Singular orbits and Baker domains

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Invariant Fatou components of rational maps

Let $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a *rational map* of degree ≥ 2.

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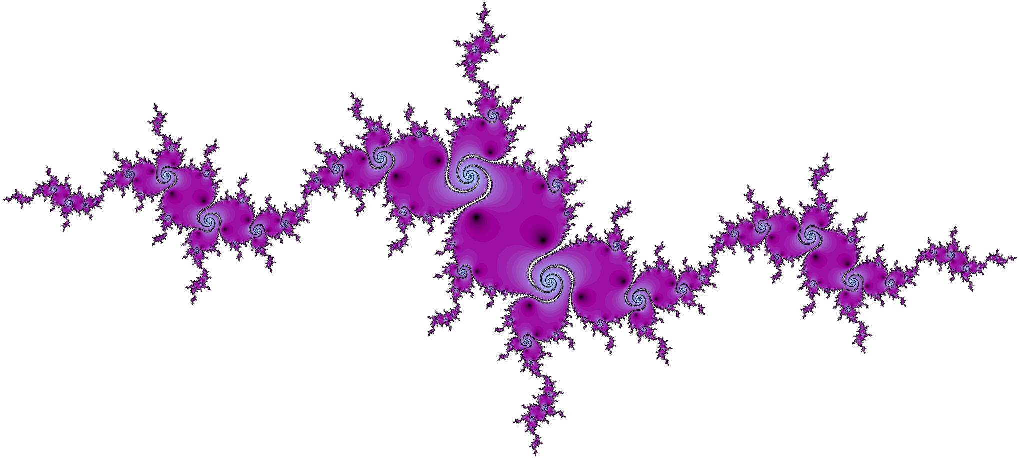

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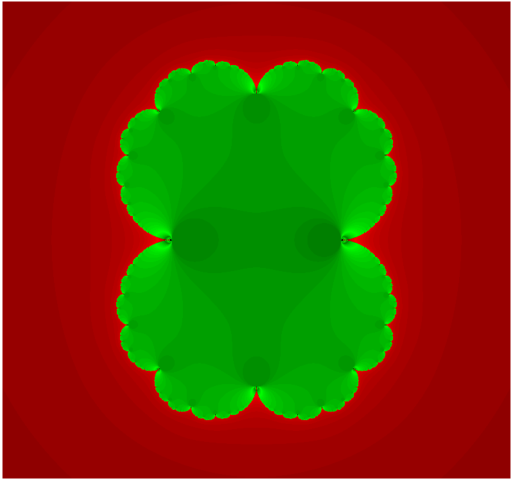

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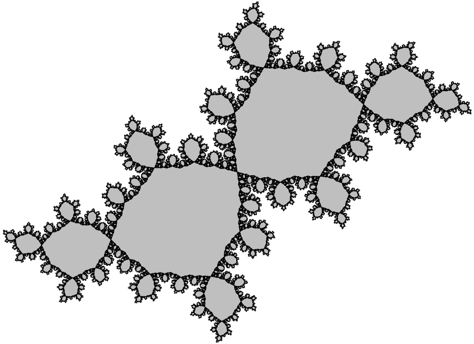

(Image by Arnaud Chéritat)

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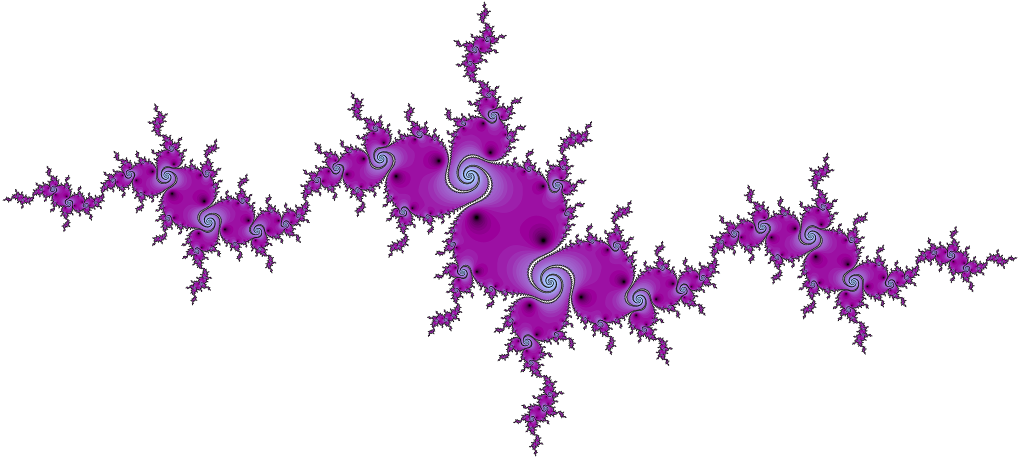

