XVA Analysis

and the Embedded Probabilistic, Risk Measure, and Machine Learning Issues

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(left) corporate and (right) bank credit spreads across the last financial crises; (top) until 2017 and (bottom) covid-19 crisis

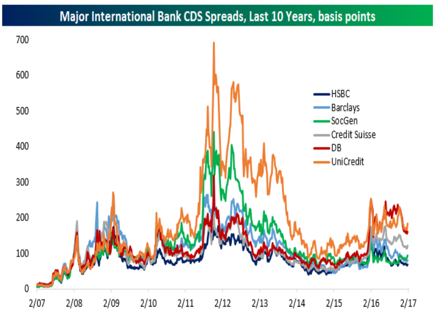
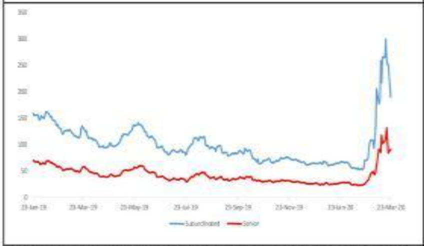
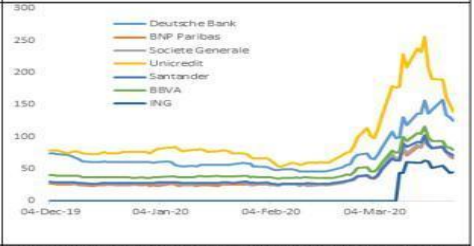


Figure 10: Itraxx indexes for senior and subordinated debt



Source: Bloomberg and ECON calculations. Note: last observation 30 March 2020

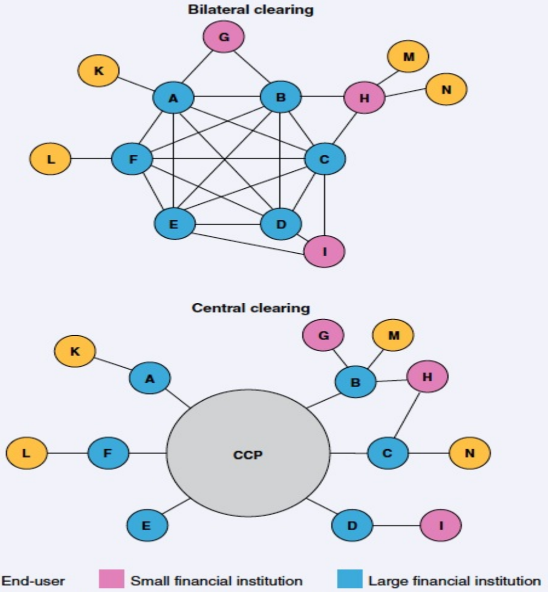
Figure 11: 5yr CDS spreads for selected European banks



Source: Bloomberg and ECON calculations. Note: last observation 30 March

- In the aftermath of the 2008-09 financial crisis, derivative markets regulators launched a major banking reform effort aimed at securing the financial system by raising collateral and capital requirements.
- Clearing of standardized derivatives through central counterparties (CCPs) was progressively enforced or strongly incentivized

1. OTC Derivatives Counterparty Relationships



Source: Reserve Bank of Australia, Central Clearing of OTC Derivatives in Australia (June 2011), available at: http://www.rba.gov.au/publications/consultations/201106-otc-derivatives/central-clearing-otc-derivatives.html

- In (counterparty credit risk) complete markets, collateral and capital requirements would be indifferent to banks.
- The quantification by banks of market incompleteness, based on various XVA metrics, emerged as the unintended consequence of the banking reform.
- XVAs: Pricing add-ons (or rebates) with respect to the counterparty-risk-free value of financial derivatives, meant to account for counterparty risk and its capital and funding implications.
- VA stands for valuation adjustment and X is a catch-all letter to be replaced by C for credit, D for debt, F for funding, M for (initial) margin, and K for capital.

- Pricing XVA add-ons at trade level
  - funds transfer price (FTP)
- But also accounting XVA entries at the aggregate portfolio level
  - In June 2011 the Basel Committee reported that

    During the financial crisis, roughly two-thirds of losses attributed counterparty credit risk were due to CVA losses and only about one-third were due to actual defaults
  - In January 2014 JP Morgan has recorded a \$1.5 billion FVA loss
  - https://www.risk.net/derivatives/7526696/fva-losses-back-in-spotlight-after-coronavirus-stress

    Banks face a new round of losses after two key inputs for calculat funding costs for uncollateralised derivatives—interest rates and fund spreads—saw wild moves last month, contributing to a combined loss almost \$2 billion at Bank of America, Goldman Sachs and JP Morga
- Individual FTP of a trade actually defined as trade portfolio incremental XVAs of the trade

Objectives of the course

- Deriving sound, principle based XVA metrics, for both bilateral and centrally cleared transactions
- Addressing the related computational challenges

References

Mainly based on:

- S. Crépey. Positive XVAs, *Working Paper* 2021.
- C. Albanese, S. Crépey, Rodney Hoskinson, and Bouazza Saadeddine. XVA Analysis From the Balance Sheet, *Quantitative Finance* 2021.
- C. Albanese, S. Crépey, and Yannick Armenti. XVA metrics for CCP optimization, *Statistics & Risk Modeling* 2020.

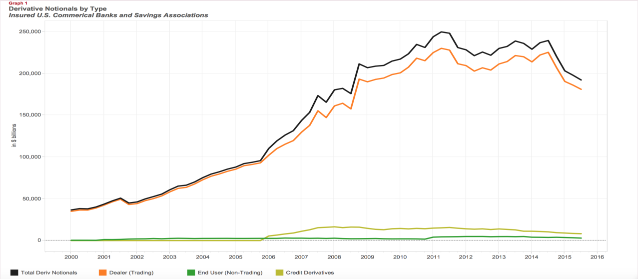
Outline

- Before coming to the technical (computational) implications, the fundamental points are to
  - understand what deserves to be priced and what does not
    - $\supseteq$ "double counting" (overlap) issues
  - by establishing, not only a pricing, but also the corresponding collateralization, accounting, and dividend policy of the bank

The Sustainable Pricing and Dividends Problem

- We want to devise a pricing, collateralization, accounting, and dividend policy for a dealer bank, sustainable in the sense of ensuring to its shareholders a constant instantaneous return rate h on their capital at risk, even in the limiting case of a portfolio held on a run-off basis, i.e. without future deals.

Ponzi scheme in the 2008–09 global financial crisis



- Moreover, the corresponding policy of the bank should satisfy several regulatory constraints.

- Firstly, the market risk of the bank should be hedged as much as possible.
    - As a result, mainly counterparty risk remains.

Secondly, reserve capital should be maintained by the bank at the level of its *expected* counterparty credit losses, along two lines:

- the credit valuation adjustment (CVA) of the bank, meant to cope with the counterparty risk of the bank clients
  - i.e. with the expected losses of the bank due to client defaults;
- the funding valuation adjustment (FVA) of the bank, meant to cope with the counterparty risk of the bank itself,
  - i.e. with its expected risky funding expenses.

- Thirdly, capital should be set at risk by the bank to deal with its *exceptional* (above expected) *losses*.
    - The above return rate h is then meant at a hurdle rate for the bank shareholders, i.e. a risk premium for their capital at risk within the bank.

- Reserve capital (RC), like capital at risk (CR) , should obviously be nonnegative.
- Furthermore, it should not decrease simply because the credit risk of the bank itself has worsened, a property which we refer to as monotonicity
    - see Section 3.1 in Albanese and Andersen (2014) for the relevant wordings from Basel Committee on Banking Supervision (2012) and Federal Register (2014)

- Further requirements on a solution to the above sustainable pricing and dividend release policy problem are
  - economic interpretability and logical consistency
    - for intellectual adhesion by market participants
  - numerical feasibility and robustness at the level of a realistic banking portfolio
    - for practicality
  - minimality in the sense of being, all things equal, as cheap as possible
    - for competitiveness

Solution Setup

The starting point of the cost-of-capital XVA solution to the sustainable pricing and dividends problem is an organizational and accounting separation between three kinds of business units within the bank: the CA (contra-assets) desks, the clean desks, and the management of the bank.

The CA desks

- are themselves split between the CVA desk and the FVA desk (or Treasury, or ALM) of the bank,
- respectively in charge of the default risk of the clients and of the risky funding expenses of the bank.
- The corresponding cash flows are collectively called the contra-assets (CA).
- The CA desks fully guarantee the trading of the clean desks against client and bank defaults, through a clean margin (CM) account of re-hypothecable collateral , which also funds the trading of the clean desks at the risk-free rate.

The clean desks

- Thanks to this work accomplished by the CA desks, the clean desks can focus on the market risk of the contracts in their respective business lines, as if there was no counterparty risk
  - even if some of their positions are liquidated, this will occur at no loss from their perspective

The management

- The management of the bank is in charge of its dividend release policy.
- We consider a level of capital at risk (CR) sufficient to make the bank resilient to a forty-year adverse event, i.e. at least as large as an economic capital (EC) defined as the expected shortfall of the losses of the bank in the next year at the confidence level $\alpha = 97.5\% = 1 - \frac{1}{40}$.
- The implementation of a sustainable dividend remuneration policy requires a dedicated risk margin (RM) account, on which bank profits are initially retained so that they can then be gradually released as dividends at a hurdle rate h on shareholder capital at risk
  - as opposed to being readily distributed as day-one profit

- Counterparty default losses, as also funding payments, are materialities for default if not paid.
- By contrast, risk margin payments, i.e. dividends, are at the discretion of the bank management, hence they do not represent an actual liability to the bank.
- As a consequence, the amount on the risk margin account (RM) is also loss-absorbing, i.e. part of capital at risk (CR).
- With minimality in view, we thus set

$$\mathrm{CR} = \max(\mathrm{EC}, \mathrm{RM}). \tag{1}$$

Physical or Risk-Neutral?

- Let there be given a physical probability measure on a σ algebra $\mathfrak{A}$ and a risk-neutral pricing measure on a financial σ algebra $\subseteq \mathfrak{A}$
  a. The risk-neutral measure is calibrated to the market (prices of fully collateralized transaction for which counterparty risk is immaterial)
  b. The physical probability measure expresses user views on the unhedgeable risk factors
  c. The risk-neutral and physical measures are assumed equivalent on the financial σ algebra
- One can think of our reference probability measure $\mathbb{Q}^*$ as the unique probability measure[1] on $\mathfrak{A}$ that coincides
  i. with the risk-neutral pricing measure *on* the financial σ algebra (and is then is calibrated to the market via a. above)
  ii. with the physical measure *conditional* on the financial σ algebra.
- Risk-free asset used as numéraire (except in the numérics)

[1] See Proposition 2.1 in Artzner, Eisele, and Schmidt (2020), building on Dybvig (1992), for a proof.

Rules of the Game

- In line with the first-edicted sustainability requirement, the portfolio is supposed to be held on a run-off basis between inception time 0 and its final maturity.
  - The bank locks its portfolio at time 0 and lets it amortize in the future,
- All bank accounts are marked-to-model, i.e. continuously and instantaneously readjusted to theoretical target levels, specifically the following balance conditions hold:

$$\mathrm{CM} = \mathrm{MtM}\,, \;\; \mathrm{RC} = \mathrm{CA} = \mathrm{CVA} + \mathrm{FVA}\,, \;\; \mathrm{RM} = \mathrm{KVA},$$

for some theoretical target levels MtM, CVA, FVA, and KVA, which will be defined later in view of yielding a solution to the sustainable pricing and dividends problem.

- At time 0:
  - The clean desks pay MtM_0 to the clients and the CA desks put an amount MtM_0 on the clean margin account if $\mathrm{MtM}_0 > 0$, whereas the clean desks put an amount $(-\mathrm{MtM}_0)$ on the clean margin account if $\mathrm{MtM}_0 < 0$;
  - The CA desks charge to the clients an amount CA_0 and add it on the reserve capital account;
  - The management of the bank charges the amount KVA_0 to the clients and adds it on the risk margin account.
- Between time 0 and the bank default time τ (both excluded), mark-to-model readjustments of all bank accounts are on bank shareholders.

The broad rule regarding the settlement of contracts following defaults is that, at the liquidation time t_c of a netting set c between two counterparties:

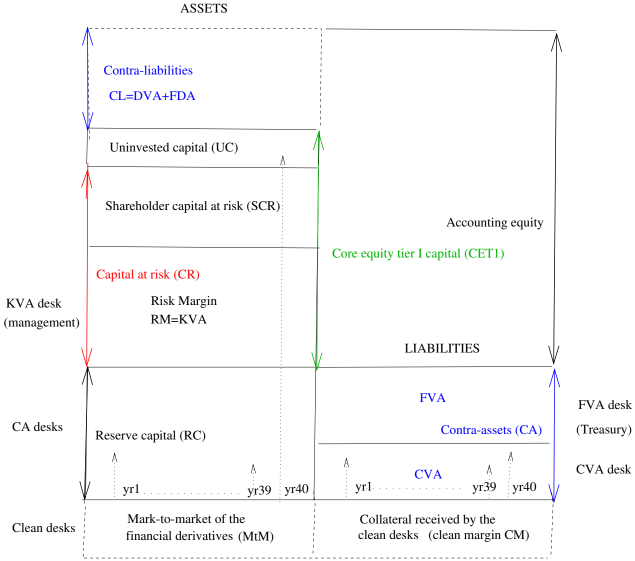
- In the case where only one of the two involved counterparties is in default at t_c, then:
  - If the debt of the counterparty in default toward the other does not exceed its posted margin, then this debt is reimbursed in totality to the other party;
  - Otherwise, this debt is only reimbursed at the level of this posted margin plus a fraction (recovery rate of the defaulted party) times the residual debt beyond the margin;

- In the case where both involved counterparties are in default at t_c, then:
  - If one is indebted to the other beyond its posted margin (as we will detail later this cannot occur for both jointly), then this counterparty transfers to the other the property of its posted margin plus its recovery rate times its residual debt beyond the margin;
  - Otherwise the debt between the two parties is fully settled.

Here debt is understood on a counterparty-risk-free basis and gross of the promised contractual cash flows unpaid during the liquidation period. Within the bank, the CVA desk is in charge of the liquidation close-out cash flows at t_c.

- If the bank itself defaults, then any residual amount on the reserve capital and risk margin accounts, as well as any remaining trading cash flows, are transferred to the creditor of the bank, who also needs to address the liquidation costs of the bank.
- These are outside the scope of the model, as is also the primary business of the clients of the bank, which motivates their deals with the bank.

The Balance Sheet Invites Itself Into Pricing



Hedging Assumptions

- For simplicity and in line with the no prop. trading Volcker rule, we assume perfect hedging by the clean desks, in the sense that their trading loss, inclusive of their hedging loss, vanishes.
- But, from a conservative XVA perspective, we assume that the CA desks do no hedge.

$\rightarrow$ The derivative portfolio and its hedge reduces to its counterparty risk related cash flows

Remark 1

- One could include further a (partial) XVA hedge
  - of the embedded market risk, as opposed to jump-to-default risk
- Conversely, one could relax the perfect clean hedge assumption
- The related extensions of the setup would change nothing to the qualitative conclusions of the paper, only implying additional terms in the trading loss L of the bank and accordingly modified economic capital and KVA figures.

Now: What are the Cash Flows??

Top floor

3. Management / KVA (−δ) ← dividends

2. Treasury / FVA (−f) ← default of the bank itself / risky funding
self

1. CVA desk clients / CVA (−c) ← defaults of the bank clients.

0. Clean desks / MtM ϑ ← contractually promised cash flows

Trading floors

counterparty credit risk

market risk

Outline

- In this section we present the main ideas of the cost-of-capital XVA approach in an elementary static one-year setup
- Assume that at time 0 a bank enters a derivative position (or portfolio) with a client.

- Deal with promised cash flows (from the client to the bank) $\mathcal{P}$
- But the bank and its client are both default prone with zero recovery.
- We denote by J and J_1 the survival indicators of the bank and its client at time 1
  - Both being assumed alive at time 0
  - With default probability of the bank $\mathbb{Q}^*(J = 0) = \gamma \in (0, 1)$

- We assume that unsecured borrowing is fairly priced as $\gamma \times$ the amount borrowed by the bank for funding its trading, which is assumed paid at time 1 by the bank, irrespective of its default status.
- We assume further that a fully collateralized back-to-back market hedge is set up by the bank in the form of a deal with a third party, with no entrance cost and a payoff to the bank $(\mathrm{MtM} - \mathcal{P})$ at time 1, irrespective of the default status of the bank and the third party at time 1.

- For simplicity in a first stage, we will ignore the possibility of using capital at risk for funding purposes, only considering in this respect reserve capital $\mathrm{RC} = \mathrm{CA}$.
- The additional free funding source provided by capital at risk will be introduced later, as well as collateral between bank and clients.

Cash flows

Lemma 1

Given the to be specified MtM *and* CA *amounts, the credit and funding cash flows* $\mathcal{C}$ *and* $\mathcal{F}$ *of the bank and its trading loss (and profit)* L *satisfy* $L = \mathcal{C} + \mathcal{F} - \mathrm{CA}$, *with*

$$\begin{aligned}
\mathcal{C} &= (1 - J_1)\mathcal{P}^+ - (1 - J)\mathcal{P}^- = J(1 - J_1)\mathcal{P}^+ \\
&\qquad - (1 - J)\big(\mathcal{P}^- - (1 - J_1)\mathcal{P}^+\big) \\
\mathcal{F} &= \gamma(\mathrm{MtM} - \mathrm{CA})^+ - (1 - J)(\mathrm{MtM} - \mathrm{CA})^+ = J\gamma(\mathrm{MtM} - \mathrm{CA})^+ \\
&\qquad - (1 - J)\big((\mathrm{MtM} - \mathrm{CA})^+ - \gamma(\mathrm{MtM} - \mathrm{CA})^+\big).
\end{aligned}$$

Proof. The trading desks of the bank pay $\mathrm{MtM} - \mathrm{CA}$ for the deal, whereas they receives on the hedge and as portfolio settlement

$$\begin{aligned}
&(\mathrm{MtM} - \mathcal{P}) + (1 - J_1)J(-\mathcal{P}^-) + J_1(1 - J)\mathcal{P}^+ + (1 - J_1)(1 - J)\,0 + J_1 J\mathcal{P}\\
&= \mathrm{MtM} + (1 - J_1)J(-\mathcal{P}^+) + J_1(1 - J)\mathcal{P}^- + (1 - J_1)(1 - J)(-\mathcal{P})\\
&= \mathrm{MtM} + (1 - J_1)J(-\mathcal{P}^+) + J_1(1 - J)\mathcal{P}^- + (1 - J_1)(1 - J)(\mathcal{P}^- - \mathcal{P}^+)\\
&= \mathrm{MtM} + (1 - J_1)(-\mathcal{P}^+) + (1 - J)\mathcal{P}^-,
\end{aligned}$$

i.e. the bank pays

$$(1 - J_1)\mathcal{P}^+ - (1 - J)\mathcal{P}^- - \mathrm{CA}$$

The funding side of the strategy yields to the bank a further cost $\gamma(\mathrm{MtM} - \mathrm{CA})^+$ and a windfall funding benefit $(1 - J)(\mathrm{MtM} - \mathrm{CA})^+$,

- money borrowed at time 0 and kept at time 1 in the case where the bank defaults
- if $\mathrm{MtM} - \mathrm{CA} < 0$ then the bank is actually lender at time 0 and reimbursed at time 1 whatever its default status at time 1 ∎

- Flipping the signs in the above, the result of the bank over the year (appreciation of its accounting equity) is rewritten as

$$
\begin{aligned}
&-\underbrace{J(1-J_1)\mathcal{P}^+}_{J\mathcal{C}} + J\mathrm{CVA} - \underbrace{J\gamma(\mathrm{MtM}-\mathrm{CA})^+}_{J\mathcal{F}} + J\mathrm{FVA} \\
&\qquad + \underbrace{(1-J)\big(\mathcal{P}^- - (1-J_1)\mathcal{P}^+\big)}_{(-(1-J)\mathcal{C})} + (1-J)\mathrm{CVA} \\
&\qquad + \underbrace{(1-J)\big((\mathrm{MtM}-\mathrm{CA})^+ - \gamma(\mathrm{MtM}-\mathrm{CA})^+\big)}_{(-(1-J)\mathcal{F})} + (1-J)\mathrm{FVA}.
\end{aligned}
$$

- However, the cash flows in the last two lines are only received by the bank if it is in default at time 1, hence they go to the estate of the defaulted bank (liquidators of the bank, sometimes dubbed bank creditors below).
- Hence, the profit-and-loss of bank shareholders reduces to the first line, i.e. the bank shareholders' trading loss is

$$\mathcal{J}L = \mathcal{J}\mathcal{C} - \mathcal{J}\mathrm{CVA} + \mathcal{J}\mathcal{F} - \mathcal{J}\mathrm{FVA}. \tag{2}$$

Remark 2

- The above derivation implicitly allows for negative equity (that arises whenever $JL > \mathrm{CET1}$) which is interpreted as recapitalization.
- In a variant of the model excluding recapitalization, the default of the bank would be modeled in a structural fashion as the event $\{L = \mathrm{CET1}\}$, where

$$L = \big((1 - J_1)\mathcal{P}^+ + \gamma(\mathrm{MtM} - \mathrm{CA})^+ - \mathrm{CA}\big) \wedge \mathrm{CET1},$$

and we would obtain, instead of the above, the bank trading loss

$$\mathbb{1}_{\{\mathrm{CET1} > L\}} L + \mathbb{1}_{\{\mathrm{CET1} = L\}} (\mathrm{CET1} - \mathcal{P}^- - (\mathrm{MtM} - \mathrm{CA})^+).$$

Shareholder valuation

- Let $\mathbb{E}^*$ and $\mathbb{E}$ denote the expectations with respect to the measure $\mathbb{Q}^*$ and the associated bank survival measure, $\mathbb{Q}$, i.e., for any random variable $\mathcal{Y}$,

$$\mathbb{E}\mathcal{Y} = (1-\gamma)^{-1}\mathbb{E}^*(J\mathcal{Y}) \tag{3}$$

  - $= \mathbb{E}J\mathcal{Y}$
  - $= \mathbb{E}^*\mathcal{Y}$ if $\mathcal{Y}$ is independent from J.

Lemma 2

For any random variable $\mathcal{Y}$ and constant Y, we have

$$Y = \mathbb{E}^*(J\mathcal{Y} + (1-J)Y) \Longleftrightarrow Y = \mathbb{E}\mathcal{Y}.$$

Proof. Indeed,

$$\begin{aligned} Y = \mathbb{E}^*(J\mathcal{Y} + (1-J)Y) &\Longleftrightarrow \mathbb{E}^*(J(\mathcal{Y} - Y)) = 0 \\ &\Longleftrightarrow \mathbb{E}(\mathcal{Y} - Y) = 0 \Longleftrightarrow Y = \mathbb{E}\mathcal{Y}, \end{aligned}$$

where the passage to the second line is justified by (51). ∎

- Clean and CA desks make their shareholder trading losses $\mathbb{Q}^*$ centered
- The clean desks pay to the client MtM such that

$$\mathbb{E}^*(J\mathcal{P} - J\mathrm{MtM}) = 0, \text{ i.e. } \mathrm{MtM} = \mathbb{E}^*\big(J\mathcal{P} + (1-J)\mathrm{MtM}\big).$$

- CA desks charge to the client CVA and FVA add-ons such that

$$\mathbb{E}^*(J\mathcal{C} - J\mathrm{CVA}) = \mathbb{E}^*(J\mathcal{F} - J\mathrm{FVA}) = 0, \tag{4}$$

i.e.

$$\mathrm{CVA} = \mathbb{E}^*\big(J\mathcal{C} + (1-J)\mathrm{CVA}\big)\,,\;\; \mathrm{FVA} = \mathbb{E}^*\big(J\mathcal{F} + (1-J)\mathrm{FVA}\big).$$

- These are MtM, CVA, and FVA equations.

