ENTIRE MAPS WITH CANTOR BOUQUET JULIA SETS

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Let $f : \mathbb{C} \to \mathbb{C}$ be an entire map.

$$f^n := f \circ \overset{n}{\cdots} \circ f$$

- **Fatou set**: set of stability. $F(f)$.

small perturbations $\rightsquigarrow$ small perturbations.

- **Julia set**: locus of chaotic behaviour. $J(f) = \mathbb{C} \setminus F(f)$.
- **escaping set**: points that escape to infinity under iteration:

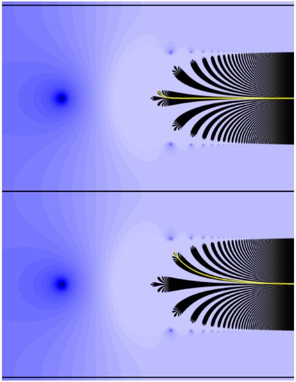
$$I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty\}.$$

In particular,

$$\boxed{J(f) = \partial I(f)}.$$

CURVES IN THE ESCAPING SET

- Fatou observed in 1926 that the escaping sets of certain functions in the sine family contain arcs to infinity.
- In the eighties, Devaney, with several co-authors, found many such curves for maps in the exponential family $f_\lambda(z) = \lambda e^z$.

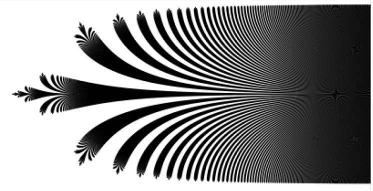


- These curves are known as (Devaney) hairs or dynamic rays.

Definition

$J(f)$ is a **Cantor bouquet** if

- Every conn. comp. of $J(f)$ is an arc to infinity, called hair;
- $J(f)$ is topologically straight, i.e., there is a homeo. $\varphi \colon \mathbb{C} \to \mathbb{C}$ such that the image of every hair is a straight horizontal line.



Theorem (Aarts–Oversteegen, '93)

The Julia set of any $\lambda \sin(z)$ with $\lambda \in (0,1)$ and $\mu \exp(z)$ with $\mu \in (1, 1/e)$ is a Cantor bouquet.

Definition (Benini-Rempe '20)

An entire function f is **criniferous** if for every $z \in I(f)$ and for all sufficiently large n, there is an arc γ_n connecting $f^n(z)$ to ∞, such that

- f maps γ_n injectively onto γ_{n+1};
- $\min_{z\in\gamma_n} |z| \to \infty$ as $n \to \infty$.

Remark: If f is criniferous, then every $z \in I(f)$ can be connected to infinity by a curve of escaping points.

The set of **singular values** $S(f)$ is the smallest closed subset of $\mathbb{C}$ such that $f\colon \mathbb{C} \setminus f^{-1}(S(f)) \to \mathbb{C} \setminus S(f)$ is a covering map.

$$S(f) = \overline{\{ \text{ asymptotic and critical values of } f \, \}}.$$

⋆ **Eremenko-Lyubich class**:

$$\mathcal{B} := \{f\colon \mathbb{C} \to \mathbb{C} \text{ transcendental entire} : S(f) \text{ is bounded}\}.$$

Remark: If $f \in \mathcal{B}$, then $I(f) \subset J(f)$.

Examples of criniferous functions:

- **Exponential family**, [Schleicher-Zimmer '03],

$$f_\lambda(z) = \lambda e^z.$$

- **Cosine family**, [Rottenfußer-Schleicher '08],

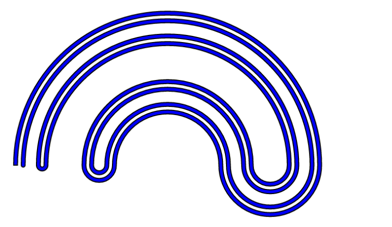
$$f_{a,b}(z) = ae^z + be^{-z}.$$

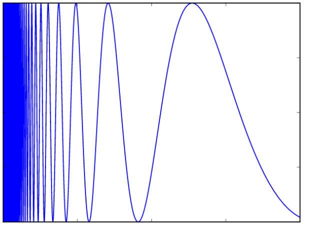
- Functions of **finite order in class $\mathcal{B}$**.
  [Barański '07], [Ruckert-Rottenfußer-Rempe-Schleicher '11].
  * *f has finite order of growth* if $\log\log|f(z)| = O(\log|z|)$.

