Dynamics of Zorich Maps

Athanasios Tsantaris

University of Nottingham

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Introduction

Definition

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. The Julia set $\mathcal{J}(f)$ is the set of all points in $\mathbb{C}$ where the family of iterates $\{f^n : n \in \mathbb{N}\}$ is not normal with respect to the spherical metric of $\overline{\mathbb{C}}$.

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Intuitively the Julia set is the set where the iterates of f behave chaotically. An alternative definition is using the blow-up property.

Blow-up property

Let f be a holomorphic map of the complex plane and $\mathcal{J}(f)$ its Julia set. Let $z \in \mathcal{J}(f)$. Then for any open neighbourhood U of z it is true that

$$\overline{\mathbb{C}} \setminus \bigcup_{n=0}^{\infty} f^n(U)$$

contains at most two points.

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The simplest and most well studied families of entire functions are the quadratic family

$$f_c : z \mapsto z^2 + c$$

for polynomials and the exponential family

$$E_\kappa : z \mapsto \kappa e^z, \ \ \kappa \in \mathbb{C} \setminus \{0\}$$

for transcendental (essential singularity at ∞) entire functions.

Studying those maps is usually the first step towards a better understanding of more general classes of maps.

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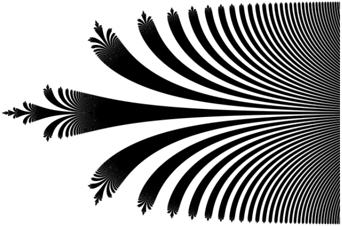
Theorem (Devaney and Krych, 1984)

For $0 < \kappa \leq 1/e$ the Julia set $\mathcal{J}(E_\kappa)$ consists of uncountably many, disjoint curves each of which has a finite endpoint and goes off to infinity.



Source: wikipedia

Quasiregular maps

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- Holomorphic maps send infinitesimally small circles to infinitesimally small circles.
- Quasiregular maps send infinitesimally small spheres to infinitesimally small ellipsoids of bounded eccentricity.
- Quasiregular maps stretch the space locally by a bounded amount.

Examples

- An easy one: $(x, y) \to (2x, y)$.
- Winding maps (i.e. in polar coordinates $(r, \theta) \to (r, k\theta)$)

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Definition

A quasiregular map in $\mathbb{R}^d$ is said to be of *polynomial type* if $\lim_{x\to\infty} f(x) = \infty$. If this limit does not exist we say that f is of *transcendental type*.

Julia sets for qr maps

Definition (Julia set for quasiregular maps, Bergweiler, 2013)

Let f be a quasiregular map on $\mathbb{R}^d$. We define the Julia set to be $\{x \in \mathbb{R}^d : \operatorname{cap}\left(\mathbb{R}^d \setminus \bigcup_{k=1}^{\infty} f^n(U)\right) = 0, U \text{ any open neighboorhood of x}\}$

This new Julia set has many of the properties of the classical Julia set

Theorem (Bergweiler, Nicks, 2014)

Let f be a quasiregular map on $\mathbb{R}^d$. Then assuming $\deg f > K$

- $\mathcal{J}(f)$ is non empty.
- $\mathcal{J}(f)$ is completely invariant.

Moreover if $\operatorname{cap} \mathcal{J}(f) > 0$ then:

- $\mathcal{J}(f)$ is perfect.
- $\mathcal{J}(f^p) = \mathcal{J}(f)$.
- $\mathcal{J}(f) = \overline{O_f^-(x)}$, where $x \in \mathcal{J}(f) \setminus E(f)$.

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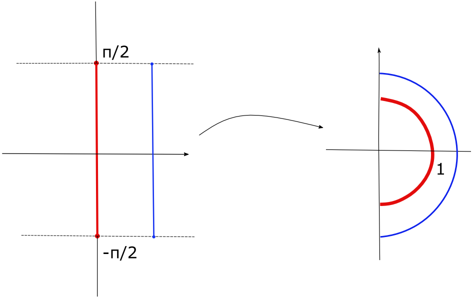

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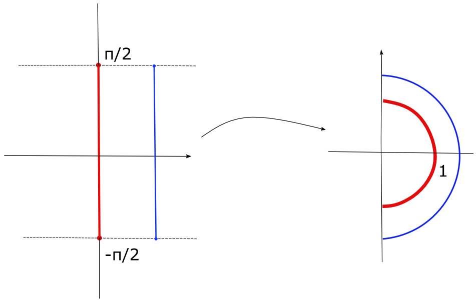

- Then extend E to the whole plane by reflecting across the boundaries of strips in the domain and across the imaginary axis in the range.

