Dynamics of Generalized Tangent maps

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A value $v \in \hat{\mathbb{C}}$ an asymptotic value of a holomorphic function $f$ if there is a path $\gamma : [0, 1)$ such that

$$\lim_{t \to 1^-} r(t) = \infty \quad \text{and} \quad \lim_{t \to 1^-} f(\gamma(t)) = v.$$
A value $\nu \in \hat{\mathbb{C}}$ an **asymptotic value** of a holomorphic function $f$ if there is a path $\gamma : [0, 1)$ such that

$$\lim_{t \to 1^-} r(t) = \infty \text{ and } \lim_{t \to 1^-} f(\gamma(t)) = \nu.$$ 

Let $\nu$ be asymptotic value of $f$. If there exists neighborhood $V$ of $\nu$ and an unbounded simply connected set $U$ such that $f : U \to V \setminus \{\nu\}$ is a universal covering, then $U$ is an asymptotic tract of $\nu$. 
The most well-known family of exponential maps

\[ f_k(z) = e^z + k \text{ or } f_\lambda(z) = \lambda e^z \]

Two asymptotic values are \( k \) and \( \infty \) (0 and \( \infty \)) and their asymptotic tracts are left and right half plane respectively.
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The most well-known family of meromorphic maps

\[ \lambda \tan z. \]

Two asymptotic values are \( \pm \lambda i \) and their asymptotic tracts are upper and lower half plane respectively.
Theorem (Rempe–Schleicher, 09)

1. The bifurcation locus is connected

2. Each hyperbolic component is unbounded and its boundary is a unbounded Jordan arc tending to $\infty$ in both directions.

L. Rempe and D. Schleicher

Bifurcations in the Space of Exponential Maps

Invent. Math. 175 (2009), No. 1, 103 - 135
Figure: Parameter space of $e^z + k$ by L. Rempe and D. Schleicher
Any map $f_\lambda = \lambda \tan z$ is an odd function. If $\{z_1, z_2, \cdots, z_k\}$ is a cycle, then $\{-z_1, -z_2, \cdots, -z_k\}$ is also a cycle with the same multiplier.

Let $\Omega_k = \{\lambda : f_\lambda \text{ has two attracting cycles of period } k\}$ and $\Omega'_k = \{\lambda : f_\lambda \text{ has one attracting cycles of period } 2k\}$. 
Parameter Space of $\lambda \tan z$

$\lambda = i\pi/2$ satisfies $f_\lambda(i\lambda) = \infty$. 
Parameter Space of $\lambda \tan z$

**Theorem (Keen-Kotus, 97)**

1. $\Omega_1$ and $\Omega'_1$ are unbounded, and other components are bounded
2. There is a pair of $\Omega_k$ and $\Omega'_k$ meeting at each solution of $f^{k-1}_\lambda(\lambda i) = \infty$.

L. Keen and J. Kotus

Dynamics of the family $\lambda \tan z$

Fagella and Keen considered $M_\infty$, a family of meromorphic maps of finite type for which $\infty$ is not an asymptotic value.

**Dynamically natural slice**: the dynamics of singular values is fixed except one asymptotic value, denoted by $v_\lambda$.

A component in the parameter space of which the free asymptotic value $v_\lambda$ is attracted by a new attracting cycle is called a **shell component**. Let $\Omega_k$ denote components where the period of the attracting cycle is $k$. 
Theorem (Fagella–Keen 2020)

1. Each shell component is simply connected.
2. Each component in $\Omega_1$ is unbounded.
3. On the boundary any shell component, there exists a $\lambda$, such that $f^k_\lambda(v_\lambda) = \infty$, called virtual cycle parameter.

N. Fagella and L. Keen
Stable components in the parameter plane of transcendental functions of finite type.
Theorem (Fagella–Keen, 2020)

1. Each shell component is simply connected.
2. Each component in $\Omega_1$ is unbounded.
3. On the boundary of any shell component, there exists a virtual cycle parameter, that is a $\lambda$, such that $f^k_\lambda(v_\lambda) = \infty$. 