- However, in terms of the bank survival expectation, Lemma 7 yields $\mathrm{MtM} = \mathbb{E}(J\mathcal{P})$ and

$$\mathrm{CVA} = \mathbb{E}(J\mathcal{C}) = \mathbb{E}((1 - J_1)\mathcal{P}^+)\,, \;\; \mathrm{FVA} = \mathbb{E}(J\mathcal{F}) = \gamma(\mathrm{MtM} - \mathrm{CA})^+$$

(as the latter is deterministic), hence by (2)

$$JL = J\mathcal{C} - J\mathrm{CVA}. \tag{5}$$

- The possibility for the clean desks to find hedge counterparties at the price MtM leads to assume that $\mathrm{MtM} = \mathbb{E}^*\mathcal{P}$

$\rightarrow$

$$(\mathbb{E}^* J)(\mathbb{E}^*\mathcal{P}) = (1 - \gamma)(\mathbb{E}\mathcal{P}) = \mathbb{E}^*(J\mathcal{P}), \tag{6}$$

by (51).

Remark 3

Even if the clean desks were able to find (clients and) hedge counterparties accepting to deal with the bank on the basis of an MtM process that would be the bank shareholder value of $\mathcal{P}$ but not its value process, the corresponding discrepancy between valuation and shareholder valuation of $\mathcal{P}$ would be an indication of extreme dependence between the derivative portfolio and the default of the bank itself, such as the bank trading its own default risk,

- note that $\mathcal{P} = \pm J$ violates (6) (having assumed $\gamma \in (0, 1)$),

which should be considered with caution.

- We have the following semi-linear equation for $\mathrm{FVA} = \mathrm{CA} - \mathrm{CVA}$:

$$\mathrm{FVA} = \gamma(\mathrm{MtM} - \mathrm{CVA} - \mathrm{FVA})^+, \tag{7}$$

which has the unique solution

$$\mathrm{FVA} = \frac{\gamma}{1+\gamma}(\mathrm{MtM} - \mathrm{CVA})^+. \tag{8}$$

- The creditors of the bank get

$$(-(1-J)L) = (-(1-J)\mathcal{C}) + (1-J)\mathrm{CVA} + (-(1-J)\mathcal{F}) + (1-J)\mathrm{FVA}$$

- Let $\mathrm{CL} = \mathrm{DVA} + \mathrm{FDA}$, where

$$\mathrm{DVA} = \mathbb{E}^*\big((-(1-J)\mathcal{C}) + (1-J)\mathrm{CVA}\big)$$
$$\mathrm{FDA} = \mathbb{E}^*\big((-(1-J)\mathcal{F}) + (1-J)\mathrm{FVA}\big)$$

  - debt valuation adjustment and funding debt adjustment

- As $\mathbb{E}^*(J\mathcal{F}) = \mathbb{E}^*(-(1-J)\mathcal{F}) = \gamma(1-\gamma)(\mathrm{MtM} - \mathrm{CA})^+$, we have

$$\begin{aligned}\mathrm{FVA} =& \mathbb{E}^*\big(J\mathcal{F} + (1-J)\mathrm{FVA}\big) = \\ & \mathbb{E}^*\big((-(1-J)\mathcal{F}) + (1-J)\mathrm{FVA}\big) = \mathrm{FDA}.\end{aligned}$$

- Writing $\mathcal{C} = J\mathcal{C} - (-(1-J)\mathcal{C})$ and $\mathcal{F} = J\mathcal{F} - (-(1-J)\mathcal{F})$, also note that the fair valuation $\mathrm{FV} = \mathbb{E}^*(\mathcal{C} + \mathcal{F})$ of counterparty credit risk satisfies

$$\begin{aligned}\mathrm{FV} =& \mathbb{E}^*\mathcal{C} = \mathbb{E}^* J\mathcal{C} - \mathbb{E}^*(-(1-J)\mathcal{C}) \\ &= \mathbb{E}^*\big(J\mathcal{C} + (1-J)\mathrm{CVA}\big) - \mathbb{E}^*\big((-(1-J)\mathcal{C}) + (1-J)\mathrm{CVA}\big) \\ &= \mathrm{CVA} - \mathrm{DVA} = \mathrm{CA} - \mathrm{CL}.\end{aligned}$$

KVA and Funds Transfer Price

- Let EC denote economic capital, i.e. the theoretical target level of capital at risk that a regulator would like to see in the bank from a structural point of view.
- For simplicity we assess EC "on a going concern" as

$$\mathrm{EC} = \mathbb{ES}(JL)$$

  - 97.5% expected shortfall of the bank shareholder trading loss JL under the bank survival measure $\mathbb{Q}$
  - nonnegative, as JL is $\mathbb{Q}^*$ centered, hence $\mathbb{Q}$ centered by (51).

- Under the cost of capital XVA approach, the bank charges to its client an additional amount (retained margin, which is loss absorbing) such that

$$\mathrm{KVA} = \mathbb{E}^*\big(Jh(\mathrm{EC} - \mathrm{KVA})^+ + (1 - J)\mathrm{KVA}\big),$$

for some so called hurdle rate parameter h (e.g. 10%),

- i.e.

$$\mathrm{KVA} = \mathbb{E}h(\mathrm{EC} - \mathrm{KVA})^+ = h(\mathrm{EC} - \mathrm{KVA})^+, \tag{9}$$

i.e.

$$\mathrm{KVA} = \frac{h}{1+h}\mathrm{EC}. \tag{10}$$

Funds Transfer Price

The all-inclusive XVA add-on aligning the entry price of the deal to shareholder interest , which we call funds transfer price (FTP), is

$$\begin{aligned}
\mathrm{FTP} &= \underbrace{\mathrm{CVA}+\mathrm{FVA}}_{\text{Expected costs } \mathrm{CA}} + \underbrace{\mathrm{KVA}}_{\text{Risk premium}} \\
&= \underbrace{\mathrm{CVA}-\mathrm{DVA}}_{\text{Fair valuation } \mathrm{FV}} + \underbrace{\mathrm{DVA}+\mathrm{FDA}}_{\text{Wealth transfer } \mathrm{CL}} + \underbrace{\mathrm{KVA}}_{\text{Risk premium}},
\end{aligned}$$

where the random variable used to size the economic capital EC in the KVA formula (9) is the bank shareholders loss-and-profit JL as per (5).

Monetizing the Contra-Liabilities?

- Let us now assume, for the sake of the argument, that the bank would be able to hedge its own jump-to-default risk through a new deal, whereby the bank would deliver a payment $(-(1-J)L)$ at time 1 in exchange of a premium fairly valued as

$$\mathrm{CL} = \mathbb{E}^*(-(1-J)L) = \mathrm{DVA} + \mathrm{FDA},$$

and deposited in the reserve capital account at time 0.

- Accounting for the new deal and assuming the client provides $\mathrm{FV} = \mathrm{CA} - \mathrm{CL}$ (instead of CA before) in the reserve capital account of the bank, the amount that needs by borrowed by the CA desk for implementing the strategy is still $\gamma(\mathrm{MtM} - \mathrm{CA})^+$ as before and the bank trading loss is now given by

$$\begin{aligned}
&\mathcal{C} + \mathcal{F} - \mathrm{FV} + (-(1 - J)L) - \mathrm{CL} = \\
&\qquad\qquad \mathcal{C} + \mathcal{F} - \mathrm{CA} + (-(1 - J)L) = L + (-(1 - J)L) = JL,
\end{aligned}$$

for JL as in (5).

- Hence, because of the new deal:
  - The client is better off by the amount $\mathrm{CA} - \mathrm{FV} = \mathrm{CL}$;
  - The creditors are left without any ressources to address the liquidation costs of the bank;
  - The shareholders are indifferent as always to the (duly priced) deal.
- Summing up, the CL originating cash flow $(-(1 - J)L)$ has been hedged out and monetized by the shareholders, which have passed the corresponding benefit to the client.

- In this situation, the bank would still charge to its client a KVA add-on $\frac{h}{1+h}\mathrm{EC}$, where EC is the same as before (as the random variable JL is the same as before).
- If the bank could also hedge its client default, then the bank trading loss and the KVA would vanish and the FTP would reduce to

$$\mathrm{FTP} = \mathrm{FV} = \mathrm{CVA} - \mathrm{DVA} = \mathrm{CA} - \mathrm{CL}.$$

Initial Margin

- In case of variation margin (VM) that would be exchanged between the bank and its client, and of initial margin that would be received (RIM) and posted (PIM) by the bank, at the height of, say for simplicity, some $\mathbb{Q}$ value-at-risk of $\pm(\mathcal{P} - \mathrm{MtM})$, then
- $\mathcal{P}^+$ needs be replaced by $(\mathcal{P} - \mathrm{VM} - \mathrm{RIM})^+$ in $\mathcal{JC}$, whence an accordingly modified (in principle: diminished) CVA.

- There would be an additional initial margin related cash flow in $J\mathcal{F}$ given as $J\gamma\mathrm{PIM}$, triggering an additional adjustment MVA in CA, where

$$\mathrm{MVA} = \mathbb{E}^*\big(J\gamma\mathrm{PIM} + (1-J)\mathrm{MVA}\big) = \gamma\mathrm{PIM};$$

- There would be additional initial margin related cash flows in $(-(1-J)\mathcal{F})$, triggering an additional adjustment $\mathrm{MDA} = \mathrm{MVA}$ in CL;
- Because of this additional MVA, the FVA formula (8) would become $\mathrm{FVA} = \frac{\gamma}{1+\gamma}(\mathrm{MtM} - \mathrm{VM} - \mathrm{CVA} - \mathrm{MVA})^+$.

Fungibility of Capital at Risk as a Funding Source

- In order to account for the additional free funding source provided by capital at risk, one would need to replace $(\mathrm{MtM} - \mathrm{CA})^{\pm}$ by $(\mathrm{MtM} - \mathrm{CA} - \max(\mathrm{EC}, \mathrm{KVA}))^{\pm}$ everywhere in the above.
  - Note that the marginal cost of capital for using capital as a funding source for variation margin is nil, because when one posts cash as variation margin, the valuation of the collateralized hedge is reset to zero and the total capital amount does not change.
  - If, instead, the bank were to post capital as initial margin, then the bank would record a "margin receivable" entry on its balance sheet, which however cannot contribute to capital since this asset is too illiquid and impossible to unwind without unwinding all underlying derivatives.
  - Hence, capital can only be used as VM, while IM must be borrowed entirely.

This would end-up in (the same modified CVA formula as above and) the following *system* for the random variable JL and the FVA and the KVA numbers (cf. (2), (7), and (10)):

$$
\begin{aligned}
JL &= J(1 - J_1)(\mathcal{P} - \mathrm{VM} - \mathrm{RIM})^{+} - \mathrm{JCVA} \\
\mathrm{KVA} &= \frac{h}{1+h}\mathbb{ES}(JL) \\
\mathrm{FVA} &= \gamma(\mathrm{MtM} - \mathrm{VM} - \mathrm{CA} - \mathrm{EC})^{+} \\
&\qquad = \frac{\gamma}{1+\gamma}(\mathrm{MtM} - \mathrm{VM} - \mathrm{CVA} - \mathrm{MVA} - \mathbb{ES}(\mathrm{JL}))^{+}.
\end{aligned}
$$

Outline

Probabilistic Pricing setup

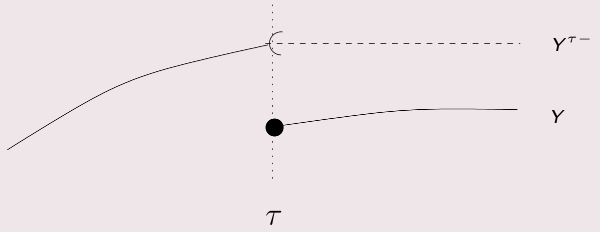
- Stochastic basis $(\mathbb{G}, \mathbb{Q}^*)$, with $\mathbb{G} = (\mathfrak{G}_t)$
- (Risk-neutral) value process of a financial cash flow stream: $(\mathbb{G}, \mathbb{Q}^*)$ conditional expectation process its of future cash flows.
    - (implicitely) discounted through our choice of the risk-free asset as a numéraire
- Portfolio first assumed held on a run-off basis, with final maturity T
    - also including the time (assumed bounded, in practice of the order of one or two weeks) of liquidating defaulted positions

- Bank default time τ with survival indicator process $J = \mathbb{1}_{[\![0,\tau[\![}$ and intensity γ, i.e. the process $dJ_t + \gamma_t dt$ is a martingale
  - additive martingale vs. multiplicative martingale $Je^{\int_0^{\cdot} \gamma_s ds}$
- For any left-limited process Y, we denote by

$$Y^{\tau-} = JY + (1-J)Y_{\tau-}$$

and ${}^{\tau-}Y = Y - Y^{\tau-}$ the processes Y stopped before and starting before the bank default time τ.

Stopping before τ



- Reduced filtration $\mathbb{F} = (\mathfrak{F}_t) \subseteq \mathbb{G}$ with $\mathbb{Q}^*(\tau > T \,|\, \mathfrak{F}_T) > 0$.
- *Reduction* of any $\mathbb{G}$ predictable (resp. optional) process
  - the unique $\mathbb{F}$ predictable (resp. optional) process on $[0, T]$ coinciding with it until (resp. before) τ
- Invariance probability measure $\mathbb{P} \sim \mathbb{Q}^*$ on $\mathfrak{F}_T$ such that
  - $(\mathbb{F}, \mathbb{P})$ local martingales on $[0, T]$ stopped before τ are $(\mathbb{G}, \mathbb{Q})$ local martingales;
  - $\mathbb{F}$ reductions of $(\mathbb{G}, \mathbb{Q})$ local martingales on $[0, \tau \wedge T]$ without jump at τ are $(\mathbb{F}, \mathbb{P})$ local martingales on $[0, T]$.
- *Clean valuation* of an $\mathbb{F}$ adapted cash flow stream, with respect to $(\mathbb{F}, \mathbb{P})$
- $(\mathfrak{G}_t, \mathbb{Q}^*)$ and $(\mathfrak{F}_t, \mathbb{P})$ conditional expectations denoted by $\mathbb{E}^*_t$ and $\mathbb{E}_t$.

Lemma 3

Given an optional, integrable process $\mathcal{Y}$ stopped at T (cumulative cash flow stream in the financial interpretation), the shareholder valuation equation of $\mathcal{Y}$: $Y_T = 0$ on $\{T < \tau\}$ and

$$Y_t = \mathbb{E}^*_t(\mathcal{Y}_{\tau-} - \mathcal{Y}_t + Y_{\tau-})\,, \quad t < \tau,$$

is equivalent, "within suitable spaces of square integrable solutions", to the clean valuation equation of $\mathcal{Y}'$

$$Y'_t = \mathbb{E}_t(\mathcal{Y}'_T - \mathcal{Y}'_t), t \leq T.$$

Proof. *(Sketched, i.e. martingale square integrability considerations aside)* Differential variations on the equations for Y and Y':

$$\begin{aligned}
&Y^{\tau-}_T = 0 \text{ on } \{T < \tau\} \text{ and, for } t \le \tau \wedge T,\\
&dY^{\tau-}_t = -d\mathcal{Y}^{\tau-}_t + d\nu_t, \qquad (11)\\
&\text{for some } (\mathbb{G}, \mathbb{Q}^*) \text{ square integrable martingale } \nu,
\end{aligned}$$

respectively

$$\begin{aligned}
&Y'_T = 0 \text{ and, for } t \le T,\\
&dY'_t = -d\mathcal{Y}'_t + d\mu_t, \qquad (12)\\
&\text{for some } (\mathbb{F}, \mathbb{P}) \text{ square integrable martingale } \mu.
\end{aligned}$$

By definition of $\mathbb{F}$ optional reductions, the terminal condition in (12) obviously implies the one in (11). Conversely, taking the $\mathfrak{F}_T$ conditional expectation of the terminal condition in (11) yields

$$0 = \mathbb{E}[Y_T^{\tau-}\mathbb{1}_{\{T<\tau\}}|\mathfrak{F}_T] = \mathbb{E}[Y_T'\mathbb{1}_{\{T<\tau\}}|\mathfrak{F}_T] = Y_T'\mathbb{Q}^*(\tau > T \,|\, \mathfrak{F}_T),$$

hence $Y_T' = 0$ (as by assumption $\mathbb{Q}^*(\tau > T \,|\, \mathfrak{F}_T) > 0$), which is the terminal condition in (12).

The martingale condition in (12) implies the one in (11), by stopping before τ and application to $\nu = \mu^{\tau-}$ of the invariance probability measure direct condition.
Conversely, the martingale condition in (11) implies that $(Y', \mu = \nu')$ satisfies the second line in (12) on $[\![0, \tau \wedge T]\!]$, hence on $[0, T]$ (by uniqueness of the reduction of the null process). Moreover, by application of the invariance probability measure converse condition, $\mu = \nu'$ is an $(\mathbb{F}, \mathbb{P})$ martingale. ∎

Assuming τ endowed with a $(\mathbb{G}, \mathbb{Q}^*)$ intensity process $\gamma = \gamma J_-$ such that $e^{\int_0^T \gamma_s ds}$ is $\mathbb{Q}^*$ integrable

- Bank survival probability measure $\mathbb{Q}$ associated with $\mathbb{Q}^*$:
  - Probability measure $\mathbb{Q}$ on $(\Omega, \mathfrak{A})$ with $(\mathbb{G}, \mathbb{Q}^*)$ density process $Je^{\int_0^\cdot \gamma_s ds}$
  - cf. Schönbucher (2004) and Collin-Dufresne, Goldstein, and Hugonnier (2004)
- Crépey and Song (2017):
  - Clean valuation $\sim$ valuation with respect to $(\mathbb{G}, \mathbb{Q})$
  - $\mathbb{P} = \mathbb{Q}_{|\mathfrak{F}_T}$
- Reduction of filtration into $(\mathbb{F}, \mathbb{P})$ is the systematic way to address "computations under the (singular) survival probability measure $\mathbb{Q}$"
- Mainstream immersion setup where

$$\mathbb{P} = \mathbb{Q}^*(= \mathbb{Q}) \text{ on } \mathfrak{F}_T$$

.

Remark 4

- For $A \in \mathfrak{A}$,

$$\mathbb{Q}^*(A \,|\, \tau > T) = \frac{\mathbb{Q}^*(A \cap \{\tau > T\})}{\mathbb{Q}^*(\{\tau > T\})} = \mathbb{E}^{\mathbb{Q}}\Big[\frac{\mathbb{1}_A e^{-\int_0^T \gamma_s' ds}}{\mathbb{E}^{\mathbb{Q}}(e^{-\int_0^T \gamma_s' ds})}\Big]$$

  where the first equality follows from Bayes' rule and the second follows from the definition of the probability measure $\mathbb{Q}$:

$$\mathbb{Q}^*(A \cap \{\tau > T\}) = \int_\Omega \mathbb{1}_{A\cap\{\tau>T\}}(\omega)\mathbb{Q}^*(d\omega)$$
$$= \int_\Omega \mathbb{1}_{A\cap\{\tau>T\}}(\omega)\frac{d\mathbb{Q}^*}{d\mathbb{Q}}(\omega)\mathbb{Q}(d\omega) = \int_\Omega \mathbb{1}_{A\cap\{\tau>T\}}(\omega)e^{-\int_0^{\tau\wedge T}\gamma_s ds}\mathbb{Q}(d\omega)$$
$$= \mathbb{E}^{\mathbb{Q}}\big[\mathbb{1}_{A\cap\{\tau>T\}}e^{-\int_0^{\tau\wedge T}\gamma_s' ds}\big] = \mathbb{E}^{\mathbb{Q}}\big[\mathbb{1}_A e^{-\int_0^T \gamma_s' ds}\big].$$

- Hence $\mathbb{Q}$ coincides with the conditional probability $\mathbb{Q}^*(\cdot \,|\, \tau > T)$ if and only if γ' is a deterministic (measurable and Lebesgue integrable) function of time.

Trading Cash-Flows

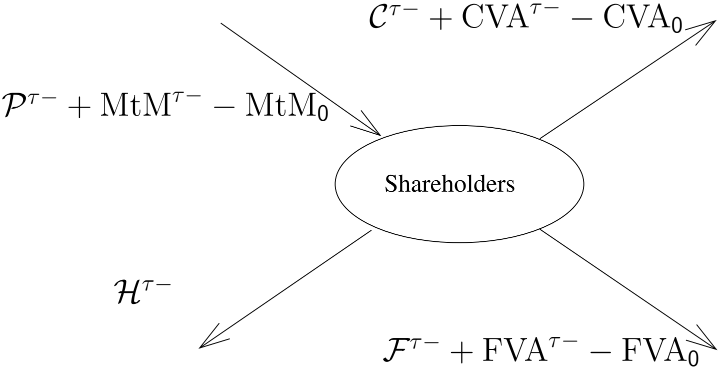
- Cash flows to the clean desks $\mathcal{P}$ and from the CVA and FVA desks $\mathcal{C}$ and $\mathcal{F}$
- $\sim$ contractually promised cash flows $\mathcal{P}$
- Counterparty credit cash flows $\mathcal{C}$
  - a finite variation process with nondecreasing component $\mathcal{C}^{\tau-}$
- Risky funding cash flows $\mathcal{F}$
  - a (zero valued) martingale with nondecreasing $\mathcal{F}^{\tau-}$ and ${}^{\tau-}(-\mathcal{F})$ components, stopped at $\tau \wedge T$.
- Hedging (of market risk) cash flows $\mathcal{H}$
  - a (zero valued) martingale with martingale $(\cdot)^{\tau-}$ component

Remark 5

Martingales with martingale $(\cdot)^{\tau-}$ component include

- martingales without jump at τ
  - in particular, continuous martingales,
- all the $\mathbb{F}$ (càdlàg) martingales in a standard progressive enlargement of filtration setup with the immersion property, provided the $\mathbb{F}$ Azéma supermartingale of τ is continuous and nonincreasing
  - see Lemma 2.1(ii) in Crépey (2015b)

Pre-bank default trading cash flows



Valuation compensates shareholder trading cash flows,

i.e. $\mathrm{MtM}_T = \mathrm{CVA}_T = \mathrm{FVA}_T = 0$ on $\{T < \tau\}$ and, for $t \leq T$,

$$0 = \mathbb{E}_t^* \int_t^T d(\mathcal{P}_s^{\tau-} + \mathrm{MtM}_s^{\tau-} - \mathcal{H}_s^{\tau-}) = \mathbb{E}_t^* \int_t^T d(\mathcal{P}_s^{\tau-} + \mathrm{MtM}_s^{\tau-})$$
$$= \mathbb{E}_t^* \int_t^T d(\mathcal{C}_s^{\tau-} + \mathrm{CVA}_s^{\tau-}) = \mathbb{E}_t^* \int_t^T d(\mathcal{F}_s^{\tau-} + \mathrm{FVA}_s^{\tau-}),$$

i.e., for $t \leq \tau \wedge T$,

$$\mathrm{MtM}_t^{\tau-} = \mathbb{E}_t^* \big(\mathcal{P}_{\tau\wedge T}^{\tau-} - \mathcal{P}_t^{\tau-} + \mathbb{1}_{\{\tau \leq T\}} \mathrm{MtM}_{\tau-}\big), \tag{13}$$
$$\mathrm{CVA}_t^{\tau-} = \mathbb{E}_t^* \big(\mathcal{C}_{\tau\wedge T}^{\tau-} - \mathcal{C}_t^{\tau-} + \mathbb{1}_{\{\tau \leq T\}} \mathrm{CVA}_{\tau-}\big), \tag{14}$$
$$\mathrm{FVA}_t^{\tau-} = \mathbb{E}_t^* \big(\mathcal{F}_{\tau\wedge T}^{\tau-} - \mathcal{F}_t^{\tau-} + \mathbb{1}_{\{\tau \leq T\}} \mathrm{FVA}_{\tau-}\big), \tag{15}$$

i.e. $\mathrm{MtM}_T = \mathrm{CVA}_T = \mathrm{FVA}_T = 0$ on $\{T < \tau\}$ and, for $t < \tau$,

$$\mathrm{MtM}_t = \mathbb{E}^*_t\big(\mathcal{P}_{\tau-} - \mathcal{P}_t + \mathrm{MtM}_{\tau-}\big), \tag{16}$$
$$\mathrm{CVA}_t = \mathbb{E}^*_t\big(\mathcal{C}_{\tau-} - \mathcal{C}_t + \mathrm{CVA}_{\tau-}\big), \tag{17}$$
$$\mathrm{FVA}_t = \mathbb{E}^*_t\big(\mathcal{F}_{\tau-} - \mathcal{F}_t + \mathrm{FVA}_{\tau-}\big), \tag{18}$$

i.e. by Lemma 3, for $t \leq T$,

$$\mathrm{MtM}'_t = \mathbb{E}_t\big(\mathcal{P}'_T - \mathcal{P}'_t\big), \tag{19}$$
$$\mathrm{CVA}'_t = \mathbb{E}_t\big(\mathcal{C}'_T - \mathcal{C}'_t\big), \tag{20}$$
$$\mathrm{FVA}'_t = \mathbb{E}_t\big(\mathcal{F}'_T - \mathcal{F}'_t\big). \tag{21}$$

Core Equity Tier I Capital

Proposition 1

The core equity of the bank satisfies

$$\mathrm{CET1} = \mathrm{CET1}_0 - L, \tag{22}$$

where L is the trading loss of the bank (i.e. of the CA desks), such that

$$L^{\tau-} = \mathcal{C}^{\tau-} + \mathcal{F}^{\tau-} + \mathrm{CA}^{\tau-} - \mathrm{CA}_0$$

is a local martingale on $[0, \tau \wedge T]$ without jump at τ; The $\mathbb{F}$ reduction L' of L is an $(\mathbb{F}, \mathbb{P})$ local martingale on $[0, T]$.

