Dynamics of Generalized Tangent maps

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Asymptotic values and tracts

1. A value $v \in \hat{\mathbb{C}}$ an **asymptotic value** of a holomorphic function f if there is a path $\gamma : [0, 1)$ such that

$$\lim_{t \to 1^-} r(t) = \infty \text{ and } \lim_{t \to 1^-} f(\gamma(t)) = v.$$

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2. Let v be asymptotic value of f. If there exists neighborhood V of v and an unbounded simply connected set U such that $f : U \to V \setminus \{v\}$ is a universal covering, then U is an asymptotic tract of v.

Examples

1. The most well-known family of exponential maps

$$f_k(z) = e^z + k \ \text{ or } f_\lambda(z) = \lambda e^z$$

Two asymptotic values are k and ∞ (0 and ∞) and their asymptotic tracts are left and right half plane respectively.

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2. The most well-known family of meromorphic maps
$$\lambda \tan z.$$
Two asymptotic values are $\pm\lambda i$ and their asymptotic tracts are upper and lower half plane respectively

Bifurcation of Exponential Maps

Theorem (Rempe–Schleicher, 09)

1. *The bifurcation locus is connected*
2. *Each hyperbolic component is unbounded and its boundary is a unbounded Jordan arc tending to ∞ in both directions.*

L. Rempe and D. Schleicher
Bifurcations in the Space of Exponential Maps
Invent. Math. 175 (2009), No. 1, 103 - 135

Bifurcation of Exponential Maps

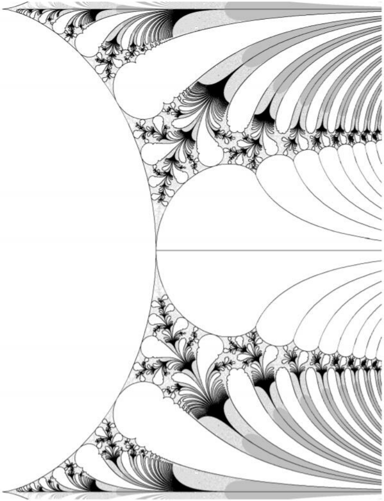


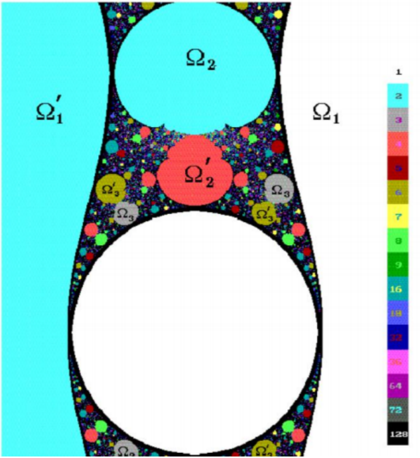
Figure: Parameter space of $e^z + k$ by L. Rempe and D. Schleicher

Parameter Space of $\lambda \tan z$

Any map $f_\lambda = \lambda \tan z$ is an odd function. If $\{z_1, z_2, \cdots, z_k\}$ is a cycle, then $\{-z_1, -z_2, \cdots, -z_k\}$ is also a cycle with the same multiplier.

Let $\Omega_k = \{\lambda : f_\lambda \text{ has two attracting cycles of period } k\}$ and $\Omega'_k = \{\lambda : f_\lambda \text{ has one attracting cycles of period } 2k\}$.

Parameter Space of $\lambda \tan z$



$\lambda = i\pi/2$ satisfies $f_\lambda(i\lambda) = \infty$.

Parameter Space of $\lambda \tan z$

Theorem (Keen-Kotus, 97)

1. *Ω_1 and Ω_1' are unbounded, and other components are bounded*
2. *There is a pair of Ω_k and Ω_k' meeting at each solution of $f_\lambda^{k-1}(\lambda i) = \infty$.*

L. Keen and J. Kotus
Dynamics of the family $\lambda \tan z$
Conform. Geom. Dyn. 1 (1997), 28-57

Meromorphic functions of finite type

1. Fagella and Keen considered M_∞, a family of meromorphic maps of finite type for which ∞ is not an asymptotic value.
2. **Dynamically natural slice**: the dynamics of singular values is fixed except one asymptotic value, denoted by v_λ.
3. A component in the parameter space of which the free asymptotic value v_λ is attracted by a new attracting cycle is called a **shell component**. Let Ω_k denote components where the period of the attracting cycle is k.

Shell components

Theorem (Fagella–Keen 2020)

1. *Each shell component is simply connected.*
2. *Each component in Ω_1 is unbounded.*
3. *On the boundary any shell component, there exists a λ, such that $f_\lambda^k(v_\lambda) = \infty$, called* **virtual cycle parameter**.

N. Fagella and L. Keen
Stable components in the parameter plane of transcendental functions of finite type.
The Journal of Geometric Analysis (2020), 1-40.

Shell components

Theorem (Fagella–Keen, 2020)

1. *Each shell component is simply connected.*
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Question/Conjecture

1. Conjecture (Fagella-Keen, 2020): Each component of $\Omega_k, k \geq 2$ is bounded.
2. For each virtual cycle parameter λ, is it on the boundary of a shell component?