Zorich maps

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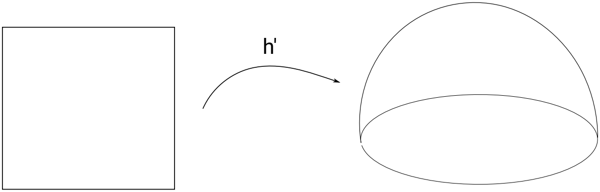
First consider an L bi-Lipschitz, sense-preserving map h' that maps the square

$$Q := \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq 1, |x_2| \leq 1\right\}$$

to the upper hemisphere

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1, x_3 \geq 0\}.$$



Then define $Z : Q \times \mathbb{R} \to \mathbb{R}^3$ as $Z(x_1, x_2, x_3) = e^{x_3} h'(x_1, x_2)$. The map Z maps the square beam $Q \times \mathbb{R}$ to the upper half-space.

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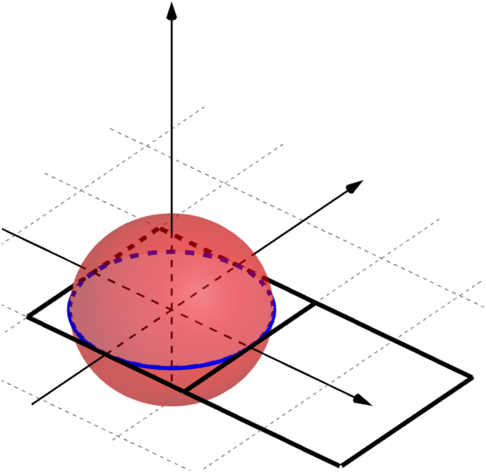
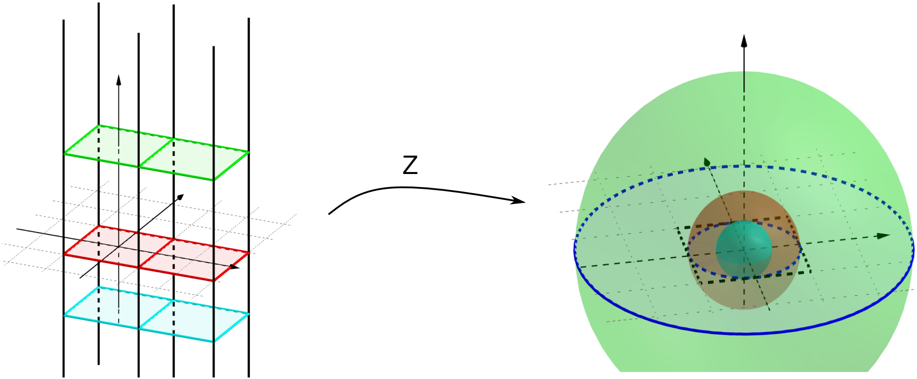
By repeatedly reflecting now, across the sides of the square beam in the domain and the $x_1 x_2$ plane in the range, we get a map $Z : \mathbb{R}^3 \to \mathbb{R}^3$.




Z

Properties:

- Z is doubly periodic, meaning that $Z(x_1+4,x_2,x_3)=Z(x_1,x_2+4,x_3)=Z(x_1,x_2,x_3)$.
- Unlike the exponential map, Z has a non empty branch set meaning that

$$B_Z := \{x \in \mathbb{R}^3 : Z \text{ is not locally homeomorphic at } x\} \neq \emptyset.$$

- Z is quasiregular, omits 0 and has an essential singularity at infinity, just like the exponential map on the plane.

Julia sets of Zorich maps

For the family of maps $Z_\nu = \nu Z$, $\nu > 0$ in analogy with the exponential family we have the following.

Theorem (Bergweiler, 2010 and Bergweiler, Nicks, 2014)

For all ν small enough, the Julia set of the Zorich map $\mathcal{J}(Z_\nu)$ consists of uncountably many, disjoint curves each of which has a finite endpoint and goes off to infinity.

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Theorem (Karpinska's paradox in $\mathbb{R}^3$) (Bergweiler, 2010)

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Question: Is it true that the Julia set of Z_ν is the entire $\mathbb{R}^3$ for large ν?