Economic capital

- Since contra-assets (not even talking about contra-liabilities) cannot be replicated, the regulator requires that capital be set at risk by the shareholders.
- The capital at risk (CR) of the bank is its resource devoted to cope with losses beyond their expected levels that are already taken care of by reserve capital $\mathrm{RC} = \mathrm{CA} = \mathrm{CVA} + \mathrm{FVA}$.
- Economic capital (EC) is the level of capital at risk that a regulator would like to see on an economic, structural basis, based on $\mathrm{CET1}$ depletions
- Recall from Proposition 1 that $\mathrm{CET1}$ depletions correspond to $L^{\tau-}$ in the present setup.
- For simplicity we assess EC on the following 'going concern" basis:

Definition 1

EC_t is the $(\mathfrak{F}_t, \mathbb{P})$ conditional 97.5% expected shortfall of $(L'_{t+1} - L'_t)$, killed at τ. ■

- Let $\mathbb{ES}_t(\ell)$ denote the $(\mathfrak{F}_t,\mathbb{P})$ conditional expected shortfall, at some level α (e.g. $\alpha = 97.5\%$), of an $\mathfrak{F}_T$ measurable, $\mathbb{P}$ integrable random variable ℓ. That is, denoting by $q_t^a(\ell)$ the $(\mathfrak{F}_t,\mathbb{P})$ conditional value at risk (left quantile) of level a of ℓ (cf. Artzner, Delbean, Eber, and Heath (1999)):

$$
\begin{aligned}
&\mathbb{ES}_t(\ell) = \mathbb{E}_t(\ell \,|\, \ell \geq q_t^a) \\
&\quad = (1-\alpha)^{-1} \int_\alpha^1 q_t^a(\ell)da \\
&\quad = \inf_{x \in \mathbb{R}} \Big((1-\alpha)^{-1} \mathbb{E}_t\big[(\ell - x)^+ + x\big] \Big) \\
&\quad = \sup \big\{ \mathbb{E}_t[\ell\chi] \,;\, \chi \text{ is } \mathfrak{F}_T \text{ measurable, } 0 \leq \chi \leq (1-\alpha)^{-1}, \mathbb{E}_t[\chi] = 1 \big\}.
\end{aligned}
$$

- For any integrable random variables ℓ_1 and ℓ_2, we have (cf. Lemma 6.10, Eq. (6.20) in Barrera et al. (2019) and its proof):

$$|\mathbb{ES}_t(\ell_1) - \mathbb{ES}_t(\ell_2)| \ \leq \ (1-\alpha)^{-1}\mathbb{E}_t[|\ell_1 - \ell_2|], \ \ 0 \leq t \leq T.$$

- Note incidentally that we will only deal with martingale loss and profit processes $L^{\tau-}$ and therefore centered loss variables ℓ, for which $\mathbb{ES}_t(\ell) \geq 0$ holds in view of its third formulation above.

Capital at risk, KVA, and dividends

Assumption 1

The risk margin is loss-absorbing, hence part of capital at risk. ∎

As a consequence, shareholder capital at risk (SCR) is only the difference between the capital at risk (CR) of the bank and the risk margin $(\mathrm{RM} = \mathrm{KVA})$, i.e.

$$\mathrm{SCR} = \mathrm{CR} - \mathrm{KVA}. \tag{23}$$

Given a positive target hurdle rate h:

Definition 2

We set

$$\mathrm{CR} = \max(\mathrm{EC}, \mathrm{KVA}), \tag{24}$$

for a KVA process such that $\mathrm{KVA}_T = 0$ on $\{T < \tau\}$ and

$$(-\mathrm{KVA}^{\tau-}) \text{ has for drift coefficient } h\mathrm{SCR} \text{ killed at } \tau, \blacksquare$$

i.e. $\mathrm{KVA}_T = 0$ on $\{T < \tau\}$ and

$$\mathrm{KVA}_t = \mathbb{E}_t^*\Big[\int_t^{\tau\wedge T} h\mathrm{SCR}_s ds + \mathrm{KVA}_{\tau-}\Big], \quad t < \tau, \blacksquare$$

i.e.

$$\mathrm{KVA}'_t = \mathbb{E}_t\Big[\int_t^T h\mathrm{SCR}'_s ds\Big],\;\; 0 \le t \le T. \tag{25}$$

Note that, in view of (23) and (24), (25) is in fact a KVA' equation, namely

$$\begin{aligned}\mathrm{KVA}'_t =&\mathbb{E}_t\Big[\int_t^T h\big(\mathrm{CR}'_s - \mathrm{KVA}'_s\big) ds\Big]\\ &= \mathbb{E}_t\Big[\int_t^T he^{-h(s-t)} \max(\mathrm{EC}'_s, \mathrm{KVA}'_s) ds\Big],\;\; 0 \le t \le T.\end{aligned} \tag{26}$$

- Continuous-time analog of the risk margin formula under the Swiss solvency test cost of capital methodology: See Swiss Federal Office of Private Insurance (2006, Section 6, middle of page 86 and top of page 88).
- Can be used either in the direct mode, for computing the KVA corresponding to a given h, or in the reverse-engineering mode, for defining the “implied hurdle rate” associated with the actual RM level on the risk margin account of the bank.

Proposition 2

Shareholder capital (i.e. equity) $\mathrm{SHC} = \mathrm{SCR} + \mathrm{UC}$ *satisfies* $\mathrm{SHC} = \mathrm{SHC}_0 + \mathcal{D}$, *where*

$$\mathcal{D} = -(L^{\tau-} + \mathrm{KVA}^{\tau-} - \mathrm{KVA}_0),$$

is a submartingale with drift coefficient $h\mathrm{SCR}$ *on* $[0, \tau \wedge T]$, *without jump at* τ.

- Cost of capital proxies have always been used to estimate return on equity (ROE). The KVA is a refinement, dynamic and fine-tuned for derivative portfolios, but the base ROE concept itself is far older than even the CVA.
- In particular, the KVA is very useful in the context of collateral and capital optimization.

Portfolio-Wide XVAs are nonnegative

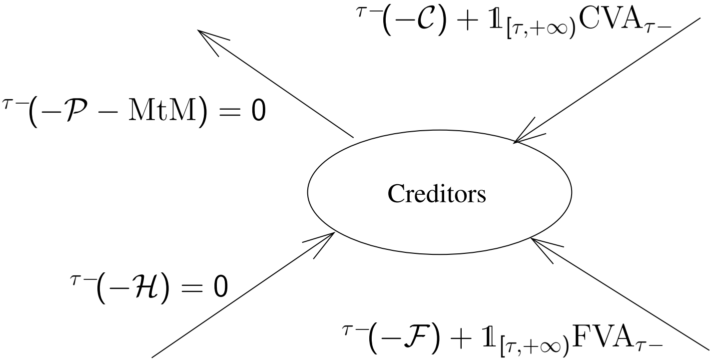
- and, even though we do crucially include the default of the bank itself in our modeling, unilateral
  - computed "under the bank survival probability measure"
- This makes them naturally in line with the regulatory requirement that capital should not diminish as an effect of the sole deterioration of the bank credit spread

Trading cash flows from bank default onward

- MtM,CVA, FVA, and KVA are so far unconstrained on $[\![\tau, +\infty[\![\bigcap \big(\{\tau \le T\} \times \mathbb{R}_+\big)$.
- We define the three XVA processes as zero there.
- As they already vanish on $[T, +\infty)$ if $T < \tau$, either of them, say Y, is in fact killed at $\tau \wedge T$, hence such that

$$^{\tau -}Y = \mathbb{1}_{[\![\tau,+\infty[\![}(Y_\tau - Y_{\tau-}) = -\mathbb{1}_{[\![\tau,+\infty[\![} Y_{\tau-} = -\mathbb{1}_{[\![\tau,+\infty[\![} Y_\tau^{\tau-}.$$

- As for MtM, we suppose, for clean valuation consistency across different banks (with hedging in mind), that it is not only the shareholder value process of $\mathcal{P}$, but also its value process in the first place.
- This determines MtM on $[0, T]$ and implies that $\mathcal{P} + \mathrm{MtM}$ is a martingale with martingale $(\cdot)^{\tau-}$ component.


$^{\tau-}(-\mathcal{C}) + \mathbb{1}_{[\tau,+\infty)}\mathrm{CVA}_{\tau-}$
$^{\tau-}(-\mathcal{P} - \mathrm{MtM}) = 0$
Creditors
$^{\tau-}(-\mathcal{H}) = 0$
$^{\tau-}(-\mathcal{F}) + \mathbb{1}_{[\tau,+\infty)}\mathrm{FVA}_{\tau-}$

Contra-liabilities

Definition 3

By contra-liability value process CL, we mean $\mathrm{CL} = \mathrm{DVA} + \mathrm{FDA}$, where

- DVA (debt valuation adjustment) is the value process of $^{\tau-}(-\mathcal{C}) + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{CVA}_{\tau-}$;
- FDA (funding debt adjustment) is the value process of $^{\tau-}(-\mathcal{F}) + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{FVA}_{\tau-}$.

By fair valuation of counterparty credit risk, we mean the value process FV of $\mathcal{C} + \mathcal{F}$.

As is then immediate by the different martingale assumptions involved:

Lemma 4

We have $\mathrm{CL} = \mathrm{DVA} + \mathrm{FDA}$, *which is the value process of both* $^{\tau-}(-L)$ *and* $(-L)$. *Moreover, before* τ,

$$\mathrm{FVA} = \mathrm{FDA}\,,\ \ \mathrm{FV} = \mathrm{CA} - \mathrm{CL} = \mathrm{CVA} - \mathrm{DVA}. \blacksquare \tag{27}$$

Wealth Transfer Analysis

- We assume that the shareholders have no other business than their involvement within the bank.
- Like the bank clients, whose business with firms other than the bank (which provides their motivation for the deals) is not present in the model, creditors have to face the liquidation costs of the bank, which are outside the scope of the model.

Definition 4

We call wealth of the bank shareholders, $\mathcal{W}^{sh}$, the sum between their accumulated cash flows and the valuation of their future cash flows. ∎

- The wealth of the shareholders before entering the portfolio (“at time $0-$”) is implicitly (and conventionally) taken as zero in this definition.
- So $\mathcal{W}^{sh}$ is in fact a wealth *transfer*, namely the wealth transferred to the shareholders by the derivative portfolio of the bank
    - without the portfolio, their wealth process in the sense of Definition 4 would vanish identically.

Definition 5

We call wealth transfer to the creditor, denoted by $\mathcal{W}^{cr}$, the sum between the cash flows that they receive from the bank and the valuation of the corresponding future cash flows. ∎

Let

$$\mathrm{KVA}_t^{sh} = \mathbb{1}_{\{t<\tau\}}\mathbb{E}_t^* \int_t^{\tau} h\big(\mathrm{EC}_s - \mathrm{KVA}_s^{\tau-}\big)^+ ds\,,\ \ \mathrm{KVA}_t^{cr} = \mathbb{1}_{\{t<\tau\}}\mathbb{E}_t^*\mathrm{KVA}_{\tau-}. \tag{28}$$

Proposition 3

The shareholder and creditor wealth transfer processes are

$$\mathcal{W}^{sh} = -(L^{\tau-} + \mathrm{KVA}^{\tau-} - \mathrm{KVA}_0) + \mathrm{KVA}^{sh}, \quad (29)$$

$$\mathcal{W}^{cr} = ^{\tau-}(-L) + \mathrm{CL} + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{KVA}_{\tau-} + \mathrm{KVA}^{cr}. \quad (30)$$

Shareholder and creditor wealth transfers are martingales starting from KVA_0^{sh} and $\mathrm{CL}_0 + \mathrm{KVA}_0^{cr}$ at time 0.

Proof. The first part follows from Definition 4 by inspection of the related cash flows, namely $\mathcal{D}$ as per (27) for shareholders and $^{\tau-}(-L) + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{KVA}_{\tau-}$ for creditors (recalling for (29) that the $(-L^{\tau-})$ component of $\mathcal{D}$ is zero-valued, as a martingale).
We have

$$\mathcal{W}^{sh} + \mathcal{W}^{cr} = \mathrm{KVA}_0 + \mathrm{CL} - L. \quad (31)$$

As seen in Lemma 4, this is a martingale. So are also $\mathcal{W}^{cr}$ and, by difference, $\mathcal{W}^{sh}$. ∎

- Should the shareholders decide to put the bank in default at time 0 right after the portfolio has been set up, they should not make any profit or loss, otherwise this would be a form of shareholder arbitrage.
- The fact that the shareholder wealth transfer martingale $\mathcal{W}^{sh}$ starts from $\mathrm{KVA}_0^{sh} > 0$ (positive initial wealth transfer to shareholders, unless the KVA vanishes) might suggest that the derivative trading of the bank entails shareholder arbitrage.
- Yet, given the rules of default settlement, upon bank default, the residual value on the (reserve capital and) risk margin account of the bank goes to creditors. So the shareholders would not monetize KVA_0^{sh} by putting the bank in default at time 0 right after the portfolio has been set up.
- The positive initial wealth transfer to shareholders does not entail any shareholder arbitrage, at least not in this sense.

- Likewise, the fact that the creditor wealth transfer martingale $\mathcal{W}^{cr}$ starts from $\mathrm{CL}_0 + \mathrm{KVA}_0^{cr} > 0$ (unless both CL and the KVA vanish) might suggest that the derivative trading of the bank entails a riskless profit to creditors.
- However, the scope of the model does not include the liquidation costs.
- For the creditors to monetize the wealth transfer triggered to them by the derivative portfolio of the bank, the bank has to default and there is a substantial cost associated to that to the creditors.

- Assume, for the sake of the argument, that the bank would be able to hedge its own jump-to-default risk by selling a new deal delivering the cumulative cash flow stream (zero valued martingale) $\mathrm{CL} - \mathrm{CL}_0 - L$.
- Accounting for the new deal and assuming that the CA desks would pass to the client (at time 0) and shareholders (through resets later on) the modified add-on $\mathrm{CA} - \mathrm{CL} = \mathrm{FV}$ (instead of CA before without the hedge), then the amount that needs by borrowed by the CA desk for implementing the strategy is the same as before and the trading loss of the bank would become

$$\begin{aligned} &\mathcal{C} + \mathcal{F} + \mathrm{FV} - \mathrm{FV}_0 + \mathrm{CL} - \mathrm{CL}_0 - L \\ &= \mathcal{C} + \mathcal{F} + \mathrm{CA} - \mathrm{CA}_0 - L = L - L = 0. \end{aligned}$$

- So $\mathrm{FV} = \mathrm{CA} - \mathrm{CL} = \mathrm{CVA} - \mathrm{DVA}$ is the cost of replicating counterparty risk in a theoretical, complete counterparty risk market;
- However:
  - the hedge of ${}^{\tau-}L$ is impossible because a bank cannot (is not even allowed) to sell credit protection on itself;
  - Hedging out $L^{\tau-}$ is not practical either, even in the case of a theoretical default-free bank, by lack of sufficiently liquid CDS instruments on the clients.
- Hence the shareholder and creditor wealth transfers can be interpreted as the wealth transferred to them by the trading of the bank, due to the inability of the bank to hedge counterparty risk.

Trade Incremental Cost-of-Capital XVA Policy

In the (realistic) case of an incremental portfolio, at each new trade, the funds transfer price (all-inclusive XVA add-on) sourced from the client is

$$\begin{aligned}\mathrm{FTP} &= \Delta\mathrm{CA} + \Delta\mathrm{KVA} = \Delta\mathrm{CVA} + \Delta\mathrm{FVA} + \Delta\mathrm{KVA} \\ &= \Delta\mathrm{FV} + \Delta\mathrm{CL} + \Delta\mathrm{KVA},\end{aligned}$$

computed on a trade incremental run-off basis.

- Meant incrementally at every new deal, the above FTP can be interpreted dynamically as the cost of the possibility for the bank to go into run-off,
  - i.e. lock its portfolio and let it amortize in the future,

  while staying in line with shareholder interest, from any point in time onward if wished.
- A "soft landing" or "anti-Ponzi" corrective pricing scheme accounting for counterparty risk incompleteness

Theorem 1

Under a trade incremental cost-of-capital XVA approach, consistently between and throughout deals: shareholder equity SHC *is a submartingale on* $\mathbb{R}_+$*, with drift coefficient* $h\mathrm{SCR}$ *killed before* τ*.*

- Hence, the preservation of the balance conditions in between and throughout deals yields a sustainable strategy for profits retention, which is already the key principle behind the Eurozone Solvency II insurance regulation.

Outline

Setup

- Bilateral netting sets c with value P^c assumed equal to their bank shareholder value
  - The case of centrally cleared derivative portfolios will be addressed later
- Default time, survival indicator, and (predictable) recovery rate of the related client denoted by τ_c, J^c, and R_c
  - Analogous data τ, J, and R regarding the bank itself, with risky funding spread process λ
  - All default times assumed positive
- Close-out periods ending at (stopping) times $\tau_c^\delta \geq \tau_c$ and $\tau^\delta \geq \tau$
  - so that the liquidation of the netting set c effectively happens at time $t_c = \tau_c^\delta \wedge \tau^\delta$, whereas margins are frozen since $s_c = \tau_c \wedge \tau$
- We consider a risky funding asset for the bank defined by $U_0 = 1$ and, for $t \geq 0$,

$$dU_t = \lambda_t U_t dt + (1 - R_t) U_{t-} dJ_t = U_{t-} d\mu_t,$$

where $\lambda = (1 - R)\gamma$ and $d\mu_t = \lambda_t dt + (1 - R_t) dJ_t$.

Trading Cash Flows to the Clean Desks and MtM

The processes $\mathcal{P}$ and MtM correponding to the above setup are $\mathcal{P}_0 = 0$, $\mathrm{MtM}_0 = \sum_c P_0^c$ and, for $t \in (0, T]$,

$$d\mathcal{P}_t = \sum_c \Big(\mathbb{1}_{\{t < s_c\}} d\mathcal{P}_t^c + \boldsymbol{\delta}_{s_c}(dt) P_{s_c -}^c + \mathbb{1}_{\{s_c \le t \le t_c\}} (d\mathcal{P}_t^c + dP_t^c) \Big), \tag{32}$$

$$d\mathrm{MtM}_t = \sum_c \Big(\mathbb{1}_{\{t < s_c\}} dP_t^c - \boldsymbol{\delta}_{s_c}(dt) P_{s_c -}^c \Big). \tag{33}$$

Proposition 4

We have

$$\mathcal{P} = \sum_{c} \left((\mathcal{P}^c)^{t_c} + \mathbb{1}_{[\![s_c, +\infty [\![}(P^c)^{t_c} \right), \tag{34}$$

$$\mathrm{MtM} = \sum_{c} P^c \mathbb{1}_{[\![0, s_c [\![}, \text{ and} \tag{35}$$

$$\mathcal{P} + \mathrm{MtM} = \sum_{c} (\mathcal{P}^c + P^c)^{t_c}, \tag{36}$$

which is a martingale. The process MtM *is together the value process and a shareholder value process of* $\mathcal{P}$*.*

Proof. Recall all default times are assumed positive.
We have $\int_0^{\cdot} \mathbb{1}_{\{s_c \le t \le t_c\}} dP_t^c = \mathbb{1}_{[\![s_c,+\infty[\![}\big((P^c)^{t_c} - P^c_{s_c-}\big)$, hence (34) follows from (32).
We have $\mathrm{MtM}_0 = \sum_c P_0^c = \sum_c P_0^c \mathbb{1}_{\{0<s_c\}}$ and $\mathbb{1}_{\{t<s_c\}} dP_t^c - \boldsymbol{\delta}_{s_c}(dt) P^c_{s_c-} = d(P_t^c \mathbb{1}_{\{t<s_c\}})$, hence (35) follows from (33).
We have $(\mathcal{P} + \mathrm{MtM})_0 = \mathrm{MtM}_0 = \sum_c P_0^c = \sum_c (\mathcal{P}^c + P^c)_0^{t_c}$. Moreover, adding (32) and (33) yields $d(\mathcal{P} + \mathrm{MtM})_t = \sum_c \mathbb{1}_{\{t \le t_c\}} d(\mathcal{P}^c + P^c)_t$.
This proves (36).