- $\mathcal{B}_{RRRS}$: functions in $\mathcal{B}$ that satisfy a **uniform head-start condition**.
  [RRRS]. UHSC

However, **not** all functions in $\mathcal{B}$ are criniferous:

- There is $f \in \mathcal{B}$ such that $J(f)$, and hence $I(f)$, contains no arc [RRRS].
- Different arc-like continua in $J(f) \cup \{\infty\}$ [Rempe '16].





- Alternative to rays: *dreadlocks* [Benini-Rempe '20].

Definition

An entire function f is of **disjoint type** if $f \in \mathcal{B}$ and every point in $S(f)$ tends to an attracting fixed point of f under iteration.

Proposition

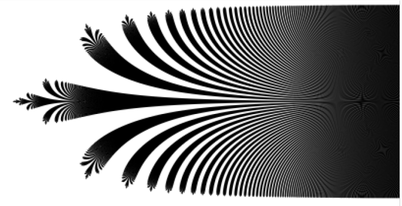
If $f \in \mathcal{B}$, then λf is of disjoint type for $|\lambda|$ sufficiently small.

- ⋆ λf is in the *parameter space* of f.
- ⋆ The dynamics of λf and f are related near infinity by some analogue of Böttcher's Theorem. [Rempe '09] Conjugacy

⋆ Aarts and Oversteegen's result generalizes to *some* disjoint type functions:

Theorem (Barański-Jarque-Rempe '12)

If $f \in \mathcal{B}$ is of finite order and of disjoint type, then $J(f)$ is a *Cantor bouquet.*



CRINIFEROUS VS CANTOR BOUQUET

We want to understand the relation between *criniferousness* and Cantor bouquets Julia sets...

$\star$ If f is disjoint type, then

$$J(f) \text{ Cantor Bouquet} \implies f \text{ criniferous.}$$

Question: f criniferous $\implies J(f)$ Cantor Bouquet?

Theorem A (P.- Rempe)

There is $f \in \mathcal{B}$ criniferous and of disjoint type such that $J(f)$ is <u>not</u> a Cantor bouquet.

Theorem

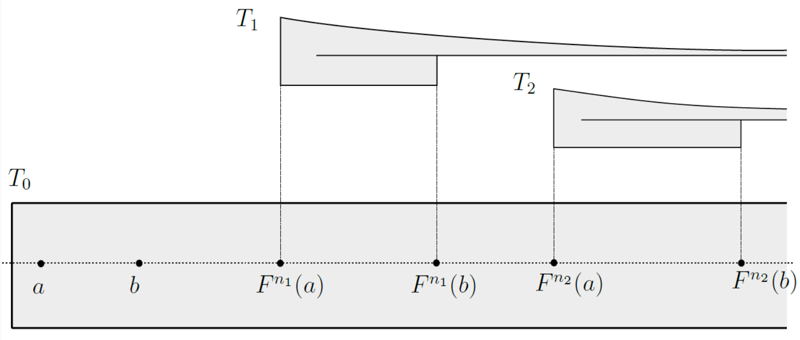
A set $X \subset \mathbb{C}$ is a *Cantor Bouquet* if and only if the following conditions are satisfied:

1. X is **closed**.
2. Every **connected component** of X is an **arc** connecting a finite endpoint to infinity.
3. For any sequence y_n converging to a point y, the arcs $[y_n, \infty)$ **converge** to $[y, \infty)$ in the Hausdorff metric.
4. The **endpoints** of X are **dense** in X.
5. If $x \in X$ is accessible from $\mathbb{C} \setminus X$, then x is an endpoint of X. (Equivalently, **every hair** of X is **accumulated** on **by other hairs** from both sides.)