Generalized Nevanlinna functions

Consider the family $\mathcal{N}_{p,q,r}$ consisting of $f = P \circ g \circ Q$, where P, Q are polynomials of degree p, q respectively, and $g \in \mathcal{N}_r$ is a meromorphic function with no critical values and r asymptotic values counted with multiplicity of asymptotic tracts. They can be characterized as

$$S_g(z) = a_{r-2}z^{r-2} + \cdots + a_0.$$

Note that $f(z) = e^{e^z}$ has 3 asymptotic values $\{0, 1, \infty\}$. However, $f \notin \mathcal{N}_3$ since both ∞ and 0 have infinitely many asymptotic tracts.

Virtual Cycle Parameters are on the boundary

Theorem (C.-Keen, 2019)

For any dynamical natural slice of $\mathcal{N}_{p,q,r}$, at every virtual cycle parameter λ, there exists a shell component Ω, such that $\lambda \in \partial\Omega$.

Generalized Tangent maps

The family $\mathcal{F}_\lambda = \{f_\lambda(z) = \lambda \tan^p z^q, p, q \in N, pq > 1,\ \lambda \neq 0\}$.

- Critical points: solutions of $z^{q-1} \tan^{p-1} z^q = 0$
- Critical Value: 0. Thus it is super-attracting.
- attracting basin: $A(0) = \{z :\ f_\lambda^n(z) \to 0 \text{ as } n \to \infty\}$
- immediate basin: $A^*(0)$ the component of $A(0)$ containing 0.
- Asymptotic value: $(\pm i)^p \lambda$
- Denote $i^p \lambda$ by v_λ

Dichotomy

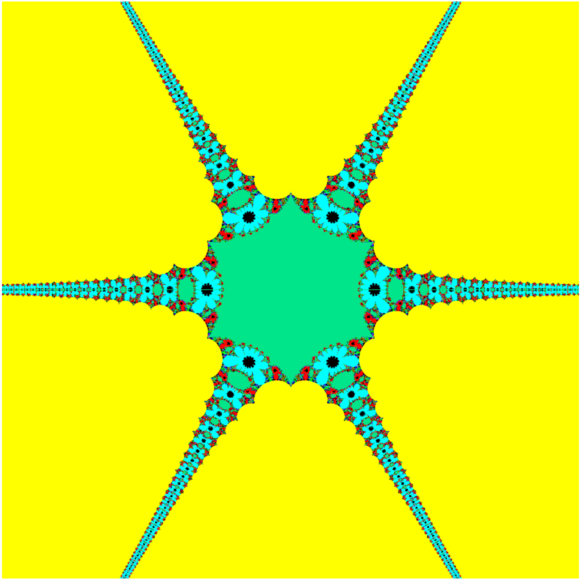
Theorem

The Julia set of f_λ is connected if and only if $v_\lambda \in A^*(0)$.

Capture and Shell Components

1. Capture components $\mathcal{C}$ consist of λ such that $f_\lambda^n(v_\lambda) \to 0$.
   $\mathcal{C}_k = \{\lambda : k \text{ is the smallest integer such that } f_\lambda^k(v_\lambda) \in A^*(0)\}$
2. Shell components $\mathcal{S}$ consist of λ such that v_λ is attracted to a nonzero attracting cycle.
   - pq is even, $\mathcal{S}_k$ consists of λ such that f_λ has an attracting cycle of period k
   - pq is odd, $\mathcal{S}_k$ consists of λ such that f_λ has an attracting cycle of period $2k$ or two attracting cycles of period k.

Parameter space of $\lambda \tan^2 z^3$



Enumerate Capture Components

Theorem (C.-Keen)

1. *$\mathcal{C}_0 \cup \{0\}$ is simply connected.*
2. *Each component of $\mathcal{C}_k$ for $k \geq 1$ is simply connected and contains a unique solution of*
$$f_\lambda^k(v_\lambda) = 0.$$

P. Roesch
Hyperbolic components of polynomials with a fixed critical point of maximal order.
Annales Scientifiques de L'Ecole Normale Supérieure, Vol 40(6), 2007, 901-949.

N. Fagella and A. Garijo
The parameter planes of $\lambda z^m e^z$, $m \geq 2$.
Communications in mathematical physics 273 (3), 2007, 755-783.

Central Capture Component

Theorem (C.-Keen)

1. *$D(0, r_1) \subset \mathcal{C}_0 \cup \{0\}$, where $r_1 = \sqrt[q]{\frac{\pi}{4}}$.*
2. *There exists a covering: $\phi : \mathcal{C}_0 \to \mathbb{D}^*$. Any periodic rational parameter ray lands at a parabolic parameter λ (That is, f_λ has a parabolic periodic point.) or a Misiurewicz point λ (That is, v_λ lands at a period cycle of f_λ.)*

Shell Components

Theorem (C.-Keen, 2019)

1. *The set $\mathcal{S}_1$ consists of $2q$ unbounded simply connected components.*
2. *For $k \geq 2$, at each solution of $f_\lambda^{k-1}(v_\lambda) = \infty$, there are $2pq$ components of $\mathcal{S}_k$.*
3. *All components of $\mathcal{C}_k$, $k \geq 0$, and $\mathcal{S}_k$, $k \geq 2$, are bounded.*

Future work

1. meromorphic functions with two asymptotic values.
2. Meromorphic functions with more critical values or asymptotic values

Thank you for your attention!