As each of the $(\mathcal{P}^c + P^c)^{t_c}$ is a martingale, so is their sum $(\mathcal{P} + \mathrm{MtM})$. Moreover $\mathrm{MtM}_T = 0$, by (35). Hence MtM is the value process of $\mathcal{P}$. Moreover, $(\mathcal{P} + \mathrm{MtM})^{\tau-} = \sum_c ((\mathcal{P}^c + P^c)^{t_c})^{\tau-} = (\sum_c (\mathcal{P}^c + P^c)^{\tau-})^{t_c}$ is a martingale, by our assumption that each of the $(\mathcal{P}^c + P^c)^{\tau-}$ is a martingale. Hence MtM is also a shareholder value process of $\mathcal{P}$. ∎

Proposition 5

We have

$$\mathcal{C} =$$

$$\sum_{c;\tau_c \le \tau^\delta} (1-R_c)\Big((P^c+\mathcal{P}^c)_{\tau_c^\delta \wedge \tau^\delta} - (\mathcal{P}^c + \mathrm{VM}^c + \mathrm{RIM}^c)_{(\tau_c\wedge\tau)-}\Big)^+ \mathbb{1}_{[\tau_c^\delta\wedge\tau^\delta,\infty)}$$

$$-(1-R)\sum_{c;\tau\le\tau_c^\delta}\Big((P^c+\mathcal{P}^c)_{\tau^\delta\wedge\tau_c^\delta} - (\mathcal{P}^c + \mathrm{VM}^c - \mathrm{PIM}^c)_{(\tau\wedge\tau_c)-}\Big)^- \mathbb{1}_{[\tau^\delta\wedge\tau_c^\delta,\infty)}$$

$$\mathcal{F} = \int_0^\cdot J_t\lambda_t\Big(\sum_c J^c(P^c-\mathrm{VM}^c) - \mathrm{CA} - \max(\mathrm{EC},\mathrm{KVA})\Big)_t^+ dt$$

$$-(1-R)\Big(\sum_c J^c(P^c-\mathrm{VM}^c) - \mathrm{CA} - \max(\mathrm{EC},\mathrm{KVA})\Big)_{\tau-}^+ \mathbb{1}_{[\tau,+\infty)}$$

$$+\int_0^\cdot J_t\lambda_t\sum_c J_t^c\mathrm{PIM}_t^c dt - (1-R)\sum_c J_{\tau-}^c\mathrm{PIM}_{\tau-}^c\mathbb{1}_{[\tau,+\infty)}.$$

Proof. For each netting set c, during the liquidation period $[\![s_c, t_c]\!]$, the CVA desk loses $P^c_{s_c-}$ at time s_c from the transfer of the property of the clean margin amount related to the netting set c, plus $(\mathcal{P}^c + P^c)_{t_c} - (\mathcal{P}^c + P^c)_{s_c-}$ channeled to the clean desks during the period, summing up to $\mathcal{P}^c_{t_c} - \mathcal{P}^c_{s_c-} + P^c_{t_c} := x_c$.
In addition, at the liquidation time t_c, the CVA desk receives the following amount from the client, depending on x_c and on whether the client of the netting set c but not the bank is in default at t_c, the bank but not the client is in default at t_c, or both are in default at t_c (the three main cases in what follows):

$$
\begin{aligned}
&\mathbb{1}_{\{\tau_c \le t_c < \tau\}}\Big(\mathbb{1}_{\{x_c \le \Gamma^c_{s_c-}\}} x_c + \mathbb{1}_{\{x_c > \Gamma^c_{s_c-}\}}\big(\Gamma^c_{s_c-} + R^c_{t_c}(x_c - \Gamma^c_{s_c-})\big)\Big) + \\
&\mathbb{1}_{\{\tau \le t_c < \tau_c\}}\Big(\mathbb{1}_{\{(-x_c) \le \bar{\Gamma}^c_{s_c-}\}} x_c - \mathbb{1}_{\{(-x_c) > \bar{\Gamma}^c_{s_c-}\}}\big(\bar{\Gamma}^c_{s_c-} + R_{t_c}(-x_c - \bar{\Gamma}^c_{s_c-})\big)\Big) + \\
&\mathbb{1}_{\{\tau \vee \tau_c \le t_c\}}\Big(\mathbb{1}_{\{x_c > \Gamma^c_{s_c-}\}}\big(\Gamma^c_{s_c-} + R^c_{t_c}(x_c - \Gamma^c_{s_c-})\big) + \\
&\qquad \mathbb{1}_{\{-\bar{\Gamma}^c_{s_c-} \le x_c \le \Gamma^c_{s_c-}\}} x_c - \\
&\qquad \mathbb{1}_{\{(-x_c) > \bar{\Gamma}^c_{s_c-}\}}\big(\bar{\Gamma}^c_{s_c-} + R_{t_c}(-x_c - \bar{\Gamma}^c_{s_c-})\big)\Big).
\end{aligned}
$$

Retrieving x_c, the total gain of the CVA desk is

$$\begin{aligned}&\mathbb{1}_{\{\tau_c\le t_c<\tau\}}\big(-(1-R^c_{t_c})(x_c-\Gamma^c_{s_c-})^+\big)+\\&\mathbb{1}_{\{\tau\le t_c<\tau_c\}}(1-R_{t_c})(x_c+\bar\Gamma^c_{s_c-})^-+\\&\mathbb{1}_{\{\tau\vee\tau_c\le t_c\}}\Big(\mathbb{1}_{\{x_c>\Gamma^c_{s_c-}\}}\big(-(1-R^c_{t_c})(x_c-\Gamma^c_{s_c-})^+\big)\\&\qquad\qquad+\mathbb{1}_{\{(-x_c)>\bar\Gamma^c_{s_c-}\}}(1-R_{t_c})(x_c+\bar\Gamma^c_{s_c-})^-\Big).\end{aligned}$$

The loss of the CVA desk is the opposite, which simplifies into

$$\mathbb{1}_{\{\tau_c\le t_c\}}(1-R^c_{t_c})(x_c-\Gamma^c_{s_c-})^+-\mathbb{1}_{\{\tau\le t_c\}}(1-R_{t_c})(x_c+\bar\Gamma^c_{s_c-})^-.$$

By summation over the netting sets c, we obtain the formula for $\mathcal{C}$.

Let $D = J\sum_c J^c(P^c - \mathrm{VM}^c)$ denote the difference between the clean margin MtM posted by the CA desks to the clean desks and the collateral received by the CA desks from the clients, and $E = J\sum_c J^c\mathrm{PIM}^c$, the initial margin posted by the bank.

The latter has to be unsecurely borrowed in totality

The variation margin VM^c received from clients and the clean margin amount P^c (cf. (35)) are fungible (this is all re-hypothecable collateral).

The amounts $\mathrm{RC} = \mathrm{CA}$ and $\mathrm{CR} = \max(\mathrm{EC}, \mathrm{KVA})$ sitting on the reserve capital and capital at risk accounts of the bank provide risk-free sources of re-hypothecable collateral. Hence the risky borrowing needs of the bank for re-hypothecable collateral, which can only be satisfied using U, reduce to $(D - \mathrm{CA} - \max(\mathrm{EC}, \mathrm{KVA})^+)$.

In sum, the funding strategy of the FVA desk of the bank consists in maintaining a short position of $\frac{((D-\mathrm{CA}-\max(\mathrm{EC},\mathrm{KVA}))^{+}+E)^{\tau-}}{U^{\tau-}}$ units of the asset U.
Given our use of the risk-free asset as numéraire, the self-financing condition on the funding strategy of the FVA desk is written as

$$d\mathcal{F}_t = \frac{((D-\mathrm{CA}-\max(\mathrm{EC},\mathrm{KVA})^{+}+E)_t^{\tau-}}{U_t^{\tau-}}dU_t \tag{37}$$

This yields the formula for $\mathcal{F}$. ∎

Reduced XVA Equations

The reduced XVA equations are, all under $(\mathbb{F}, \mathbb{P})$ from now on and dismissing the ' notation to alleviate the equations, and distinguishing hereafter between FVA and MVA as the respective costs of funding variation and initial margin:

$L_0 = 0$ and, for $t \in (0, T]$,

$$
\begin{aligned}
dL_t &= \sum_c (1 - R_c)\Big((P^c + \mathcal{P}^c)_{\tau_c^\delta} - (\mathcal{P}^c + \mathrm{VM}^c + \mathrm{RIM}^c)_{\tau_c -}\Big)^+ \boldsymbol{\delta}_{\tau_c^\delta}(dt) \\
&+ \lambda_t \Big(\sum_c J^c (P^c - \mathrm{VM}^c) - \mathrm{CA} - \max(\mathrm{EC}, \mathrm{KVA})\Big)_t^+ dt \qquad (38) \\
&+ \lambda_t \sum_c J_t^c \mathrm{PIM}_t^c dt \\
&+ d\mathrm{CA}_t,
\end{aligned}
$$

where, for $0 \leq t \leq T$,

$$\mathrm{CA}_t = \underbrace{\mathbb{E}_t \sum_{t<\tau_c^\delta} (1 - R_c)\Big((P^c + \mathcal{P}^c)_{\tau_c^\delta} - (\mathcal{P}^c + \mathrm{VM}^c + \mathrm{RIM}^c)_{\tau_c-}\Big)^+}_{\mathrm{CVA}_t}$$

$$+ \underbrace{\mathbb{E}_t \int_t^T \lambda_s \Big(\sum_c J^c(P^c - \mathrm{VM}^c) - \mathrm{CA} - \max(\mathrm{EC}, \mathrm{KVA})\Big)_s^+ ds}_{\mathrm{FVA}_t} \quad (39)$$

$$+ \underbrace{\mathbb{E}_t \int_t^T \lambda_s \sum_c J_t^c \mathrm{PIM}_s^c ds,}_{\mathrm{MVA}_t}$$

$$\mathrm{KVA}_t = h\mathbb{E}_t \int_t^T e^{-h(s-t)} \max\left(\mathrm{EC}_s, \mathrm{KVA}_s\right) ds. \quad (40)$$

The XVA(′) BSDEs are well posed,

- including in the anticipated, "McKean" case where capital can be used for funding variation margin so that also FVA depends on L (via EC).
  - Theorem 4.1 in Crépey, Sabbagh, and Song (2020)
- Let CVA and MVA be as above, $L^{(0)} = \mathrm{KVA}^{(0)} = 0$, and

$$\mathrm{FVA}_t^{(0)} = \mathbb{E}_t \int_t^T \lambda_s \Big(\sum_c J^c (P^c - \mathrm{VM}^c) - \mathrm{CA}^{(0)} \Big)_s^+ ds, 0 \le t \le T,$$

  where $\mathrm{CA}^{(0)} = \mathrm{CVA} + \mathrm{FVA}^{(0)} + \mathrm{MVA}$
  - FVA accounting only for the re-hypothecation of the variation margin received on hedges, and reflecting the possible use of reserve (but not risk) capital as VM

Convergent Picard iteration

For $k \geq 1$, writing explicitly $\mathrm{EC} = \mathrm{EC}(L)$ to emphasize the dependence of EC on L: $L_0^{(k)} = 0$ and, for $t \in (0, T]$

$$dL_t^{(k)} = \sum_c (1 - R_c)\Big((P^c + \mathcal{P}^c)_{\tau_c^\delta} - (\mathcal{P}^c + \mathrm{VM}^c + \mathrm{RIM}^c)_{\tau_c -}\Big)^+ \boldsymbol{\delta}_{\tau_c^\delta}(dt)$$

$$+ \lambda_t \Big(\sum_c J^c (P^c - \mathrm{VM}^c) - \mathrm{CA}^{(k-1)} - \max\left(\mathrm{EC}(L^{(k-1)}), \mathrm{KVA}^{(k-1)}\right)\Big)_t^+ dt$$

$$+ \lambda_t \sum_c J_t^c \mathrm{PIM}_t^c dt + d\mathrm{CA}_t^{(k-1)}$$

$$\mathrm{KVA}_t^{(k)} = h\mathbb{E}_t \int_t^T e^{-h(s-t)du} \max\left(\mathrm{EC}_s(L^{(k)}), \mathrm{KVA}_s^{(k)}\right) ds$$

$$\mathrm{CA}_t^{(k)} = \mathrm{CVA}_t + \mathrm{FVA}_t^{(k)} + \mathrm{MVA}_t \text{ where } \mathrm{FVA}_t^{(k)} =$$

$$\mathbb{E}_t \int_t^T \lambda_s \Big(\sum_c J^c (P^c - \mathrm{VM}^c) - \mathrm{CA}^{(k)} - \max\left(\mathrm{EC}(L^{(k)}), \mathrm{KVA}^{(k)}\right)\Big)_s^+ ds. \tag{41}$$

- Numerically, one iterates in (41) as many times as is required to reach a fixed point within a preset accuracy.
- Two to three iterations ($k = 2$ or 3) found sufficient in our case studies.

- More iterations do not bring significant changes as, in the above:
  - the FVA feeds into economic capital only through FVA volatility and the economic capital feeds into FVA through a capital term which is typically not FVA dominated
  - in most cases we have that $\mathrm{CR} = \mathrm{EC}$. The inequality only stops holding when the hurdle rate *h* is very high and the term structure of EC starts very low and has a sharp peak in a few years, which is quite unusual for a portfolio held on a run-off basis, as considered in XVA computations, which amortizes in time.

Outline

- First generation XVA metrics, i.e. simple CVA and over-simplified FVA metrics, can be computed without simulating the defaults of the clients, only relying on their credit curves
  - and the one of the bank itself, in the case of the FVA

Contra-assets valuation in the case of a single netting set

- Consider a bank engaged into bilateral trading with a single client, with promised cash flows process D and final maturity of the portfolio T.
- Let R_c denote the recovery rate of the client in case it defaults at time τ_c.

- Let VM denote the variation margin
    - collateral guarantee tracking the value of the client portfolio of the bank,
    - counted positively when received by the bank
- Let PIM and RIM denote the initial margins posted and received by the bank on its client portfolio
    - collateral guarantees set on top of VM against gap risk
- Let λ denote the risky funding spread of the bank and forget the feedback of CR into FVA for simplicity.

$\rightarrow$ $\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA} + \mathrm{MVA}$, where

$$\mathrm{CVA}_t = \mathbb{E}_t\Big[\mathbb{1}_{\{t<\tau_c\le T\}}(1-R_c)\big(\mathrm{MtM}_{\tau_c} + D_{\tau_c} - D_{\tau_c-} - \mathrm{VM}_{\tau_c} - \mathrm{RIM}_{\tau_c}\big)^+$$

$$\mathrm{FVA}_t = \mathbb{E}_t\int_t^{\tau_c\wedge T}\lambda_s\Big(\mathrm{MtM}_s - \mathrm{VM}_s - \mathrm{CA}_s\Big)^+ ds.$$

Gap Risk

- The jump process $(D_{\tau_c} - D_{\tau_c-})$ of the contractually promised cash flows contributes to the CVA exposure of the bank

$$Q = \big(\mathrm{MtM} + D - D_- - \mathrm{VM} - \mathrm{RIM}\big), \tag{42}$$

because it fails to be paid by the client if it defaults.

- Assuming deterministic interest rates, the time 0 CVA can be rewritten as

$$\mathrm{CVA}_0 = (1 - R_c) \int_0^T \mathrm{EPE}(t)\mathbb{P}(\tau_c \in dt), \qquad (43)$$

for an expected positive exposure (EPE) defined as

$$\mathrm{EPE}(t) = \mathbb{E}(Q_s^+ | s = \tau_c)_{|\tau_c = t}.$$

XVA Exposure-Based Computational Approaches

- First, compute the mark-to-market cube
  - counterparty-risk free valuation of each netting set (only one in this chapter) in any scenario and future time point
- Then, for each netting set, integrate in time the ensuing exposure (EPE) profile "against the client CDS curve" in order to obtain the corresponding CVA as per (43).
- A similar approach can be applied to the computation of simplified FVA metrics, using the funding curve of the bank itself as integrator of the related funding exposure profile

- This approach is popular with practitioners (mainstream in most banks) as
  - it decouples the credit and the market sides of the problem
  - it avoids the simulation of client defaults
    - variance reduction
  - it siloes the computations by netting set
    - "divide and conquer"

- However, as we will see in more detail later, FVA computations siloed by netting set implies to work with over-simplified FVA metrics
- Moreover, an exposure-based XVA approach essentially assumes independence between the market and credit sides of the problem
  - Beyond various patches such as the ones proposed in Pykhtin (2012), Hull and White (2012), Li and Mercurio (2015), or Iben Taarit (2018), it is hard to extend rigorously to wrong-way risk
    - Risk of adverse dependence between the credit risk of the counterparty (or the bank itself) and the market exposure

- In addition, an exposure-based approach is static, whereas a dynamic perspective is required for:
  - (even partial, but rigorous) XVA hedging purposes,
  - properly accounting for the feedback effects between different XVAs (e.g. from the CVA into the FVA),
  - economic capital and KVA computations.
- Last, an exposure-based XVA approach comes with little error control

Trade Incremental XVA Computations

- An MtM store-and-reuse approach to trade incremental XVA computations circumvents repeated valuations at the cost of disk memory.
- It exploits the trade additivity of clean valuation by recording the MtM paths of the initial portfolio on a disk.
- For every new deal, the augmented portfolio exposure is obtained by adding, along the paths of the risk factors, the mark-to-market of the initial portfolio and of the new deal.
- This augmented portfolio exposure is then plugged into the XVA engine.

- An optimally implemented MtM store-and-reuse approach brings down trade incremental XVA computations to the time of generating the clean price process of the trade itself, instead of the one of the augmented portfolio as a whole.
- Another advantage of this approach is its compliance with desk segregation.
  - As far as clean valuation is concerned, the XVA desks just use the pricers of the clean desks.
  - Hence, the MtM process plugged into the XVA computations is consistent with the one used for producing the market risk hedging sensitivities.

- We denote by $\widehat{Y}$ a suitable estimation operator of a process Y at all (outer) nodes of a Monte Carlo XVA engine.
  - In particular, $\widehat{\mathrm{MtM}}$ is the fully discrete counterpart of the MtM process of a portfolio, namely the clean value of the portfolio at future exposure dates in a time grid and for different scenario paths.
- The conditions for a straightforward and satisfactory application of the MtM store-and-reuse approach to a given XVA metric are as follows, referring by indices *init*, *incr*, and *augm* to the initial portfolio, the new deal, and the augmented portfolio, and by X to the underlying risk factors, with drivers Z

  1. (Lagged market data) $\widehat{\mathrm{MtM}}^{incr}$ should be based on the same time, say 0, and initial condition X_0 (including, modulo calibration, market data), as $\widehat{\mathrm{MtM}}^{init}$. This condition ensures a consistent aggregation of $\widehat{\mathrm{MtM}}^{init}$ and $\widehat{\mathrm{MtM}}^{incr}$ into $\widehat{\mathrm{MtM}}^{augm}$.

2. (Common random numbers) $\widehat{\text{MtM}}^{incr}$ should be based on the same paths of the drivers as $\widehat{\text{MtM}}^{init}$. Otherwise, numerical noise (variance) would arise during $\widehat{\text{MtM}}$ aggregation;
3. (No nested simulation of the portfolio exposure required?) The formula for the corresponding (portfolio-wide, time-0) XVA metric should be estimatable without nested simulation, only based on the portfolio exposure rooted at $(0, X_0)$; A priori, additional simulation level makes nonpractical the MtM store-and-reuse idea of swapping execution time against storage;

These conditions have the following implications:

1. induces a lag between the market data (of the preceding night) that are used in the computation of $\widehat{\mathrm{MtM}}^{incr}$ and the exact MtM^{incr} process; when the lag on market data becomes unacceptably high (because of time flow and/or volatility on the market), a full reevaluation of the portfolio exposure is required.
2. implies to store the paths of the driver Z that were simulated for the purpose of obtaining $\widehat{\mathrm{MtM}}^{init}$; it also puts a bound on the accuracy of the estimation of MtM^{incr}, since the number of Monte Carlo paths is imposed by the initial run. The XVA desks may want to account for some wrong way risk dependency between the portfolio exposure and counterparty credit risk: this is doable provided the trajectories of the drivers and/or risk factors are shared between the clean and XVA desks;

3. seems to ban second order generation XVAs, such as CVA in presence of initial margin, but these can in fact be included with the help of regression techniques;

Embedding of an MtM store-and-reuse approach into the trade incremental XVA engine of a bank.

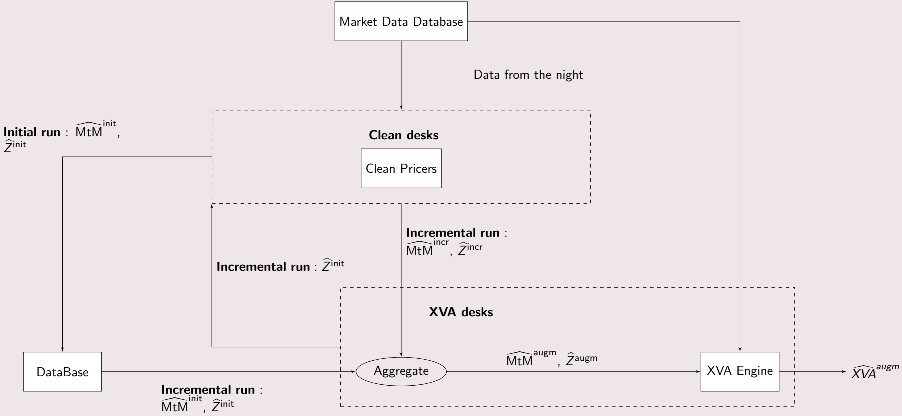
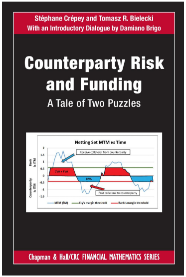


Figure: MtM store-and-reuse implementation of a trade incremental XVA engine with drivers Z.

Common Shock Model of Portfolio Credit Risk

- The numerical solution of more refined XVA equations requires to simulate client defaults.
- This requires a dynamic credit portfolio model
- In later chapters, the common shock model of



will be our reference setup for modeling client defaults in the context of multiple-netting set XVA computations.

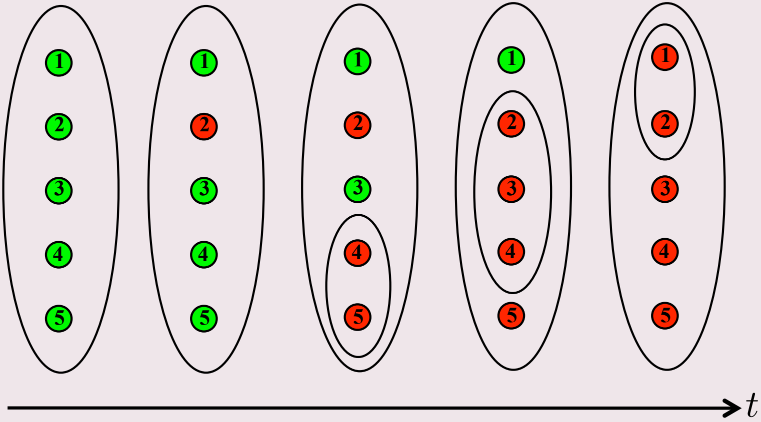
- Define a family $\mathcal{Y}$ of "shocks", i.e. subsets $Y \subseteq \{1, \ldots, n\}$ of clients, usually consisting of the singletons $\{1\}$, $\{2\}$, $\ldots$, $\{n\}$ and a few "common shocks" representing simultaneous defaults
  - Having already removed from the model the default time of the bank itself, assumed an invariance time
- Define, for $Y \in \mathcal{Y}$, affine (as well as their sums) intensity processes γ_t^Y driven by independent BM W_t^Y under the measure $\mathbb{P}$, and

$$\tau_Y = \inf\{t > 0; \int_0^t \gamma_s^Y ds \geq \epsilon_Y\}$$

  for $\mathbb{P}$ independent standard exponential random variables ϵ_Y
- We then define, for every $i \in N$,

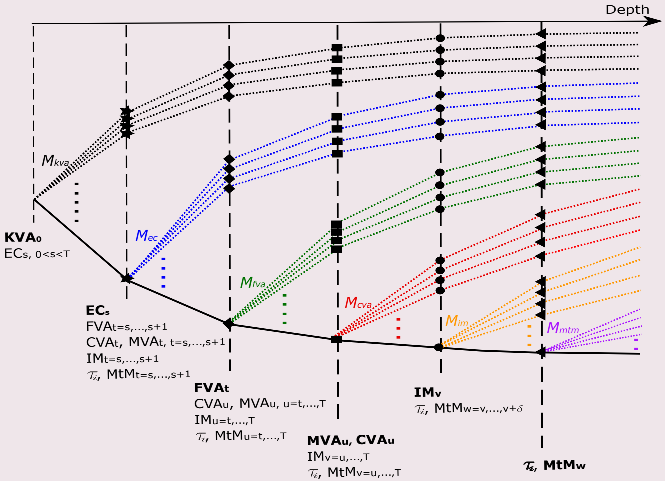
$$\tau_i = \bigwedge_{Y \in \mathcal{Y};\, Y \ni i} \tau_Y$$

Example: $n = 4$ and
$\mathcal{Y} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{4, 5\}, \{2, 3, 4\}, \{1, 2\}\}$.



Outline

XVA dependence tree, from the most outer layer to the most inner one. The sub-tree rooted at the lowest node on each inner layer should be duplicated starting from each node above on the same layer.



Assuming n netting sets and one funding set:

- nonlinear CVA terminal payoffs, hence the CVA can only be computed at the level of each netting set
- semilinear FVA equation, hence the FVA should be computed at the level of the overall portfolio and involves time backwardation
- semilinear KVA equation also involving a time backwardation at the level of the overall portfolio, fed by future conditional risk measures of the trading loss process of the bank, which itself involves future fluctuations of other XVAs, as these are part of the bank liabilities
- Moreover, if capital is deemed fungible with variation margin, coupled dependence between FVA and L

FVA (inner) backwardations. The yellow pavings symbolize regressions. The fine blue paths denote inner resimulated paths.