If f is a disjoint type and criniferous, then

1. $J(f)$ is closed.
2. Every connected component of $J(f)$ is an arc connecting a finite endpoint to infinity.
3. For any sequence $y_n \to y$, the arcs $[y_n, \infty)$ converge to $[y, \infty)$ in the Hausdorff metric. ??
4. The endpoints of $J(f)$ are dense in $J(f)$.
5. Every hair of $J(f)$ is accumulated on by other hairs from both sides.

-We construct $f \in \mathcal{B}$ with "hooked hairs".



Definition

We say that a subset $A \subset J(f)$ is **absorbing** if it is forward-invariant, every escaping point eventually enters A; i.e.

$$I(f) \subset \bigcup_{n=0}^{\infty} f^{-n}(A),$$

and if $\gamma \subset A$ is an arc to infinity, so is $f(\gamma)$.

Theorem B (P.-Rempe)

Let $f \in \mathcal{B}$ be of disjoint type. The following are equivalent.

(a) $J(f)$ is a Cantor bouquet.

(b) There is $R > 0$ and a Cantor bouquet $X \subset J(f)$ such that

$$X \supset J_R(f) := \{z \in J(f) : |f^n(z)| \geq R \text{ for all } n \geq 1\}.$$

(c) There is an absorbing Cantor bouquet $X \subset J(f)$.

Definition

We say that $f \in \mathcal{B}$ belongs to the **class $\mathcal{CB}$** if $J(\lambda f)$ is a Cantor bouquet for $|\lambda|$ sufficiently small.

Remark: $\mathcal{B}_{RRRS} \subset \mathcal{CB}$.

Theorem C (P. '19)

All functions in $\mathcal{CB}$ are criniferous.

Theorem D (P.-Rempe)

Let $f \in \mathcal{B}$. Then $f \in \mathcal{CB}$ if and only if $J(f)$ contains an absorbing Cantor bouquet. Moreover, $\mathcal{B}_{RRRS} \subsetneq \mathcal{CB}$.

THANKS FOR YOUR ATTENTION!

$$J_R(f) := \{z \in J(f) : |f^n(z)| \geq R \text{ for all } n \geq 1\}.$$

Theorem (Rempe '09)

Let $f \in \mathcal{B}$ and let $g := \lambda f$ be of disjoint type. Then there exist a constant $R > 0$ and a continuous map $\vartheta \colon J_R(g) \to J(f)$ such that

$$\vartheta \circ g = f \circ \vartheta$$

and is a homeomorphism onto its image. Moreover,

$$J_{e^2 R}(f) \subset \vartheta(J_R(g)).$$

Remark: The map ϑ extends to a quasiconformal map $\vartheta : \mathbb{C} \to \mathbb{C}$.

CONJUGACY NEAR INFINITY

Disjoint type

*Pictures by Rempe.

Definition (Uniform head-start condition)

Let $f \in \mathcal{B}$. We say that f satisfies a **uniform head-start condition (with respect to $|z|$) on its Julia set** if there is an upper semicontinuous function $\varphi\colon [0,\infty) \to [0,\infty)$ with the following properties for all points z and w belonging to the same component of $J(f)$.

(i) If $|w| > \varphi(|z|)$, then $|f(w)| > \varphi(|f(z)|)$.

(ii) If $z \neq w$, then there is $n \geq 0$ such that either $|f^n(w)| > \varphi(|f^n(z)|)$ or $|f^n(z)| > \varphi(|f^n(w)|)$.

Note that the conditions imply, in particular, that $\varphi(t) > t$ for all t.

Criniferous