3$.

Conjecture for the outer roots

Conjecture

Let α be a root of p such that it lies on the boundary of convex hull of roots. Then α has a channel with width at least $\pi / \log 3$ (a main channel).

- Compare with the Proposition for real roots.
- Theoretical explanation for why Newton finds ∞-exposed roots better.
- Improves theoretical estimates by $\log d$.

Conclusion

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- Two viewpoints on why Newton is especially efficient in finding ∞-exposed roots.
- Good example how theory and practice interplay and support each other.
- More experiments are needed. Several potential optimization can be implemented.

Thanks for the attention!