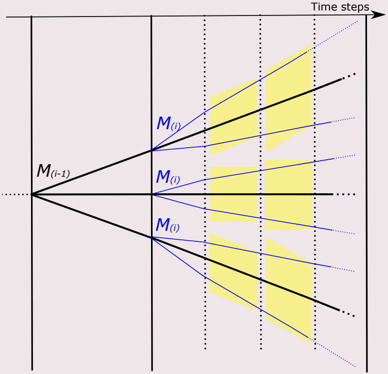


Figure: FVA Inner Backwardations.

- Holistic computations encompassing all the derivative contracts of the bank
- Driven, to a large extent, by tail default scenarios
- Yet need accuracy so that incremental XVA computations are not in the numerical noise of the machinery

- In this chapter, which is mainly based on Abbas-Turki, Diallo, and Crépey (2018), we explore a full simulation, nested Monte Carlo (NMC) XVA computational approach, optimally implemented on GPUs.
  - higher XVA layers are launched first and trigger nested simulations on-the-fly whenever required in order to compute an item from a lower XVA layer

XVA NMC Design Parameterization

- Assuming the same variance created through the different layers of the tree, the mean square error (MSE) of an $M_{(0)} \otimes M_{(1)} \otimes \ldots \otimes M_{(i)} = M_{(0)} \otimes M_{(0)} \otimes \ldots \otimes M_{(0)}$ NMC is the same as the one of an $M_{(0)} \otimes \sqrt{M_{(0)}} \otimes \ldots \otimes \sqrt{M_{(0)}}$ NMC
  - $\mathrm{O}(M_{(0)}^{-1})$
  - Result based on a uniform control of the moments of the error and regularity assumptions that are needed to justify the application of Taylor formula
  - cf. Assumption 1 in Gordy and Juneja (2010) and Rainforth, Cornish, Yang, Warrington, and Wood (2017).

Proof in the XVA context *(sketched)*. Denoting unbiased Monte Carlo estimators of the (time 0) XVAs and of the MtM by $\widehat{\cdot}$ and referring to the dependence between XVAs or MtM in functional form, e.g. $\mathrm{CVA}(\mathrm{MtM})$, we have:

$$\begin{aligned}\mathrm{MSE}^2_{cva} &= \mathbb{E}\big(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}}) - \mathrm{CVA}(\mathrm{MtM})\big)^2 \\ &= \mathbb{E}(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}}) - \mathbb{E}(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})))^2 + [\mathbb{E}(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})) - \mathrm{CVA}(\mathrm{MtM})]^2\end{aligned}$$

In the context of an outer CVA Monte Carlo, the first term is a variance in $\mathrm{O}(\frac{1}{M_{cva}})$. The second term is the square of a bias that can be Taylor expanded as follows (as $\mathbb{E}(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})) = \mathbb{E}(\mathrm{CVA}(\widehat{\mathrm{MtM}}))$):

$$
\begin{aligned}
&\mathbb{E}[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})] - \mathrm{CVA}(\mathrm{MtM}) = \partial_{\mathrm{MtM}}\mathrm{CVA} \times (\mathbb{E}\widehat{\mathrm{MtM}} - \mathrm{MtM}) \\
&+ \frac{1}{2}\partial^2_{\mathrm{MtM}^2}\mathrm{CVA} \times \mathbb{E}(\widehat{\mathrm{MtM}} - \mathrm{MtM})^2 + \mathrm{O}\Big(\big(\mathbb{E}(\widehat{\mathrm{MtM}} - \mathrm{MtM})^2\big)^2\Big),
\end{aligned}
$$

where the first line vanishes, because $\widehat{\mathrm{MtM}}$ estimates without bias MtM, whereas $\mathbb{E}(\widehat{\mathrm{MtM}} - \mathrm{MtM})^2$ is a variance in $\mathrm{O}(\frac{1}{M_{mtm}})$.

In conclusion, we obtain

$$
\mathrm{MSE}^2_{cva} = \mathrm{O}(\frac{1}{M_{cva}} + \frac{1}{M^2_{mtm}}).
$$

In the case of a multiply nested XVA computation, say an outer FVA Monte Carlo, we have:

$$\begin{aligned}\mathrm{MSE}^2_{fva} &= \mathbb{E}(\widehat{\mathrm{FVA}}(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})) - \mathrm{FVA}(\mathrm{CVA}(\mathrm{MtM})))^2\\ &= \mathbb{E}(\widehat{\mathrm{FVA}}(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})) - \mathbb{E}(\widehat{\mathrm{FVA}}\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}}))))^2 +\\ &\quad [\mathbb{E}(\widehat{\mathrm{FVA}}\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}}))) - \mathrm{FVA}(\mathrm{CVA}(\mathrm{MtM}))]^2.\end{aligned}$$

The term in the first line is a variance like $\mathrm{O}(\frac{1}{M_{fva}})$. The second line is the square of a bias that can be Taylor expanded as follows (as $\mathbb{E}(\widehat{\mathrm{FVA}}(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}}))) = \mathbb{E}(\mathrm{FVA}(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}}))))$:

$$
\begin{aligned}
&\mathbb{E}(\widehat{\mathrm{FVA}}(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}}))) - \mathrm{FVA}(\mathrm{CVA}(\mathrm{MtM})) = \\
&\quad \partial_{\mathrm{CVA}}\mathrm{FVA} \times (\mathbb{E}(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})) - \mathrm{CVA}(\mathrm{MtM})) \\
&\quad + \frac{1}{2}\partial^2_{\mathrm{CVA}^2}\mathrm{FVA} \times \mathbb{E}(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}}) - \mathrm{CVA}(\mathrm{MtM}))^2 \\
&\quad + \mathrm{O}\Big(\big(\mathbb{E}(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}}) - \mathrm{CVA}(\mathrm{MtM}))^2\big)^2 \Big),
\end{aligned}
$$

where $\mathbb{E}(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})) - \mathrm{CVA}(\mathrm{MtM})$ was seen before to be $\mathrm{O}(\frac{1}{M_{mtm}})$, whereas $\mathbb{E}(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}}) - \mathrm{CVA}(\mathrm{MtM}))^2$ is MSE^2_{cva}.

Hence

$$
\mathrm{MSE}^2_{fva} = \mathrm{O}(\frac{1}{M_{fva}} + \frac{1}{M^2_{cva}} + \frac{1}{M^2_{mtm}}). \blacksquare
$$

- Moreover, the variances (at least, the corresponding constants) are not homogeneous with respect to the stages.

Accordingly, the design of our XVA NMC algorithm reads as follows:

- Select layers of choice in a XVA NMC sub-tree of choice, with corresponding tentative number of simulations denoted by $M_{(0)}, \ldots, M_{(i)}$, for some $1 \leq i \leq 5$ (we assume at least one level of nested simulation).
- By dichotomy on $M_{(0)}$, reach a target relative error (in the sense of the outer confidence interval) for $M_{(0)} \otimes M_{(1)} \ldots \otimes M_{(i)}$ NMCs with $M_{(1)} = \ldots = M_{(i)} = \sqrt{M_{(0)}}$.
- For each j decreasing from i to 1, reach by dichotomy on $M_{(j)}$ a target bias (in the sense of the impact on the outer confidence interval) for $M_{(0)} \otimes M_{(1)} \otimes \ldots \otimes M_{(j)} \otimes \ldots \otimes M_{(i)}$ NMCs.

- For instance, considering the overall 5-layered XVA NMC, for M_{kva} of the order of $1e3$ (which can be enough on a simple portfolio in order to ensure a 5% confidence interval at a 95% confidence level), the above approach may lead to M_{mtm}, M_{im}, and M_{cva} somewhere between $1e2$ and $1e3$.

- As the FVA is obtained from the resolution of a BSDE that involves preconditioning, M_{fva} can be even smaller than $1e2$ without compromising the accuracy.

- Due to the approximation of the conditional expected shortfall risk measure involved in economic capital computations, M_{ec} has to be bigger than $1e3$ but usually can be smaller than $1e4$.

Multi-Level Monte Carlo

- Bourgey, De Marco, Gobet, and Zhou (2019) show how to improve further the performance of nested Monte Carlo by a multi-level approach as per Giles (2008).
- The different levels correspond to increasing sample sizes of the inner simulations.

Case Study

- Albanese, Caenazzo, and Crépey (2017)
- Representative banking portfolio with about 2,000 counterparties, 100,000 fixed income trades including swaps, swaptions, FX options, inflation swaps and CDS trades.
- Noncollat ($\mathrm{VM} = \mathrm{IM} = 0$).

Computational Strategy

- Risk factors are simulated forward on CPUs
- Backward MtM pricing task is performed by fast matrix exponentiation in floating arithmetics on GPUs.
- Nested Monte Carlo simulations used for computing the CA metrics at all nodes of an outer simulation (via MtM interpolation) and simulating the loss process L required as input data in the economic computations.
- Unconditional economic capital, hence KVA, term structure approximation
- Picard iteration accounting for the impact on the FVA of the funding sources provided by reserve capital and economic capital

- For comparison purposes, we initialize the Picard iteration by

$$\mathrm{FVA}_t^{(0)} = \mathbb{E}_t \int_t^T \beta_t^{-1} \beta_s \bar{\lambda}_s \Big(\sum_i J_s^i (\mathrm{MtM}_s^i - \mathrm{VM}_s^i)\Big)^+ ds,$$

  which corresponds to the FVA accounting only for the re-hypothecation of the variation margin received on hedges, but ignores the FVA deductions reflecting the possible use of reserve and economical capital as VM.

Modeling and Hardware/Software Choices

- Market and credit portfolio models of Albanese, Bellaj, Gimonet, and Pietronero (2011) calibrated to the relevant market data.
- 20,000 primary scenarios up to 50 years in the future run on 100 underlying time points, with 1,000 secondary scenarios starting from each primary simulation node, which amounts to a total of two billion scenarios.

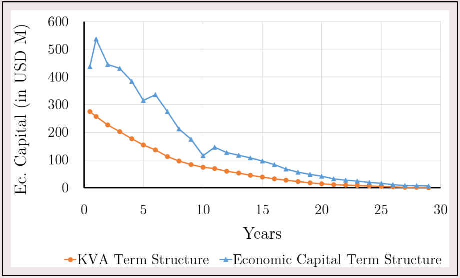
Modeling and Hardware/Software Choices (cont'd)

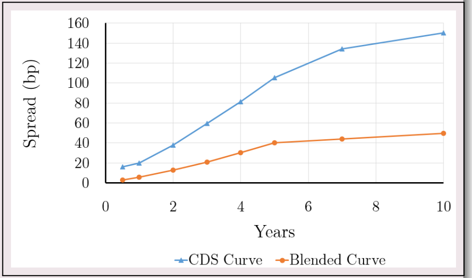
- All the computations are run using a 4-socket server for Monte Carlo simulations, Nvidia GPUs for algebraic calculations and Global Valuation Esther as simulation software.
- Using this super-computer and GPU technology the whole calculation takes a few minutes for building the models, followed by a nested simulation time in the order of about an hour for processing a billion scenarios on the bank portfolio.

Representative banking portfolio XVA values.

XVA	$Value
CVA_0	242 M
$FVA_0^{(0)}$	126 M
FVA_0	62 M
KVA_0	275 M
FTDCVA	194 M
FTDDVA	166 M

Left: Term structure of economic capital compared with the term structure of KVA.
Right:FVA blended funding curve computed from the ground up based on capital projections.





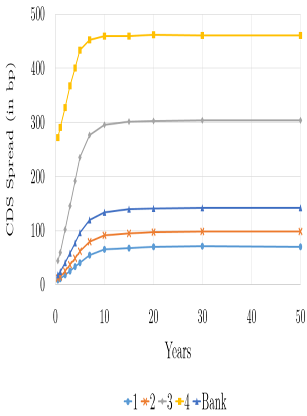
Trade incremental XVA Computations

In order to illustrate trade incremental XVA computations, using the same models and computational strategies as before, we now consider the following portfolio of ten USD currency fixed-income swaps on the date of 11 January 2016 (without initial margins, i.e. for $\mathrm{IM} = 0$).

Toy portfolio of swaps (the nominal of each swap is $\$10^4$)

Mat.	Receiver Rate	Payer Rate	i
10y	Par – 6M	LIBOR – 3M	3
10y	LIBOR – 3M	Par – 6M	2
5y	Par – 6M	LIBOR – 3M	2
5y	LIBOR – 3M	Par – 6M	3
30y	Par – 6M	LIBOR – 3M	2
30y	LIBOR – 3M	Par – 6M	1
2y	Par – 6M	LIBOR – 3M	1
2y	LIBOR – 3M	Par – 6M	4
15y	Par – 6M	LIBOR – 3M	1
15y	LIBOR – 3M	Par – 6M	4

Credit curves of the bank and its four conterparties



Introducing financial contracts one after the other in one or the reverse order in a portfolio at time 0 results in the same aggregated incremental FTP amounts for the bank, equal to the "time 0 portfolio FTP", but in different FTPs for each given contract and counterparty.

Toy portfolio. *Left*: XVA values and standard relative errors (SE). *Right*: Respective impacts when Swaps 5 and 9 are added last in the portfolio.

	\$Value	SE
CVA_0	471.23	0.46%
$FVA_0^{(0)}$	73.87	1.06%
FVA_0	3.87	4.3%
KVA_0	668.83	N/A
$FTDCVA_0$	372.22	0.46%
$FTDDVA_0$	335.94	0.51%

	Swap 5	Swap 9
ΔCVA_0	155.46	-27.17
$\Delta FVA_0^{(0)}$	-85.28	-8.81
ΔFVA_0	-80.13	-5.80
ΔKVA_0	127.54	-52.85
$\Delta FTDCVA_0$	98.49	-23.83
$\Delta FTDDVA_0$	122.91	-80.13

Outline

Benchmark NMC Approach *(previous chapter)*

- Nested Monte Carlo optimally implemented on GPUs
- Exponential complexity in the number of XVA layers
- Linear complexity in the number of pricing time steps
  - as XVA backwardation is addressed by linear regression

Simulation/Regression Approach *(this chapter)*

- Calculate the mark-to-market cube using GPU computing and the XVA cube using nonparametric neural net regression and quantile regression methods
- Albanese, Crépey, Hoskinson, and Saadeddine (2021)
- No nested Monte Carlo or conditional repricing, each successive layer (from right to left) beyond the base MtM layer in the XVA dependence tree is "learned" instead
- Linear complexity in the number of XVA layers and pricing time steps

Nonparametric, neural net quantile regression

- Value-at-risk ($\mathbb{VaR}$) and expected shortfall ($\mathbb{ES}$)
- For (X, ξ) a random vector in $\mathbb{R}^d \times \mathbb{R}$ with ξ integrable and d "large", approximate the function

$$\mathbb{R}^d \ni x \mapsto (q(x), s(x)) := (\mathbb{VaR}(\xi|X = x), \mathbb{ES}(\xi|X = x)) \in \mathbb{R}^2$$

- $\mathbb{VaR}$ *elicitable*, $\mathbb{ES}$ not, but $(\mathbb{VaR}, \mathbb{ES})$ *jointly elicitable*
  - Fissler and Ziegel (2016)

- For a suitable choice of functions f, g including $f(z) = z$ and $g = -\ln(1 + e^{-\cdot})$, the pair of the conditional value-at-risk and expected shortfall functions is the minimizer, over all measurable pair-functions $(q(\cdot), s(\cdot))$, of the error

$$\mathbb{E}\rho(q(\cdot), s(\cdot); X, \xi),$$

where

$$\rho(q(\cdot), s(X)(\cdot); X, \xi) = (1-\alpha)^{-1}\left(f(\xi) - f(q(X))\right)^{+} + f(q(X)) + \\ g(s(X)) - \dot{g}(s(X))\left(s(X) - q(X) - (1-\alpha)^{-1}(\xi - q(X))^{+}\right).$$

- In practice, one minimizes numerically the error (44), based on m independent simulated values of (X, ξ), over a parametrized family of functions $(q, e)(x) \equiv (q, e)_\theta(x)$.
- Dimitriadis and Bayer (2019) restrict themselves to multilinear functions.
- In our case we use a feedforward neural network parameterization

The Neural Net Regression XVA Algorithm

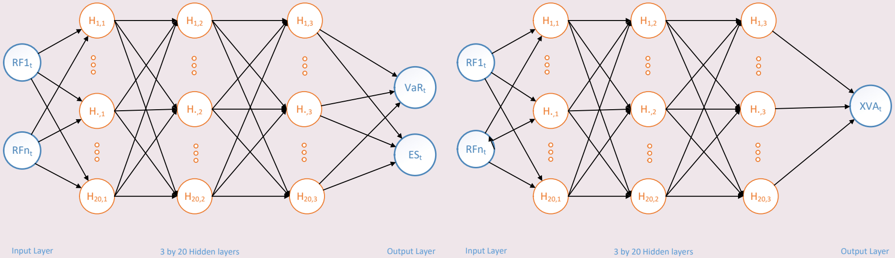
Pseudocode for the path-wise calculation of the XVAs and of the embedded dynamic initial margin and economic capital

After a forward simulation of the risk factor processes:
for each pricing time step, going backward **do**
    compute by NN regression as explained above the conditional risk measures, but also the conditional expectations (with least square loss functions)[a], involved in the XVA suite
end for

[a] cf. also Huré, Pham, and Warin (2020), Huge and Savine (2020).

At each pricing time, perform the relevant NN regression and/or quantile regression by solving an optimization (training) problem of the following form:

Pathwise XVA and Embedded Risk Measures Neural Nets



Joint value-at-risk/expected shortfall neural networkwith error (44) *(left)*: Features are state variables, labels are pathwise XVA item (e.g. loss function 1 year increment) output is joint estimate of pathwise conditional ES and VaR of the label given the features at selected alpha level. Hyperbolic tangent activation,100 epochs, full batch training, 3 by 20 hidden.

XVAs neural network with quadratic norm error *(right)*: Features are state variables, labels are pathwise XVA item (e.g. CVA) output is estimate of pathwise conditional mean of the label given the features at selected alpha level.ReLU activation,100 epochs, full batch training, 3 by 20 hidden.

Hierarchical Simulation Setup

- To achieve full efficiency in a whole bank balance sheet context, however, the XVA deep learning algorithm requires a specific simulation trick
- Abbas-Turki, Crépey, and Saadeddine (2021)
- When applied at the level of a realistic banking portfolio, a large variance of the estimated loss function can jeopardize the benefit of using a deep learning approach.
- To overcome this, we over-simulate the random variables with the most significant impact on the variance of the sampled payoff, i.e. default configurations

- Defaults X, market risk factors Y
- We are in a setting where X contributes more to the variance of cash-flows than Y
- In addition, simulating X given Y is much faster than simulating Y

$\rightarrow$ Simulating several (many) default paths X for each market path Y, even if this means giving up the independence of the simulation setup

  - hierarchical simulation setup

- Different from importance sampling that favors particular events, e.g. defaults vs. survival
- In an XVA setup, what we need is more richness regarding both default and survival, which is precisely what over-simulation provides

- Hierarchical simulation technique of independent interest, not restricted to the XVA space
    - Whenever defaults need be simulated
        - Credit portfolio applications
        - Insurance
    - Could also be used for pricing a Bermudan cliquet option in a rough volatility model, with rough (fractional Brownian motion driven) volatility Y of a stock X
- Detailed below for a quadratic loss function but also applicable to the quantile regression loss functions.

Deep learning algorithm with over-simulation (for a given time step)

1. Simulate $\{(X_i^{k,l}, Y_i^k, \xi^{k,l})\}, \begin{smallmatrix} 1 \le k \le \nu \\ 1 \le l \le \omega \end{smallmatrix}$ according to the previous over-simulation scheme;
2. Train a NN to regress $(\xi^{k,l}), \begin{smallmatrix} 1 \le k \le \nu \\ 1 \le l \le \omega \end{smallmatrix}$ against $\{(X_i^{k,l}, Y_i^k)\}, \begin{smallmatrix} 1 \le k \le \nu \\ 1 \le l \le \omega \end{smallmatrix}$, i.e.

$$\text{find } \widehat{\theta}_i \in \operatorname{argmin}_{\theta \in \Theta} \sum_{k=1}^{\nu} \sum_{l=1}^{\omega} (\xi^{k,l} - \varphi_\theta(X_i^{k,l}, Y_i^k))^2;$$

3. Use $(x, y) \mapsto \varphi_{\widehat{\theta}_i}(x, y)$ as an estimator for $\mathbb{E}[\xi | X_i = x, Y_i = y]$.

Choosing the over-simulation factor

- Assume that simulating Y_i costs κ times more than $X_i|Y_i$;
- ω chosen so as to minimize, under a budget constraint $\tau = \nu(\omega + \kappa)$, the variance of the training loss $\frac{1}{m}\sum_{k=1}^{\nu}\sum_{l=1}^{\omega} f_\theta(X_i^{k,l}, Y_i^k)$, where

$$f_\theta(X_i, Y_i) = (\xi - \varphi_\theta(X_i, Y_i))^2;$$

- Let $\alpha_i^\theta = \mathbb{E}[(f_\theta(X_i^{1,1}, Y_i^1))^2] - \mathbb{E}[f_\theta(X_i^{1,1}, Y_i^1)f_\theta(X_i^{1,2}, Y_i^1)]$ and $\beta_i^\theta = \mathbb{E}[f_\theta(X_i^{1,1}, Y_i^1)f_\theta(X_i^{1,2}, Y_i^1)] - (\mathbb{E}[f_\theta(X_i^{1,1}, Y_i^1)])^2$.

Proposition

$$\mathbb{V}\mathrm{ar}\left(\frac{1}{m}\sum_{k=1}^{\nu}\sum_{l=1}^{\omega} f_\theta(X_i^{k,l}, Y_i^k)\right) = \frac{\beta_i^\theta}{\tau}\left(\frac{1}{\omega}\left(\omega - \sqrt{\frac{\alpha_i^\theta \kappa}{\beta_i^\theta}}\right)^2 + \left(\sqrt{\frac{\alpha_i^\theta}{\beta_i^\theta}} + \sqrt{\kappa}\right)^2\right)$$

Generalization Bound (in Finite Parameter Space Case)

- We aim to establish a rule for ν too, based on concentration inequalities à la Shapiro, Dentcheva, and Ruszczynski (2014).
- Assume Θ is finite, let $0 < \delta < \epsilon$ and define:

$$
\begin{array}{lcl}
\vartheta^{\star} & = & \min_{\theta \in \Theta} \mathbb{E}[f_{\theta}(X, Y)] \\
\widehat{\vartheta}_{\nu,\omega} & = & \min_{\theta \in \Theta} \frac{1}{m} \sum_{k=1}^{\nu} \sum_{l=1}^{\omega} f_{\theta}(X_i^{k,l}, Y_i^k) \\
S^{\epsilon} & = & \{\theta \in \Theta : \mathbb{E}[f_{\theta}(X, Y)] \leq \vartheta^{\star} + \epsilon\} \\
\widehat{S}^{\delta}_{\nu,\omega} & = & \left\{\theta \in \Theta : \frac{1}{m} \sum_{k=1}^{\nu} \sum_{l=1}^{\omega} f_{\theta}(X_i^{k,l}, Y_i^k) \leq \widehat{\vartheta}_{\nu,\omega} + \delta\right\}
\end{array}
$$

- Assume $\Theta \setminus S^{\epsilon} \neq \emptyset$ and let $u : \Theta \setminus S^{\epsilon} \to \Theta$ such that $\mathbb{E}[f_{u(\theta)}(X, Y)] \leq \mathbb{E}[f_{\theta}(X, Y)] - \epsilon^{\star}$ for all $\theta \in \Theta \setminus S^{\epsilon}$, for some $\epsilon^{\star} \geq \epsilon$, and define $g_{\theta}(X, Y) = f_{u(\theta)}(X, Y) - f_{\theta}(X, Y)$.