Question/Conjecture

1. Conjecture (Fagella-Keen, 2020): Each component of $\Omega_k$, $k \geq 2$ is bounded.
2. For each virtual cycle parameter $\lambda$, is it on the boundary of a shell component?
Theorem (Fagella–Keen, 2020)

1. Each shell component is simply connected.
2. Each component in $\Omega_1$ is unbounded.
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Question/Conjecture

1. Conjecture (Fagella-Keen, 2020): Each component of $\Omega_k$, $k \geq 2$ is bounded.
2. For each virtual cycle parameter $\lambda$, is it on the boundary of a shell component?
Consider the family $\mathcal{N}_{p,q,r}$ consisting of $f = P \circ g \circ Q$, where $P$, $Q$ are polynomials of degree $p$, $q$ respectively, and $g \in \mathcal{N}_r$ is a meromorphic function with no critical values and $r$ asymptotic values counted with multiplicity of asymptotic tracts. They can be characterized as

$$S_g(z) = a_{r-2}z^{r-2} + \cdots + a_0.$$ 

Note that $f(z) = e^{ez}$ has 3 asymptotic values $\{0, 1, \infty\}$. However, $f \notin \mathcal{N}_3$ since both $\infty$ and 0 have infinitely many asymptotic tracts.
Virtual Cycle Parameters are on the boundary

Theorem (C.-Keen, 2019)

For any dynamical natural slice of $\mathcal{N}_{p,q,r}$, at every virtual cycle parameter $\lambda$, there exists a shell component $\Omega$, such that $\lambda \in \partial \Omega$. 
The family $\mathcal{F}_\lambda = \{f_\lambda(z) = \lambda \tan^p z^q, \ p, q \in \mathbb{N}, pq > 1, \ \lambda \neq 0\}$.

- Critical points: solutions of $z^{q-1} \tan^{p-1} z^q = 0$
- Critical Value: 0. Thus it is super-attracting.
- Attracting basin: $A(0) = \{z : f_\lambda^n(z) \to 0 \text{ as } n \to \infty\}$
- Immediate basin: $A^*(0)$ the component of $A(0)$ containing 0.
- Asymptotic value: $(\pm i)^p \lambda$
- Denote $i^p \lambda$ by $v_\lambda$
Theorem

The Julia set of $f_\lambda$ is connected if and only if $\nu_\lambda \in A^*(0)$. 
Capture and Shell Components

1. Capture components $C$ consist of $\lambda$ such that $f_{\lambda}^n(v_{\lambda}) \to 0$. 
   \[ C_k = \{ \lambda : k \text{ is the smallest integer such that } f_{\lambda}^k(v_{\lambda}) \in A^*(0) \} \]

2. Shell components $S$ consist of $\lambda$ such that $v_{\lambda}$ is attracted to a nonzero attracting cycle.
   - $pq$ is even, $S_k$ consists of $\lambda$ such that $f_{\lambda}$ has an attracting cycle of period $k$.
   - $pq$ is odd, $S_k$ consists of $\lambda$ such that $f_{\lambda}$ has an attracting cycle of period $2k$ or two attracting cycles of period $k$. 
Parameter space of $\lambda \tan^2 z^3$
Theorem (C.-Keen)

1. $C_0 \cup \{0\}$ is simply connected.
2. Each component of $C_k$ for $k \geq 1$ is simply connected and contains a unique solution of

$$f^k_\lambda(v_\lambda) = 0.$$ 

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P. Roesch

Hyperbolic components of polynomials with a fixed critical point of maximal order.

N. Fagella and A. Garijo

The parameter planes of $\lambda z^m e^z$, $m \geq 2$.
Theorem (C.-Keen)

1. $D(0, r_1) \subset C_0 \cup \{0\}$, where $r_1 = \sqrt{\frac{\pi}{4}}$.

2. There exists a covering: $\phi : C_0 \to \mathbb{D}^*$. Any periodic rational parameter ray lands at a parabolic parameter $\lambda$ (That is, $f_\lambda$ has a parabolic periodic point.) or a Misiurewicz point $\lambda$ (That is, $v_\lambda$ lands at a period cycle of $f_\lambda$.)
Theorem (C.-Keen, 2019)

1. The set $S_1$ consists of $2q$ unbounded simply connected components.

2. For $k \geq 2$, at each solution of $f_k^{k-1}(v_\lambda) = \infty$, there are $2pq$ components of $S_k$.

3. All components of $C_k$, $k \geq 0$, and $S_k$, $k \geq 2$, are bounded.
Future work

1. Meromorphic functions with two asymptotic values.
2. Meromorphic functions with more critical values or asymptotic values.
Thank you for your attention!