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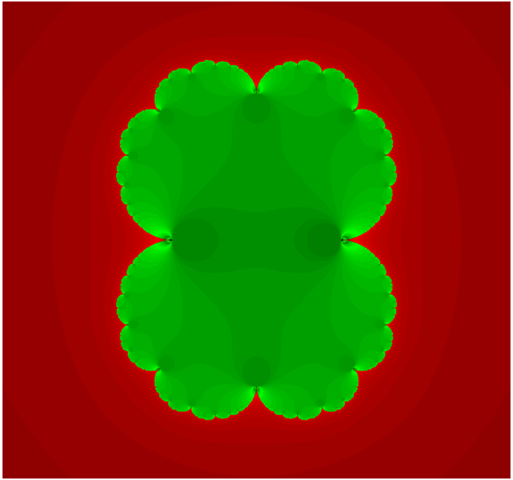

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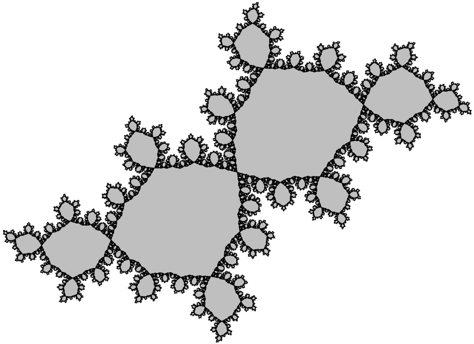

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Meromorphic functions

Now let

$$f\colon \mathbb{C} \to \hat{\mathbb{C}}$$

be a transcendental meromorphic function.

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The role of critical values is now played by

$$\begin{aligned}\operatorname{sing}(f^{-1}) &= \{\text{"singularities of } f^{-1}\text{"}\} \\ &= \{\text{critical values}\} \cup \{\text{asymptotic values}\}.\end{aligned}$$

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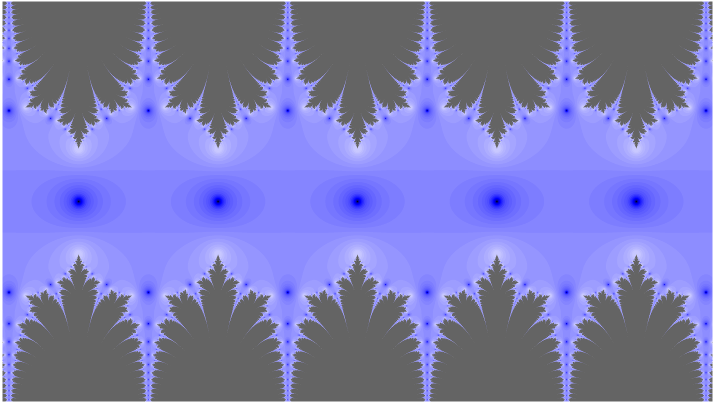
$z \mapsto \sin(z)/2$



attracting basin

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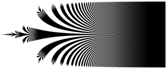
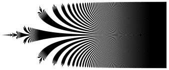
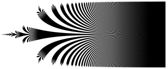
$z \mapsto e^z - 1$







parabolic basin

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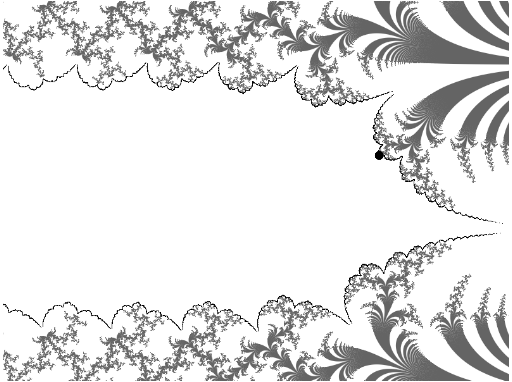

Siegel disc

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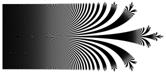

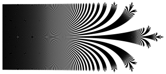

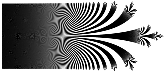

Baker domain

Singular values and Baker domains

In his famous 1993 survey on the *iteration of meromorphic functions*, Bergweiler asks:

Question 4. Let f be a meromorphic function with a cycle of Baker domains that does not contain a point of $\operatorname{sing}(f^{-1})$. Is there some relation between $\operatorname{sing}(f^{-1})$ and the boundaries of the domains of this cycle?

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Question 5. Is it possible that a meromorphic function f has Baker domains if $O^+(z)$ is bounded for all $z \in \operatorname{sing}(f^{-1})$?

A Baker domain despite finite singular orbits

Theorem (R., 2021)

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Postsingular points near Baker domains

Concerning *Question 4*, Bergweiler showed in 1995 that there exists a transcendental *entire* function f having a Baker domain U such that $\operatorname{dist}(\mathcal{P}(f), U) > 0$. On the other hand, he proved the following.

Theorem (Bergweiler, 1995)

Let f be a transcendental entire function with a Baker domain U disjoint from $\operatorname{sing}(f^{-1})$. Then there is a sequence (p_n) in $\mathcal{P}(f)$ such that

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Given $\rho > 0$, the function f can be chosen such there is a sequence $R_j \to \infty$ with

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