- In particular, $\forall \theta \in \Theta \setminus S^{\epsilon}, \mathbb{E}[g_\theta(X, Y)] \leq -\epsilon^\star$ and we have:

$$\mathbb{P}\left(\widehat{S}^{\delta}_{\nu,\omega} \not\subseteq S^{\epsilon}\right) \leq |\Theta| e^{-\nu \tau_\omega(\delta,\epsilon)}$$

  where $\tau_\omega(\delta, \epsilon) = \min_{\theta \in \Theta \setminus S^{\epsilon}} I_{\theta,\omega}(-\delta)$,
  $I_{\theta,\omega}(a) = \sup_{t \in \mathbb{R}} \left\{ at - \log \mathbb{E}\left[M_\theta\left(\frac{t}{\omega} | Y\right)^{\omega} \right] \right\}$ and
  $M_\theta(z|Y) = \mathbb{E}[e^{z g_\theta(X,Y)} | Y]$.
- We have $I_{\theta,\omega}(-\delta) > 0$ and for δ close to $-\mathbb{E}[g_\theta(X, Y)]$:

$$\begin{aligned} I_{\theta,\omega}(-\delta) &\approx \frac{(\delta + \mathbb{E}[g_\theta(X,Y)])^2}{2\left(\frac{1}{\omega}\mathbb{E}[\mathrm{var}(g_\theta(X,Y)|Y)] + \mathrm{var}(\mathbb{E}[g_\theta(X,Y)|Y])\right)} \\ &\geq \frac{(\epsilon^\star - \delta)^2}{2\left(\frac{1}{\omega}\mathbb{E}[\mathrm{var}(g_\theta(X,Y)|Y)] + \mathrm{var}(\mathbb{E}[g_\theta(X,Y)|Y])\right)} \end{aligned}$$

Transfer Learning

- Possible to have non-smooth paths $\{\varphi_{\theta_i}(x,y)\}_{0\le i\le n}$ even when the true paths $\mathbb{E}[\xi|X_i = x, Y_i = y]$ are smooth;
- Learnings on each time step t_i are being done independently of each other;
- Start the learning at T and then at every time step reuse the previous solution as an initialization of the training algorithm;
- Local minima associated with each time step will be close to each other
  $\rightarrow$ paths $\{\varphi_{\theta_i}(x,y)\}_{0\le i\le n}$ are smooth;
- A form of transfer learning, also helps accelerate the convergence of the learning procedure.

Case Study–Risk Factor Simulation

- Common shock credit model with CIR intensities
- Hybrid credit modelling: default intensity for nested pathwise calculations and default events for counterparty survival
- Risk factors: 10 IR HW1F, 9 FX GBM, 11 CR CIR (common shock drivers), up to 40 risk factors used as deep learning features (including default indicators)
- GPU-based monte carlo simulation in 100,000 paths
- 10 time steps per year risk factor evolution, 2 portfolio pricings per year.

Case Study–Trades and Counterparties

- Portfolio of 10,000 randomly generated swap trades:
  - Swap rates uniformly distributed on $[0.005, 0.05]$ - so already ITM or OTM
  - Number of six-monthly coupon resets uniform on $[5 \ldots 60]$
  - Trade currency and counterparty both uniform on $[1, 2, 3 \ldots, 10]$
  - Notional Uniform on $[10000, 20000, \ldots, 100000]$
  - Direction: Bernoulli - Asset heavy bank 75% likely to pay fixed, liability-heavy bank 75% likely to receive fixed

Case Study–Collateralization Schemes

- Let $\Delta_t^c = \mathcal{P}_t^c - \mathcal{P}_{(t-\delta)-}^c$
- We consider both "no CSA" netting sets c, with $\mathrm{VM} = \mathrm{RIM} = \mathrm{PIM} = 0$, and "(VM/IM) CSA" netting sets c, with $\mathrm{VM}_t^c = \mathrm{P}_t^c$ and, for $t \leq \tau_c$,

$$\mathrm{RIM}_t^c = \mathbb{V}\mathrm{a}\mathbb{R}_t\Big((P_{t^\delta}^c + \Delta_{t^\delta}^c) - P_t^c\Big)\,,\ \mathrm{PIM}_t^c = \mathbb{V}\mathrm{a}\mathbb{R}_t\Big(-(P_{t^\delta}^c + \Delta_{t^\delta}^c) + P_t^c\Big), \tag{44}$$

  for some PIM/RIM quantile levels a_{pim}/a_{rim}, and where $t^\delta = t + \delta$, for a suitable margin period of risk (MPoR) δ (e.g. two weeks)
- IM - posted (pledged) at 99% gap risk VaR, received (secured) covers 75% gap risk leaving excess as residual gap CVA

Proposition 6

In a common shock default model of the clients and the bank itself, with pre-default intensity processes γ^c of the clients, and assuming continuous market risk factors, then $\mathrm{CVA} = \mathrm{CVA}^{nocsa} + \mathrm{CVA}^{csa}$*, where, for* $t < \tau$,

$$\mathrm{CVA}_t^{nocsa} = \sum_{c\ nocsa} \mathbb{1}_{t<\tau_c}(1-R_c)\mathbb{E}_t \int_t^T (P^c_{s^\delta} + \Delta^c_{s^\delta})^+ \gamma^c_s e^{-\int_t^s \gamma^c_u du} ds$$
$$+ \sum_{c\ nosca} \mathbb{1}_{\tau_c<t<\tau_c^\delta}(1-R_c)\mathbb{E}_t(P^c_{\tau_c^\delta} + \Delta^c_{\tau_c^\delta})^+,$$
$$\mathrm{CVA}_t^{csa} = \sum_{c\ csa} \mathbb{1}_{t<\tau_c}(1-R_c)(1-a_{rim}) \times$$
$$\mathbb{E}_t \int_t^T (\mathbb{ES}_s - \mathbb{V}\mathrm{a}\mathbb{R}_s)\left((P^c_{s^\delta} + \Delta^c_{s^\delta}) - P^c_s\right) \gamma^c_s e^{-\int_t^s \gamma^c_u du} ds$$
$$+ \sum_{c\ csa} \mathbb{1}_{\tau_c<t<\tau_c^\delta}(1-R_c)\mathbb{E}_t \left((P^c_{\tau_c^\delta} + \Delta^c_{\tau_c^\delta}) - (P^c_{\tau_c} + \mathrm{RIM}^c_{\tau_c})\right)^+,$$

where $(\mathbb{ES}_s - \mathbb{V}\mathrm{a}\mathbb{R}_s)$ *in* (45) *is computed at the* a_{rim} *confidence level.*

Proposition 7

Assuming its posted initial margin borrowed unsecured by the bank, then $\mathrm{MVA} = \mathrm{MVA}^{csa}$, *where*

$$\mathrm{MVA}_t^{csa} = \sum_{c \quad csa} J_t^c \mathbb{E}_t \int_t^T (1-R)\gamma_s \mathrm{PIM}_s^c e^{-\int_t^s \gamma_u^c du} ds. \qquad (45)$$

Tuning the XVA Learning Engine

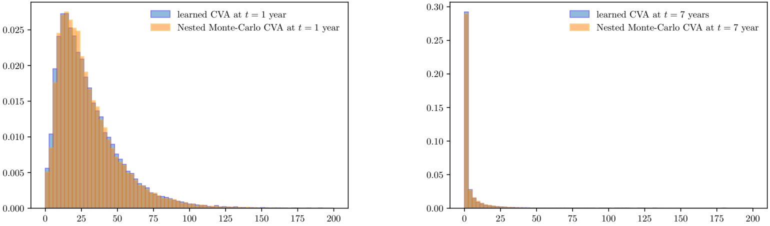
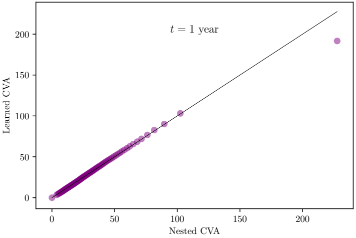


Figure: Random variables CVA_1^c and CVA_7^c (in the case of a no CSA netting set c, respectively observed after 1 and 7 years) obtained by learning (blue histogram) versus nested Monte Carlo (orange histogram). All histograms are based on out-of-sample paths.



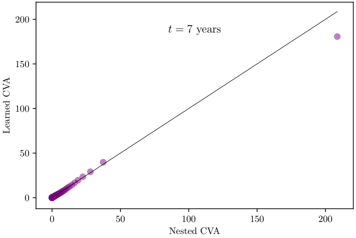


Figure: QQ-plot of learned versus nested Monte Carlo CVA for the random variables CVA_1^c (*left*) and CVA_7^c (*right*). Paths are out-of-sample.

- Recall the equivalence of optimising the mean quadratic error
  - between the NN learned estimator $h(X)$ and the labels Y ("MSE"), $\mathbb{E}\left[(h(X)-Y)^2\right]$, and
  - between the NN learned estimator and the conditional expectation $\mathbb{E}[Y|X]$ (in our case estimated by NMC), $\mathbb{E}\left[(h(X)-\mathbb{E}[Y|X])^2\right]$.
- The equivalence stems from the identities

$$\mathbb{E}\left[(h(X)-Y)^2\right] = \mathbb{E}\left[(h(X)-\mathbb{E}[Y|X])^2\right] + \mathbb{E}\left[(\mathbb{E}[Y|X]-Y)^2\right] + 2\mathbb{E}\big[(h(X)-\mathbb{E}[Y|X])(\mathbb{E}[Y|X]-Y)\big], \tag{46}$$

where

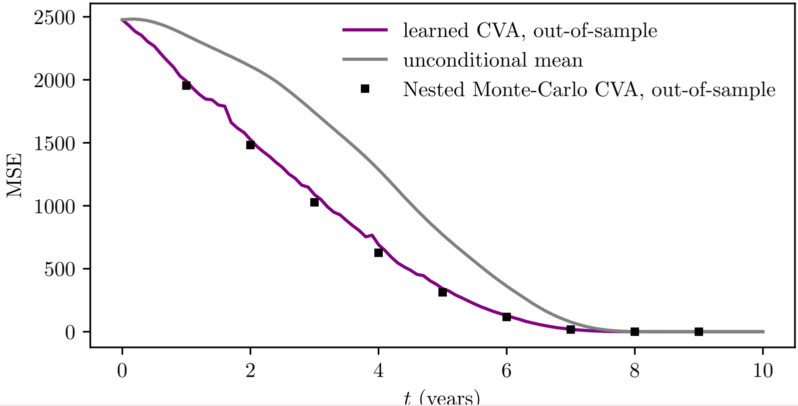
$$\begin{aligned}&\mathbb{E}\big[(h(X)-\mathbb{E}[Y|X])(\mathbb{E}[Y|X]-Y)\big]\\&\qquad=\mathbb{E}\Big[\mathbb{E}\big[(h(X)-\mathbb{E}[Y|X])(\mathbb{E}[Y|X]-Y)\mid X\big]\Big]=0\end{aligned} \tag{47}$$

and

$$\begin{aligned}\mathbb{E}\left[(\mathbb{E}[Y|X]-Y)^2\right] &= \mathbb{E}(Y^2) - \mathbb{E}\Big[\big(\mathbb{E}[Y|X]\big)^2\Big]\\ &= \mathbb{E}\Big[\mathbb{E}\left[Y^2|X\right] - \big(\mathbb{E}[Y|X]\big)^2\Big] = \mathbb{E}[\mathbb{V}\mathrm{ar}(Y|X)]\end{aligned} \tag{48}$$

does not depend on h.

Empirical quadratic loss of each CVA estimator at all coarse time-steps. The lower, the closer to the true conditional expectation (cf. (48)).

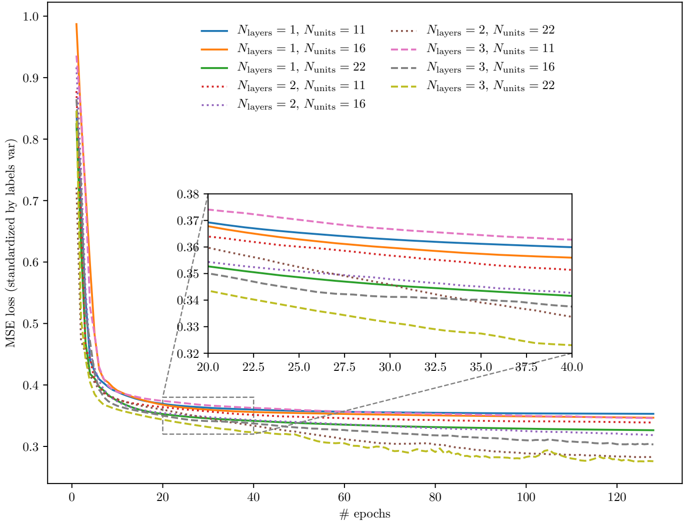


Accuracy and computation times for the NMC estimation of a CVA at a given pricing time-step (32768 outer paths). The MSE here is between the nested Monte Carlo estimator and the labels (projection error).

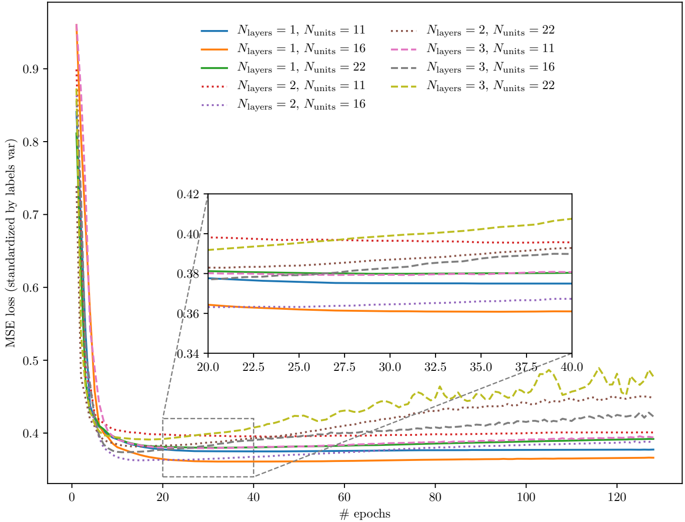
# of inner paths	MSE (vs labels)	Computational time (seconds)
2	0.523	37.562
4	0.427	37.815
8	0.393	37.819
16	0.370	38.988
32	0.360	40.707
64	0.353	57.875
128	0.348	157.536
256	0.349	301.406
512	0.348	584.475
1024	0.348	1213.756

Note that, in the presence of a multiple number of XVA layers, the complexity of a (multiply) nested Monte Carlo approach is exponential in the number of XVA layers, while the computational complexity of the

In-sample empirical quadratic loss during CVA learning at time-step $t = 5$ years, standardized by the variance of the labels.



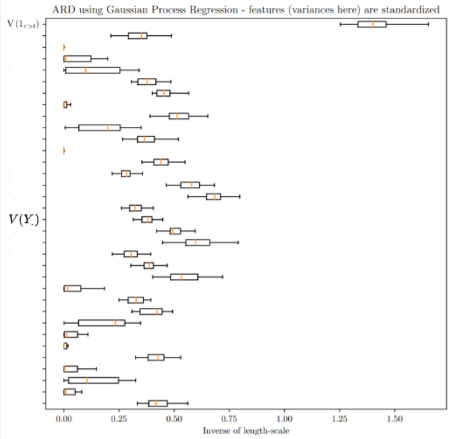
Out-of-sample empirical quadratic loss during CVA learning at time-step $t = 5$ years, standardized by the variance of the labels.



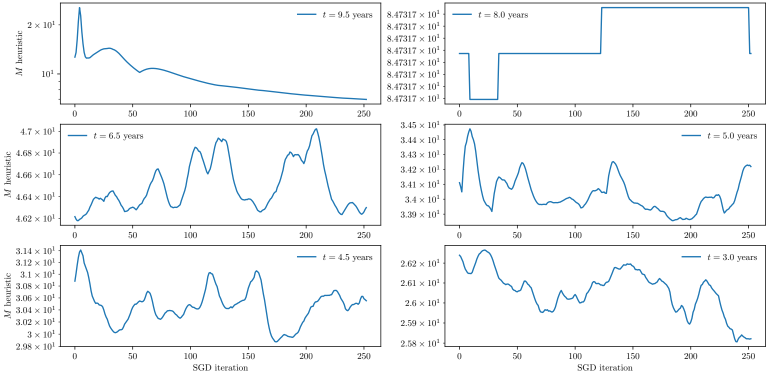
Tuning the Over-Simulation Setup

- Were it for the CVA only (and for the MVA alike), the best learning scheme (with minimal variance) would obviously be the one based on the default intensities of the clients.
- But a default intensity based reformulation is unavailable for the higher XVA layers, for which client defaults need be simulated
- One then need to adopt a hierarchical simulation setup
- We tune the over-simulation factor in the CVA case, playing with the two formulations of the latter, intensity-based (above) versus default-indicator-based (below), so that the previous results can be used for tuning our over-simulation setup.

Box-plot of the inverse length-scales obtained by randomized Gaussian process regressions of the conditional variance of a CVA default-indicator-based cash flows against the conditional variance of the risk factors, where conditional here is in reference to the parameters of the model treated as a random vector with a postulated distribution.

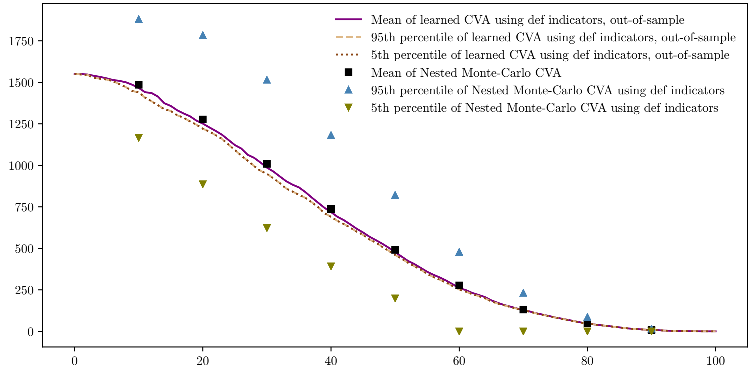


Optimal over-simulation factor $\sqrt{\frac{\alpha_i^\theta \kappa}{\beta_i^\theta}}$ at different time steps and SGD iterations plotted for for $\kappa = 1$

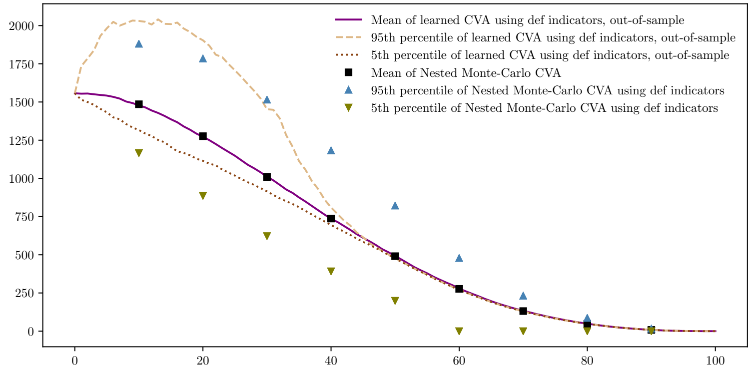


- Quite stable, a few tens
- What is plotted above is for $\kappa = 1$, it is therefore necessary to multiply what is displayed by $\sqrt{\kappa}$
  - e.g. if a market simulation is 100 times slower than a default simulation, then the factors displayed in the figure must be multiplied by 10
- Hence we expect an optimal over-simulation factor of the order of a few hundreds
- Next slides: CVA computed with 1, 32, 64, 128, 256, 512, ∞ (i.e. using default intensities) default paths per market paths
  - X axis pricing times, Y axis CVA levels
  - Since the nested Monte Carlo method is computationally expensive, it was carried out only once every 10 pricing time-steps.

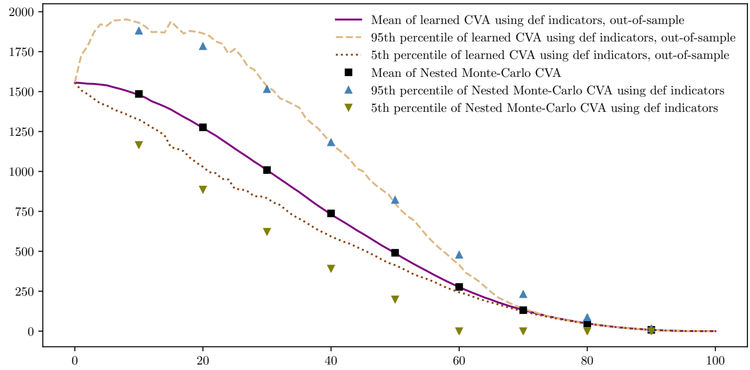
CVA computed with 1 default path per market path (x axis pricing times, y axis CVA levels).



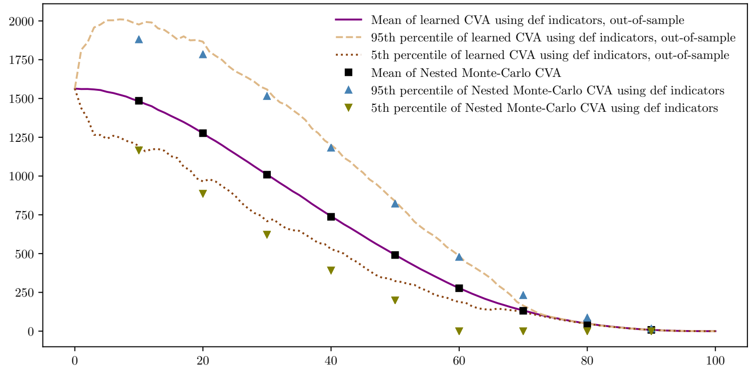
CVA computed with 32 default path per market path (x axis pricing times, y axis CVA levels).



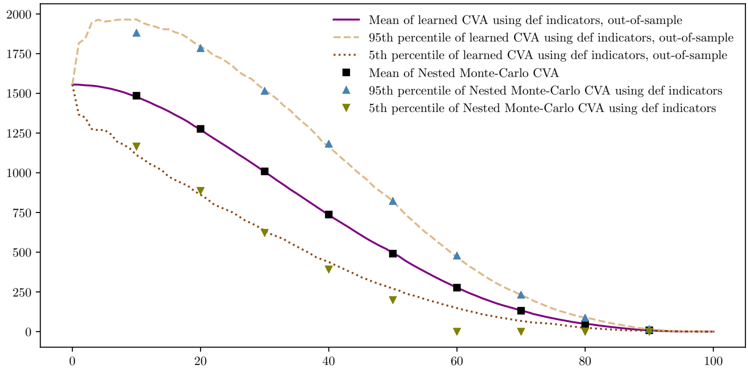
CVA computed with 64 default path per market path (x axis pricing times, y axis CVA levels).



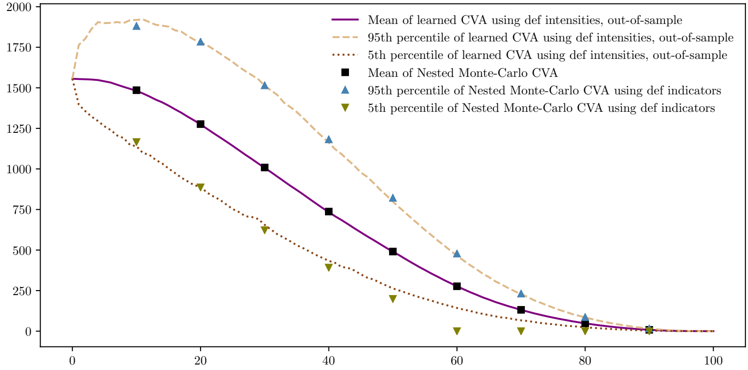
CVA computed with 128 default path per market path (x axis pricing times, y axis CVA levels).



CVA computed with 512 default path per market path (x axis pricing times, y axis CVA levels).