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We require that our map $h'(x_1, x_2) = (h'_1(x_1, x_2), h'_2(x_1, x_2), h'_3(x_1, x_2))$ must satisfy

- The images of the diagonal lines of the initial square stay on the planes $x_1 = x_2$ and $x_1 = -x_2$.

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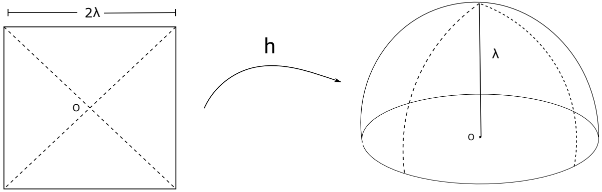
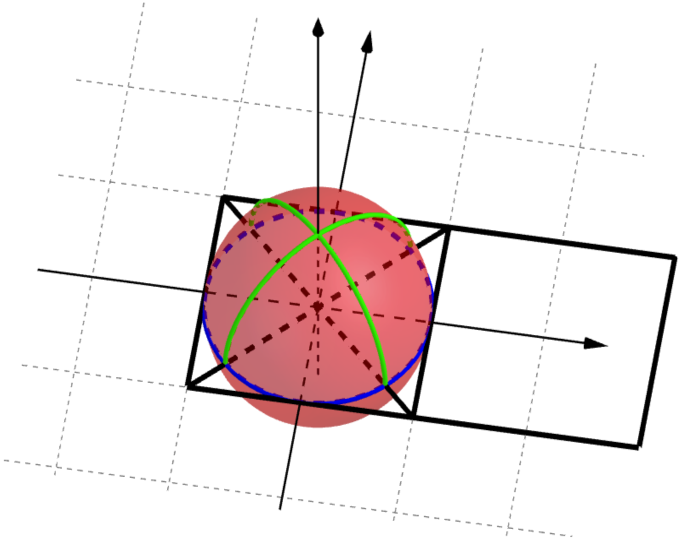
The technical condition is that:

- $h'_1(x_1, x_1) = h'_2(x_1, x_1)$
- $h'_1(x_1, -x_1) = -h'_2(x_1, -x_1)$.

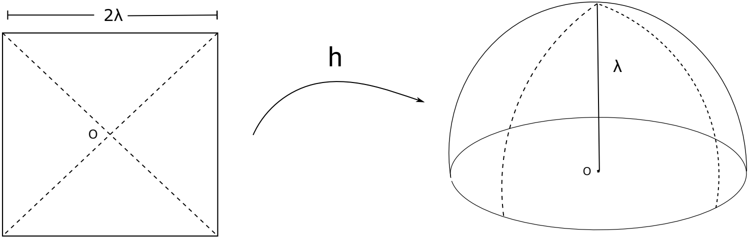




Second, we scale things by a factor $\lambda > 1$ and define

$$h(x_1, x_2) = \lambda h'\left(\frac{1}{\lambda}(x_1, x_2)\right),\ (x_1, x_2) \in \lambda Q.$$

Define now the Zorich maps as before using this h instead. We denote those Zorich maps by $\mathcal{Z}$.
$\mathcal{Z} : \lambda Q \times \mathbb{R} \to \mathbb{R}^3$ as $\mathcal{Z}(x_1, x_2, x_3) = e^{x_3} h(x_1, x_2)$ and reflect across sides and the plane $x_1 x_2$.



Julia sets of Zorich maps

Theorem 1 (T., 2020)

Let $\lambda > L^5$. Then for all $\nu > \sqrt{\frac{2L}{\lambda}}$ the Zorich map $\mathcal{Z}_\nu$ we get using this scale factor λ has as its Julia set the whole $\mathbb{R}^3$.

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- Note that we can not use normality arguments which are essential in Misiurewicz's proof.
- Our proof is in this sense more elementary and thus also more complicated.

Density of periodic points

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However we only have $\mathcal{J}(f) \subset \partial I(f)$ in the quasiregular setting.

Connectedness of the escaping set

Theorem (Devaney and Krych, 1984)

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Theorem 3 (T. 2020)

Let $\lambda > L^5$. Then for all $\nu > \sqrt{\frac{2L}{\lambda}}$ the escaping set $I(\mathcal{Z}_\nu)$ is a connected and dense subset of $\mathbb{R}^3$.

Further Questions

Question 1: What happens for smaller values of the scale factor λ?

Question 2: What is the typical orbit of the Zorich maps $\mathcal{Z}_\nu$ we constructed?

Question 3: Do higher dimensional indecomposable continua exists related with the dynamics of Zorich maps?

Thank you!