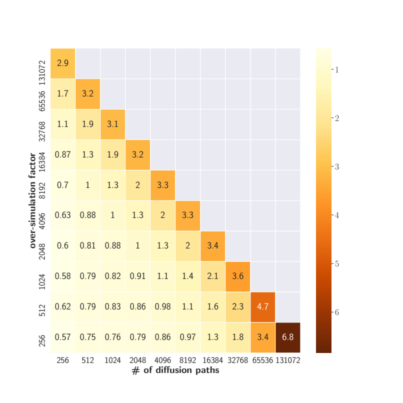
CVA computed with "∞" default path per market path (x axis pricing times, y axis CVA levels).

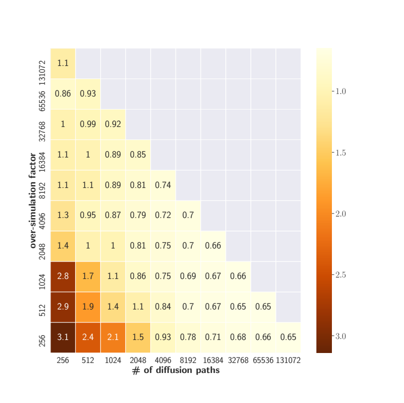


CVA Execution times

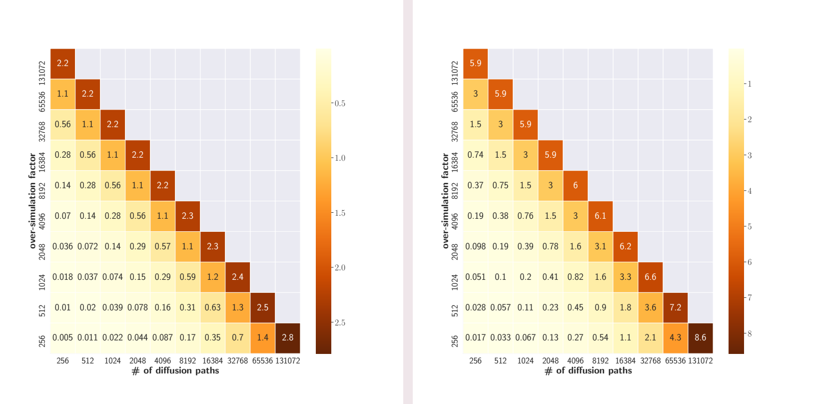
Over-simulation factor	Learning approach (training)	Nested Monte Carlo
1	37s	$\sim$ 5h27min
32	54s	
64	88s	
128	165s	
512	676s	

Simulation times (left) and (right) RMSE of the prediction against a NMC benchmark at the pricing time-step $t = 5$years, for different combinations of the number of diffusion paths and the over-simulation factor.

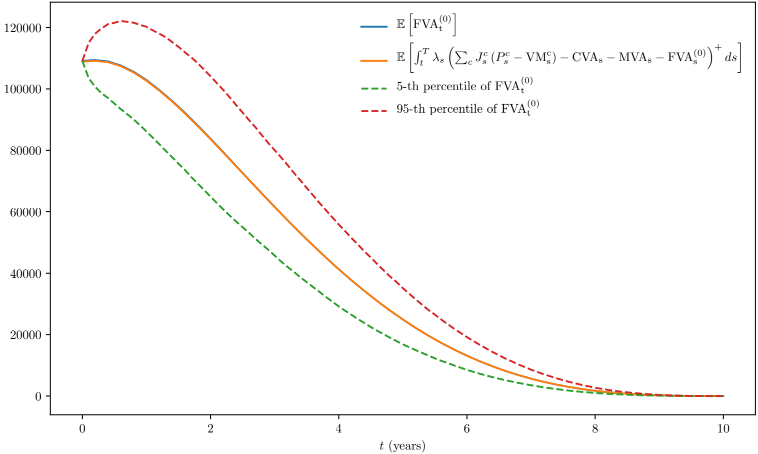




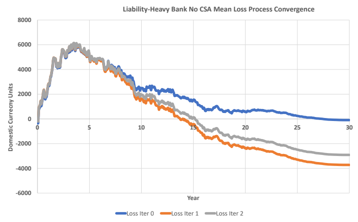
GPU (left) and CPU (right) memory usages during simulation for different combinations of the number of diffusion paths and the over-simulation factor.

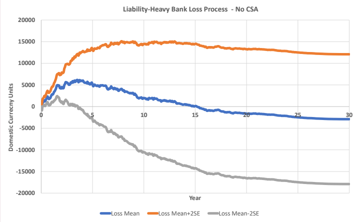


Learned FVA$^{(0)}$.

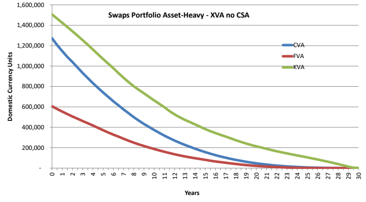


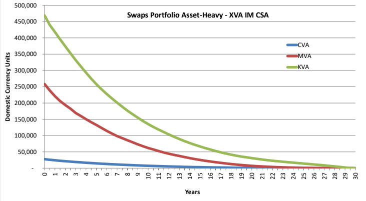
(Left) Profiles of the processes $L^{(k)}$, for $k = 1, 2, 3$; *(Right)* Mean $\pm$ 2 stdev profiles of the process $L^{(3)}$.

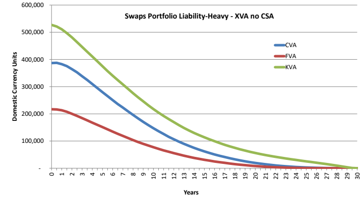


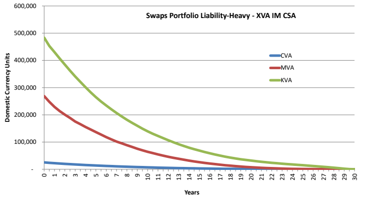


(Top left) Asset-heavy portfolio, no CSA. *(Top right)* Asset-heavy portfolio, VM/IM CSA. *(Bottom left)* Liability–heavy portfolio, no CSA. *(Bottom right)* Liability-heavy portfolio, VM/IM CSA.







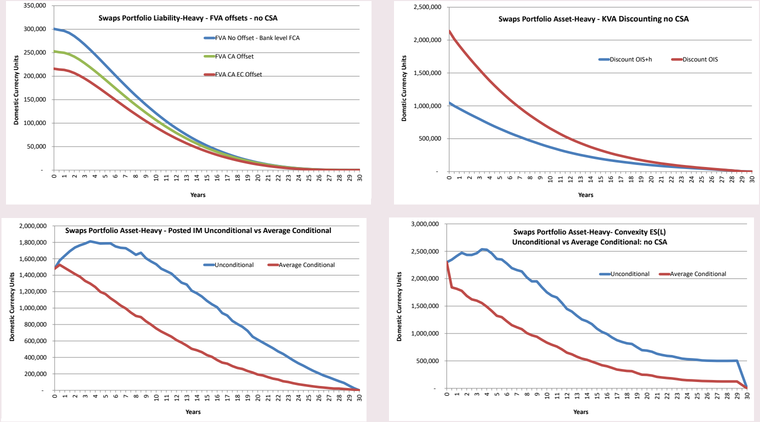


Portfolio-Wide Path-wise XVAs Take-Aways

- Extensive collateralisation transfers counterparty credit risk to liquidity funding risk
  - However FVA/MVA risk is ignored in current derivatives capital regulation
- Capital as funding has a materially reducing impact on the FVA.
- Treating KVA as a risk margin gives a huge discounting impact.
- Deep learning detects material capital convexity
  - may be material in the case of asset-heavy, no CSA portfolios
- Deep learning detects IM convexity
  - may be material in the case of asset-heavy, IM CSA portfolios

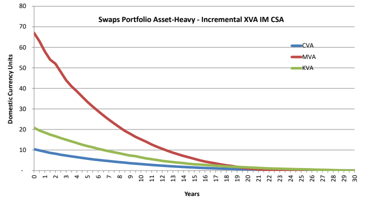
.

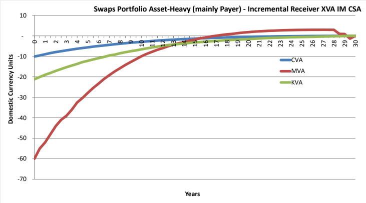
(Top left) FVA ignoring the off-setting impact of RC and CR (blue), FVA accounting for the off-setting impact of RC but ignoring the one of CR (green), FVA accounting for both impacts (red). *(Top right)* KVA ignoring the off-setting impact of RM, KVA including it (blue). *(Bottom left)* Unconditional PIM profile (blue), vs. path-wise PIM profile, i.e. mean of the path-wise PIM process (red). *(Bottom right)* Unconditional EC profile vs. path-wise EC profile (red).

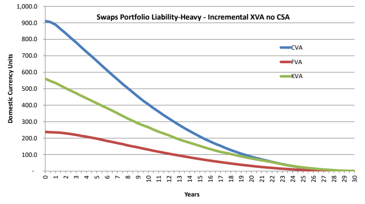


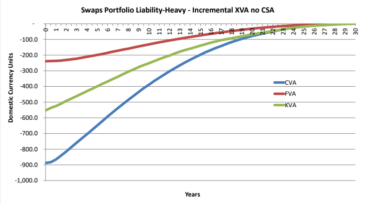
Trade Incremental XVA Profiles

(Top left) Asset-heavy portfolio, no CSA. Incremental receive fix trade. *(Top right)* Liability-heavy portfolio, no CSA. Incremental pay fix trade. *(Bottom left)* Asset-heavy portfolio under CSA. Incremental Pay Fix Trade. *(Bottom right)* Liability-heavy portfolio under CSA. Incremental receive fix trade.









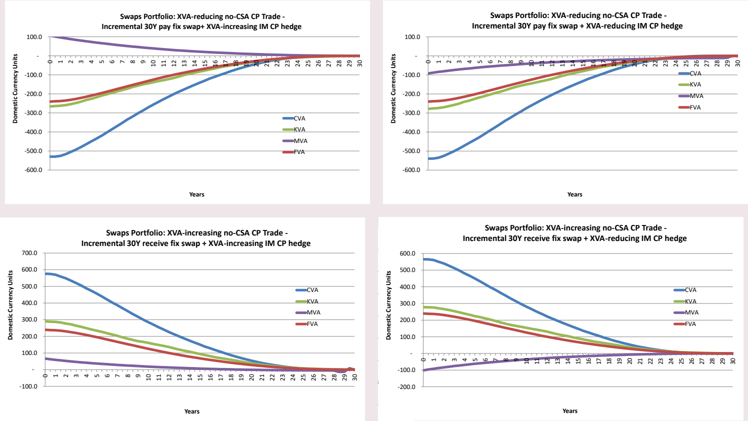
XVA Incremental Trade+Hedge Examples

Our model assumes market risk of trades to be fully hedged. Our bank XVA can include the XVA on the market risk hedges by including the relevant bilateral hedge CPs in the model

- For cleared hedges see the next chapter

- 10 counterparties: 8 no CSA and 2 bilateral IM CSA hedge counterparties
- Portfolio of 5,000 randomly generated swap trades as before, plus 5,000 corresponding hedge trades
- The 8 no CSA counterparties are primarily asset or liability heavy
- One bilateral IM CSA hedge counterparties is asset-heavy and one liability-heavy
- Incremental trade - par 30 year swap 100k notional, corresponding hedge trade

(Top left) XVA-reducing trade + XVA-increasing bilateral hedge *(Top right)* XVA-increasing trade + XVA-increasing bilateral hedge. *(Bottom left)* XVA-reducing trade + xva-reducing bilateral hedge *(Bottom right)* XVA-increasing trade + XVA-reducing bilateral hedge.



Scalability

GPU Simulation + CNTK Learning Indicative Timings

Computation Time (seconds)

	10 CP 40 Risk Factors		20 CP 80 Risk Factors	
	No CSA	IM CSA	No CSA	IM CSA
Initial Risk Factor &Trade Pricing Simulation Cuda	118.0	118.0	134.0	134.0
Counterparty Learning Calculations	386.7	1,890.0	435.8	2,211.1
Bank Level Learning Calculations	525.3	156.0	687.4	229.9
Total Initial Batch	1,030.0	2,164.0	1,257.2	2,574.9
Re-simulate 1 counterparty trade pricing Cuda	19.0	19.0	25.0	25.0
Counterparty Learning Calculations	38.7	189.0	43.6	221.1
Bank Level Learning Calculations	525.3	156.0	687.4	229.9
Total Incremental Trade	583.0	364.0	756.0	476.0

- Timings taken on Lenovo P52 laptop with NVidia Quadro P3200 GPU @ 5.5 Teraflops peak FP32 performance, 14 streaming multiprocessors.
- Around 80 – 90% Cuda achieved occupancy in our deep learning calculations (the GPU acceleration factor is of the order of 15)
- NVidia Tesla V100 should be at least 6-10 times faster.
- Cache computations and scale over multiple V100s for acceptable incremental deal pricing performance.

CNTK Implementation

- Open source unified deep learning Microsoft Cognitive Toolkit (`https://cntk.ai`)
- Neural network computational steps described via a directed graph.
- Easily realize and combine DNNs, CNNs, RNNs/LSTMs and other types.
- Implements stochastic gradient descent (SGD, error backpropagation) learning with automatic differentiation on CPU and GPU
- Parallelization across multiple GPUs and servers.
- Core in C++/CUDA, wrappers for Python, C#, Java
- Why CNTK for XVA?
  - ✗ Tensorflow - automatic differentiation in Python means no C++ training
  - ✓ CNTK - automatic differentiation in C++/CUDA means C++ training
  - ✓ CNTK - C++ usual for XVA - do AI/ML within XVA process, and leverage GPU.
- Currently for C++ API, the included Boost unit tests *are* the docs!
- Open source

CNTK Implementation—Define Joint ES/VaR NN Loss Function in C++

```
inline FunctionPtr RiskMeasureLearning::VaRES_Loss(const Variable& prediction, const
    Variable& labels, const double q, const std::wstring& name)
{
//    Define Placeholder variables
Variable labelPlaceholder = PlaceholderVariable(L"label");
Variable quantilePlaceholder = PlaceholderVariable(L"quantile");
Variable predictionPlaceholderVaR = PlaceholderVariable(L"predictionVaR");
Variable predictionPlaceholderES = PlaceholderVariable(L"predictionES");
//Slice the 2-element NN prediction tensor into VaR and ES
Variable predictionVaR = Slice(prediction, { Axis(0) }, { 0 }, { 1 });
Variable predictionES = Slice(prediction, { Axis(0) }, { 1 }, { 2 });

FunctionPtr one = OnesLike(labelPlaceholder);
FunctionPtr Ind = ElementTimes(one, Less(labelPlaceholder, predictionPlaceholderVaR));
FunctionPtr alpha = ElementTimes(quantilePlaceholder, one);
FunctionPtr VaRP1 = ElementTimes(Minus(Ind, alpha), predictionPlaceholderVaR);
FunctionPtr VaRP2 = ElementTimes(Ind, labelPlaceholder);
//g1(z)=z; g2(z)=exp(z)/(1+exp(z)), gcurly2=ln(1+e^z)
FunctionPtr ESP1 = Minus(predictionPlaceholderES, predictionPlaceholderVaR);
FunctionPtr ESP2 = ElementTimes(Minus(predictionPlaceholderVaR, labelPlaceholder), Ind);
FunctionPtr ESP3 = ElementDivide(ESP2, quantilePlaceholder);
FunctionPtr ESP4 = ElementDivide(Exp(predictionPlaceholderES), Plus(one, Exp(
    predictionPlaceholderES)));
FunctionPtr ESP5 = Log(Plus(one, Exp(predictionPlaceholderES)));
FunctionPtr ESP6 = Plus(ElementTimes(alpha, labelPlaceholder), Log(Plus(one, Exp(
    labelPlaceholder))));
FunctionPtr ESVaR = Plus(Minus(ElementTimes(ESP4, Plus(ESP1, ESP3)), ESP5), ESP6);

FunctionPtr jointVaRESloss = Plus(Minus(VaRP1, VaRP2), ESVaR);
//Bind placeholders and return loss function
return AsBlock(std::move(jointVaRESloss), { {predictionPlaceholderVaR, predictionVaR}, {
```

CNTK Implementation—train Joint Conditional ES/VaR NN in C++

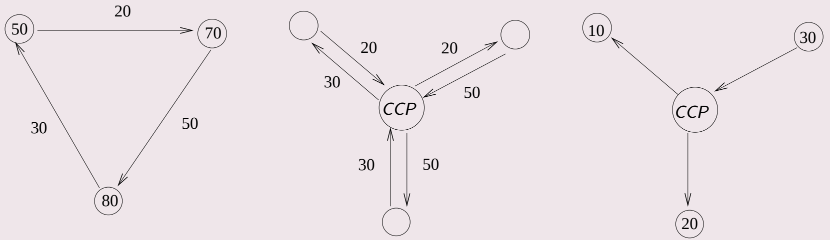
```
void RiskMeasureLearning::VaRESDeepLearning(const MatrixXd& X, const MatrixXd& Y, const
      double quantile, const DeviceDescriptor& device)
{
rescaleXYData(X, Y);
auto inputVar = InputVariable({ m_inputDim }, DataType::Float, L"features");
auto labelsVar = InputVariable({ m_outputDim }, DataType::Float, L"Labels");
NDShape inputShape({ m_inputDim });
ValuePtr inputValue = Value::CreateSequence(inputShape, m_inputData, device, true);
NDShape labelShape({ m_outputDim });
ValuePtr labelValue = Value::CreateSequence(labelShape, m_labelData, device, true);

auto trainingOutput = FullyConnectedFeedForwardRegressionNet(inputVar, m_outputDim + 1,
      m_hiddenLayersDim, m_numHiddenLayers, device, m_nonLinearity, L"trainingOutput");
auto trainingLoss = ReduceSum(VaRES_Loss(trainingOutput, labelsVar, quantile,L"
      LossFunction"), Axis::AllAxes(), L"LossFunction");
m_prediction = trainingOutput;

ProgressWriterPtr pw = MakeSharedObject<MyProgressWriter>(0, 0, 0, 0, 0, 0);
m_learner=MomentumSGDLearner(trainingOutput->Parameters(),m_learningRate,m_Momentum,
      true);
m_trainer=CreateTrainer(trainingOutput, trainingLoss, m_prediction, vector<LearnerPtr>({
       m_learner }), { pw });
for (size_t i = 0; i < m_iterationCount; ++i)
        m_trainer->TrainMinibatch({ {inputVar, inputValue}, {labelsVar, labelValue} },
              device);

EvaluateSequence(m_trainer->EvaluationFunction(), m_inputData, learningRM::JointVaRES,
      false, device);
```

Pytorch Implementation

- last MS CNTK release 2.7 April 2019
- Alternative GPU implementation combining
  - Python$\rightarrow$CUDA (through Numba) implementation of the simulations (including MtM computations)
  - learning with PyTorch (for its proximity to the CUDA programming model)

Outline

In a centrally cleared setup, a central counterparty (CCP) becomes "the buyer to every seller and the seller to every buyer"



- Since the global financial crisis of 2008-09, central clearing has become mandatory (or strongly incentivized) for standardized vanilla derivatives

Collateral

Collateral

- Variation margin (VM) tracking the mark-to-market of the members' portfolios on a $\sim$ daily basis
- Initial margin (IM) set as a barrier against gap risk, i.e. the slippage between the portfolio and the variation margin of a member during its portfolio liquidation of length δ
  - $\delta \sim$ one week
  - Dynamically updated at a frequency analogous to the one used for variation margin
  - Value-at-risk or expected shortfall of the P&L of each member over a period of length δ
  - "Initial safety cushion" at the time of default of a member, possibly eroded by gap risk during the liquidation period

Default fund

Centrally cleared IM quantile levels are typically lower than in bilateral SIMM transactions, but there are also mutualized default fund contributions:

- Funded default fund contributions, meant to protect the system under "extreme but plausible" scenarios
  - tantamount to IM add-ons, but computed on a collective basis
  - EMIR "cover two" default fund allocated proportionally to the IMs or the stress-test-loss-over-IMs (STLOIMs) of the clearing members
- Unfunded default fund contributions
  - additional refills that can be required by the CCP, often (but not even always) up to some cap in principle (without bounds in our model), in case the funded default fund contributions of the surviving members are not enough
  - capital at risk of the clearing members

Qualitative Insights

See e.g. Gregory (2014), Menkveld and Vuillemey (2020).

	Pros	Cons
Counterparty Credit Risk	• Reduced CCR of the CCP itself and reduced "domino effects" between members	• Concentration risk if a major CCP were to default • about 30 major CCPs today and only a few prominent ones (CME, LCH, Eurex, Ice,..) • Joint membership and feedback liquidity issues

Netting	• Multilateral netting benefit	• Loss of bilateral netting across asset classes

	Pros	Cons

Costs	• Default Resolution cheaper
	  ◦ Bilateral trading means a completely arbitrary transaction network.
	  ◦ An orderly default procedure cannot be done manually. It requires an IT network, whether it is CCPs, blockchain technology, SIMM reconciliation appliances, ...
		• High cost of raising funding initial margins, at least if funded by unsecured borrowing

	Pros	Cons
Information	• Better information of the CCP and the regulator	• Opacity of the default fund for the clearing members, which are not in a position of estimating their XVA metrics with accuracy

XVA Implications of CCPs

- With centrally cleared trading we are unveiling the other face of the XVA coin, i.e. the XVA implications of the hedging portfolio of the bank
- The client deals of the bank, once cleaned of their counterparty risk by the XVA desks, are hedged by the clean desks in terms of market risk.
- The (residual) bilateral market exposure of a bank can be hedged
  - either dynamically, typically through futures-style instruments traded on exchanges (platforms),
  - or by means of offsetting derivative transactions concluded with a CCP.

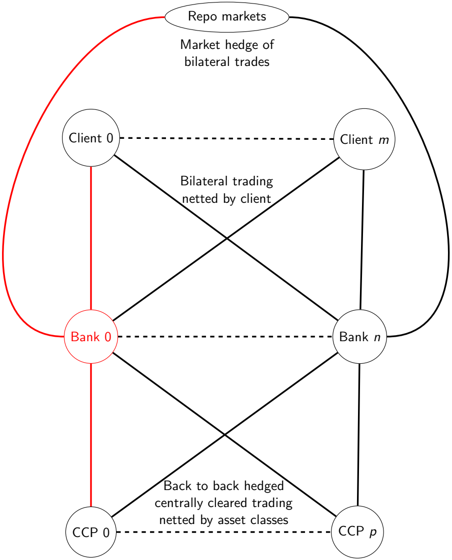


Figure: Financial network of clients, banks, and CCPs. Solid edges represent cash flows. Bilateral trades correspond to the upper part of the picture (banks and above) and centrally cleared trades to the lower part (banks and below). The trades of the bank with the CCPs are market risk hedges of client trades. The next figure provides a focus on the red part of the graph.

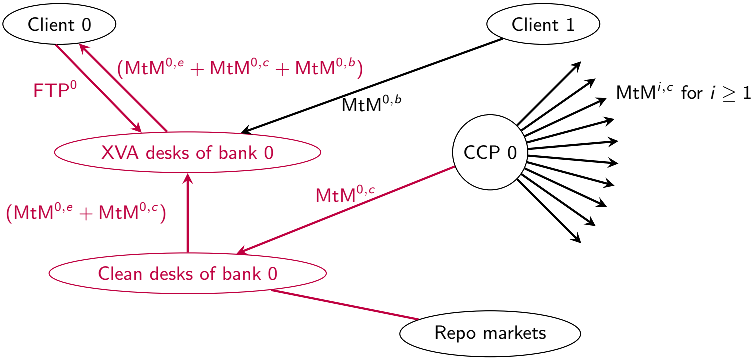


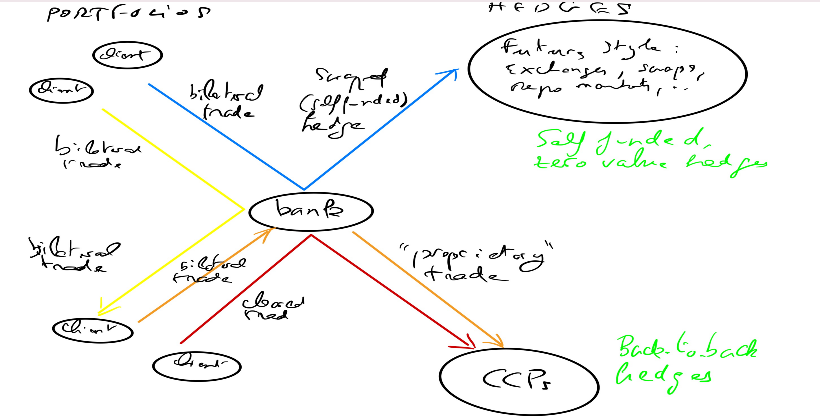
Figure: Zoom on a reference bank, labeled by 0, focusing on its transactions with client 0 and CCP 0, corresponding to the red part in the previous figure. The arrows represent the direction of deal entry payments between the bank, its clients and the CCP, under the convention that the reference clearing member bank 0 “buys” assets from its corporate clients, at an FTP (all-inclusive XVA rebate) deducted price with respect to their “clean valuation” ignoring counterparty risk. The client trades of the bank labeled by $^{\cdot,e}$, $^{\cdot,c}$ and $^{\cdot,b}$ are respectively hedged by futures-style instruments traded on exchanges, or by derivatives through the CCP or another bilateral counterparty. The trading of the CCP 0 with the other clearing members is suggested by the arrows $\mathrm{MtM}^{i,c}$, for $i = 1, \ldots, n$. The trading of the CCP clears, i.e. $\sum_{i=0}^{n} \mathrm{MtM}^{i,c} = 0$.

- The futures-style component of the hedging portfolio of the bank generates instant profit and losses, hence has no XVA implications
    - which require a positive and, in fact, long time-to-maturity to develop
- The cleared derivatives component, instead, has XVA implications
- Bearing in mind that
    - "the size of the hedge is the same as the one of the portfolio itself"
    - standardized derivatives have to be traded through CCPs,

  the XVA footprint of centrally cleared trading should be significant and has to be analyzed in detail, which is the topic of this chapter.

- The centrally cleared trades of a dealer bank are partitioned between
    - proprietary trades
        - hedges of bilateral trades
    - cleared client trades
        - outside the scope of bilateral trading
        - the client provides all the related variation margin
        - the clearning member bank does not post any initial margin to the client
- The proprietary trading and client trading of a clearing member bank are two netting sets of deals between the bank and the CCP, giving rise to two separate lines of variation margin initial margin that needs to be posted by the bank to the CCP

Hedging by the bank comes in “3.5 different ways”



- A specificity of the CCP as a counterparty is that, as long as it is nondefault, it completes all its deals with its clearing members to their contractually promised values (assuming no cap on the unfunded default fund).
- However, a CCP is nothing but the collection of its clearing members.
- It has no resources of its own.
- As long as it is nondefault, i.e. as long as at least one of the clearing members is nondefault, it can only ensure the cash flow completions rendered necessary by the defaults of some clearing members by redirecting the corresponding losses on the surviving clearing members.

- This participation of the clearing members to the losses triggered by the defaults of the other members corresponds in our setup to the usage by the CCP of their default fund contributions, both funded and unfunded.
- The former refers to a predefined amount, computed on a "cover two" basis and allocated between the clearing members through a blend of fairness and practical considerations.
  - an additional, mutualized level of initial margin
- The latter corresponds to additional refills that can be required by the CCP, often up to some cap in practice, without bounds in our model, in case the funded default fund contributions of the surviving members are not enough
  - capital at risk of the clearing members

- One can then proceed by application to the thus-understood financial network of the XVA first principles of Chapter 1.
- With respect to the XVA equations of previous chapters where there were no CCPs, we expect to obtain
    - further cost centers for the bank corresponding to, for each CCP that the bank is a clearing member of:
        - cleared client trades defaults,
        - other clearing member defaults,
        - initial margin and funded default fund funding expenses for the cleared client and proprietary trading of the bank with the CCP ,
    - corresponding CVA and MVA terms in the overall reserve capital (CA) of the bank,
    - corresponding trading loss components of the bank in its overall trading loss process L,
        - where CA should now be understood everywhere in its completed form,

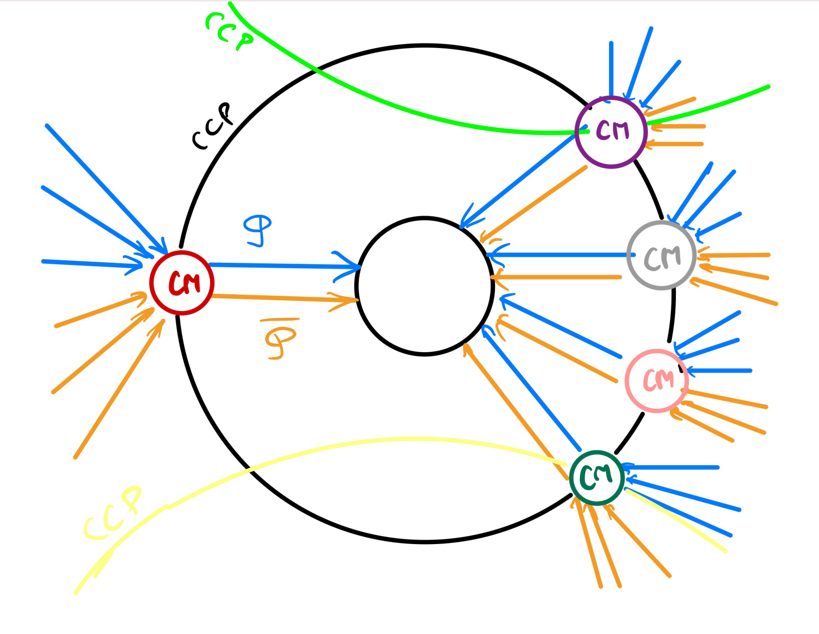
    which then feeds as usual the EC and KVA of the bank.

One-Period Model

- We consider a fix and finite set of so called participants of the CCPs, susceptible to act as clearing members and/or clients.
- Derivative transactions can then be concluded between two participants, or between a clearing member and the CCP.
- CCPs are typically siloed into different services, each devoted to a specific class of derivatives. We first consider a setup with a single CCP with a single service. The extension to several is then provided.

- The client and proprietary trading of a clearing member bank give rise to two netting sets of deals (collateralized independently from each other) between the bank and the CCP.
- See the next figure, where $\mathcal{P}$ and $\overline{\mathcal{P}}$ represent the cash flows contractually promised from cleared and bilateral clients to a reference clearing member, dubbed the bank hereafter,
  - hence promised, in successive turns, from the bank to the CCP, from the CCP to other clearing members, and from the latter to their own clientss
- As a consequence the CCP is flat in terms of market risk, as are also each of the clearing members.
- But market participants are defaultable.

Contractually promised cash flows.



For any derivative contract (or set of contracts) indexed by ι, with (aggregated) contractually promised cash flows $\mathcal{P}_\iota$, we denote their counterparty-risk-free value by $\mathrm{MtM}_\iota = \mathbb{E}^*\mathcal{P}_\iota$. Given disjoint sets of indices $I \ni 0$, C, and B, we denote by:

- $J = J_0$ and J_i, $i \in I \setminus \{0\}$, the survival indicator random variables of the bank, with default probability $\mathbb{Q}^*(J = 0) = \gamma$, and of the other clearing members, at time 1;
- $\mathcal{J} = \max_i J_i$, the survival indicator random variable of the CCP (i.e. of at least one clearing member),
- $\mathcal{P}_i$, MtM_i, IM_i, and DF_i, the contractually promised cash flows, variation margin, initial margin, and funded default fund contribution from the clearing member i to the CCP corresponding to the cleared client trading netting set of the member i;

- $\overline{\mathcal{P}}_i$, $\overline{\mathrm{MtM}}_i$, $\overline{\mathrm{IM}}_i$, and $\overline{\mathrm{DF}}_i$, the contractually promised cash flows, variation margin, initial margin, and funded default fund contribution from the clearing member i to the CCP corresponding to the proprietary trading netting set of the clearing member i;
- J_c, $c \in C$, the survival indicator random variable of the client of the cleared trading netting set c of the bank, and $\mathcal{P}_c$, MtM_c, and IM_c, the associated contractually promised cash flows, variation margin, and initial margin from the corresponding client to the bank;
- J_b, $b \in B$, the survival indicator random variable of the client of the bilaterally netting set b of the bank, and $\mathcal{P}_b$, VM_b, and IM_b, the associated contractually promised cash flows, variation margin, and initial margin from the corresponding client to the bank;
- $\mathcal{L}$, the overall loss (refill requirements) of the CCP;
- $\mu = J\mu$, the proportion of the CCP losses allocated to the reference clearing member.

- Note that $\sum_c \mathcal{P}_c = \mathcal{P}_0$, but not necessarily $\sum_b \mathcal{P}_b = \overline{\mathcal{P}}_0$, as part of the bilateral derivative trading of the bank is hedged via futures-style instruments on exchanges, i.e. via hedging instruments that can be re-hypothecated and on which the bank bears no XVA exposure.
- Namely, on the hedge side of the bilateral derivative trading of the bank, $\overline{\mathcal{P}}_0$ is promised to the CCP and $\sum_b \mathcal{P}_b - \overline{\mathcal{P}}_0$ to the exchange, in exchange of a payment of $\mathbb{E}^* \overline{\mathcal{P}}_0$ from the CCP and $\mathbb{E}^*(\sum_b \mathcal{P}_b - \overline{\mathcal{P}}_0)$ minus a suitable liquidity (e.g. repo) basis from the exchange.
  - Indexing by e in a finite set E disjoint from $I \cup C \cup B$ the deals with the exchange, so $\sum_b \mathcal{P}_b - \overline{\mathcal{P}}_0 = \sum_e \mathcal{P}_e$, we might assume an exchange liquidity basis of the form

$$\mathrm{LVA} = \alpha \sum_e |\mathrm{MtM}_e|.$$

  - For simplicity the liquidity basis is ignored hereafter, i.e. we set $\alpha = 0$.

Lemma 5

We have

$$JC = \sum_c (1 - J_c)(\mathcal{P}_c - \mathrm{MtM}$$

$$+ \sum_b (1 - J_b)(\mathcal{P}_b - \mathrm{VM}_b - \mathrm{IM_b})^+,$$

where

$$\mathcal{L} = \sum_i (1 - J_i)\big((\mathcal{P}_i - \mathrm{MtM}_i - \mathrm{IM}_i - \mathrm{DF}_i)^+ + (\overline{\mathcal{P}}_i - \overline{\mathrm{MtM}}_i - \overline{\mathrm{IM}}_i - \overline{\mathrm{DF}}_i)^+\big). \quad (50)$$

Proof. On the CCP survival event $\{\mathcal{J}=1\}$, for any member i that defaults, the CCP must complete to $\mathcal{P}_i$ the member to CCP default cash flow worth

$$\mathcal{P}_i \wedge (\mathrm{MtM}_i + \mathrm{IM}_i + \mathrm{DF}_i) + \overline{\mathcal{P}}_i \wedge (\overline{\mathrm{MtM}}_i + \overline{\mathrm{IM}}_i + \overline{\mathrm{DF}}_i),$$

which costs $(\mathcal{P}_i - \mathrm{MtM}_i - \mathrm{IM}_i - \mathrm{DF}_i)^+ + (\overline{\mathcal{P}}_i - \overline{\mathrm{MtM}}_i - \overline{\mathrm{IM}}_i - \overline{\mathrm{DF}}_i)^+$. This proves (51).

On the bank survival event $\{J=1\}$, the bank receives from its clients

$$\sum_c \Big(J_c\mathcal{P}_c + (1-J_c)\big(\mathcal{P}_c \wedge (\mathrm{MtM}_c + \mathrm{IM}_c)\big)\Big) \\ + \sum_b \Big(J_b\mathcal{P}_b + (1-J_b)\big(\mathcal{P}_b \wedge (\mathrm{VM_b} + \mathrm{IM_b})\big)\Big)$$

and pays $\sum_c \mathcal{P}_c + \overline{\mathcal{P}}_0$ to the CCP and $\sum_b \mathcal{P}_b - \overline{\mathcal{P}}_0$ to the exchange, so to the CCP and to the exchange

$$\sum_c \big(J_c\mathcal{P}_c + (1-J_c)\ \mathcal{P}_c\big) + \sum_b \big(J_b\mathcal{P}_b + (1-J_b)\ \mathcal{P}_b\big).$$

In total the bank pays

$$\sum_c (1 - J_c)(\mathcal{P}_c - \mathrm{MtM}_c - \mathrm{IM}_c)^+ + \sum_b (1 - J_b)(\mathcal{P}_b - \mathrm{VM}_b - \mathrm{IM_b})^+,$$

plus its contribution $\mu\mathcal{L}$ to the CCP default losses. ∎

Lemma 6

We have

$$J\mathcal{F} = \gamma(\mathrm{IM} + \mathrm{DF} + \overline{\mathrm{IM}} + \overline{\mathrm{DF}}) + \sum_{\mathrm{b}} \gamma \mathrm{IM}_{\mathrm{b}}$$
$$+ \gamma\Big(\sum_{b} (\mathrm{MtM}_{b} - \mathrm{VM}_{b}) - \mathrm{CA} - \mathsf{max}(\mathrm{EC}, \mathrm{KVA})\Big)^{+}.$$

Proof. On the bilateral trades of the bank and their hedges, the Treasury of the bank receives $\sum_b \mathrm{VM}_b$ of variation margin from the clients and has to raise an aggregated amount $\sum_b \mathrm{MtM}_b$ of variation margin. Note that the latter holds whatever the split of the hedges of the trades belonging to the netting sets b between proprietary cleared trades of the bank, for which the Treasury needs to raise the corresponding mark-to-market amount so that the clean desks of the bank can post it as variation margin to the CCP, and hedges on exchanges, for which the Treasury needs to raise the corresponding mark-to-market amount and deposit it on the clean margin account of the clean desks.

As explained in earlier chapters, the bank can freely use the amounts CA and $\max(\mathrm{EC}, \mathrm{KVA})$ on its reserve capital and capital at risk accounts for its variation margin posting purposes, whereas the initial margin must be borrowed entirely. This yields the result.

- Let $\mathbb{E}$ denote the expectation with respect to the bank survival measure $\mathbb{Q}$ associated with $\mathbb{Q}^*$, i.e., for any random variable $\mathcal{Y}$,

$$\mathbb{E}\mathcal{Y} = (1-\gamma)^{-1}\mathbb{E}^*(J\mathcal{Y}).$$

(expectation of $\mathcal{Y}$ conditional on the survival of the bank).

- As already seen in a purely bilateral static setup:

Lemma 7

For any random variable $\mathcal{Y}$ and constant Y, we have

$$Y = \mathbb{E}^*(J\mathcal{Y} + (1-J)Y) \Longleftrightarrow Y = \mathbb{E}\mathcal{Y}.$$

- Under a cost-of-capital XVA approach, the CVA desk and the Treasury (funding desks) of the bank charge their future losses to the clients of the bank at a CA level making $J(\mathcal{C}+\mathcal{F}-\mathrm{CA})$, their shareholder trading loss, $\mathbb{Q}^*$ centered.
- In addition, the management of the bank ensures to the shareholders expected dividends equal to a certain hurdle rate h times their capital at risk $(\mathrm{EC}-\mathrm{KVA})^+$, where EC is modeled as $\mathbb{ES}\big(J(\mathcal{C}+\mathcal{F}-\mathrm{CA})\big)$, the expected shortfall of the bank trading loss and profit computed under its survival measure at a quantile level of 97.5%.
- Accordingly:

Definition 6

$$\mathrm{CA} = \mathrm{CCVA} + \mathrm{CMVA} + \mathrm{BCVA} + \mathrm{BMVA} + \mathrm{FVA}, \text{ where}$$

$$\mathrm{CCVA} = \mathbb{E}^*\Big(J\big(\sum_c (1 - J_c)(\mathcal{P}_c - \mathrm{MtM}_c - \mathrm{IM}_c)^+ + \mu\mathcal{L}\big) + (1 - J)\mathrm{CCVA}\Big),$$

$$\mathrm{CMVA} = \mathbb{E}^*\big(J\gamma(\mathrm{IM} + \mathrm{DF} + \overline{\mathrm{IM}} + \overline{\mathrm{DF}}) + (1 - \mathrm{J})\mathrm{CMVA}\big),$$

$$\mathrm{BCVA} = \mathbb{E}^*\big(J\sum_b (1 - J_b)(\mathcal{P}_b - \mathrm{VM}_b - \mathrm{IM_b})^+ + (1 - \mathrm{J})\mathrm{BCVA}\big),$$

$$\mathrm{BMVA} = \mathbb{E}^*\big(J\sum_b \gamma\mathrm{IM_b} + (1 - \mathrm{J})\mathrm{BMVA}\big),$$

$$\mathrm{FVA} = \mathbb{E}^*\Big(J\big(\sum_b (\mathrm{MtM}_b - \mathrm{VM}_b) - \mathrm{CA} - \max(\mathrm{EC}, \mathrm{KVA})\big)^+ + (1 - J)\mathrm{FVA}\Big),$$

$$\mathrm{KVA} = \mathbb{E}^*\big(Jh(\mathrm{EC} - \mathrm{KVA})^+ + (1 - J)\mathrm{KVA}\big), \text{ where}$$

$$\mathrm{EC} = \mathbb{ES}\big(J(\mathcal{C} + \mathcal{F} - \mathrm{CA})\big).$$

Proposition 8

$$\mathrm{CCVA} = \mathbb{E}\Big[\sum_c (1 - J_c)(\mathcal{P}_c - \mathrm{MtM}_c - \mathrm{IM}_c)^+ + \mu\mathcal{L}\Big], \textit{ where}$$

$$\mathcal{L} = \mathbb{E}\Big[\mu\sum_i (1 - J_i)\big((\mathcal{P}_i - \mathrm{MtM}_i - \mathrm{IM_i} - \mathrm{DF_i})^+ + (\overline{\mathcal{P}}_i - \overline{\mathrm{MtM}}_i - \overline{\mathrm{IM}}_i - \overline{\mathrm{DF}}_i)^+\big)\Big],$$

$$\mathrm{CMVA} = \gamma(\mathrm{IM} + \mathrm{DF} + \overline{\mathrm{IM}} + \overline{\mathrm{DF}}),$$

$$\mathrm{BCVA} = \mathbb{E}\Big(\sum_b (1 - J_b)(\mathcal{P}_b - \mathrm{VM}_b - \mathrm{IM_b})^+\Big),$$

$$\mathrm{BMVA} = \mathbb{E}\Big(\sum_b \gamma\mathrm{IM_b}\Big).$$

Proof. These formulas directly follow from Definition 6 and Lemma 7 by definition of the involved cash flows in Lemma 5. ∎

Proposition 9

$$\mathrm{EC} = \mathbb{ES}(J(\mathcal{C} + \mathcal{F} - \mathrm{CA})) = \mathbb{ES}(J(\mathcal{C} - \mathrm{CVA})), \text{ where}$$

$$J(\mathcal{C} - \mathrm{CVA}) = J\Big(\sum_c (1 - J_c)(\mathcal{P}_c - \mathrm{MtM}_c - \mathrm{IM}_c)^+ + \mu\mathcal{L} - \mathrm{CCVA}$$
$$+ \sum_b (1 - J_b)(\mathcal{P}_b - \mathrm{VM}_b - \mathrm{IM}_\mathrm{b})^+ - \mathrm{BCVA}\Big),$$

$$\mathrm{FVA} = \frac{\gamma}{1+\gamma}\Big(\sum_b (\mathrm{MtM}_b - \mathrm{VM}_b)$$
$$- (\mathrm{CCVA} + \mathrm{CMVA} + \mathrm{CVA} + \mathrm{MVA}) - \mathrm{EC}\Big)^+,$$

$$\mathrm{KVA} = \frac{h}{1+h}\mathrm{EC}.$$

Proof. Lemma 7 implies that
$\mathrm{KVA} = \mathbb{E}\big(h(\mathrm{EC} - \mathrm{KVA})^+\big) = h(\mathrm{EC} - \mathrm{KVA})^+$. As h is nonnegative, this KVA semilinear equation is equivalent to the KVA formula in the proposition.
In particular (for $h \le 1$), $\mathrm{KVA} \le \mathrm{EC}$, i.e. $\max(\mathrm{EC}, \mathrm{KVA}) = \mathrm{EC}$. This and Lemma 7 yield $\mathrm{FVA} = \mathbb{E}\Big(\big(\sum_b(\mathrm{MtM}_b - \mathrm{VM}_b) - \mathrm{CA} - \mathrm{EC}\big)^+\Big) = \big(\sum_b(\mathrm{MtM}_b - \mathrm{VM}_b) - \mathrm{CA} - \mathrm{EC}\big)^+$. As $\mathrm{CA} = \mathrm{CCVA} + \mathrm{CMVA} + \mathrm{BCVA} + \mathrm{BMVA} + \mathrm{FVA}$, this is an FVA semilinear equation, which, as γ is nonnegative, is equivalent to the FVA formula

$$\mathrm{FVA} = \frac{\gamma}{1+\gamma}\big(\sum_b(\mathrm{MtM}_b - \mathrm{VM}_b) - (\mathrm{CCVA} + \mathrm{CMVA} + \mathrm{CVA} + \mathrm{MVA}) - \mathrm{EC}\big)^+.$$

Last, we have $\mathrm{EC} = \mathbb{ES}(J(\mathcal{C} + \mathcal{F} - \mathrm{CA}))$, where the formula for $J(\mathcal{C} + \mathcal{F} - \mathrm{CA})$ in the proposition is obtained by substituting the already derived XVA formulas in Lemmas 5 and 6.

- In the realistic case where the reference bank is a clearing member of several services of one or several CCPs, we index all the CCP related quantities in the above by an additional index *ccp* in a finite set disjoint from $I \cup C \cup B \cup E$.
- The only changes to the cash flow results of Lemmas 5 and 6 are that the centrally cleared trading default losses and initial margin borrowing requirements must be summed over the various CCPs in which the bank is involved as a clearing member, i.e. turned into

$$\sum_{ccp}\Big(\sum_{c}(1-J_c)(\mathcal{P}_c^{ccp}-\mathrm{MtM}_c^{ccp}-\mathrm{IM}_c^{ccp})^+ + \mu^{ccp}\mathcal{L}^{ccp}\Big)$$

and

$$\sum_{ccp}(\mathrm{IM}^{\mathrm{ccp}}+\mathrm{DF}^{\mathrm{ccp}}+\overline{\mathrm{IM}}^{\mathrm{ccp}}+\overline{\mathrm{DF}}^{\mathrm{ccp}}).$$

The rest of the analysis proceeds as before.

Hence we have the exact same formulas as before, except for:

Proposition 10

$$\mathrm{CCVA} = \mathbb{E}\Big[\sum_{c,ccp}\big((1-J_c)(\mathcal{P}_c^{ccp} - \mathrm{MtM}_c^{ccp} - \mathrm{IM}_c^{ccp})^+ + \mu^{ccp}\mathcal{L}^{ccp}\big)\Big], \textit{ where}$$

$$\mathcal{L}^{ccp} = \mathbb{E}\Big[\mu\sum_{i,ccp}(1-J_i)\big((\mathcal{P}_i^{ccp} - \mathrm{MtM}_i^{ccp} - \mathrm{IM}_\mathrm{i}^\mathrm{ccp} - \mathrm{DF}_\mathrm{i}^\mathrm{ccp})^+ + (\overline{\mathcal{P}}_i^{ccp} - \overline{\mathrm{MtM}}_i^{ccp} - \overline{\mathrm{IM}}_i^{ccp} - \overline{\mathrm{DF}}_i^{ccp})^+\big)\Big],$$

$$\mathrm{CMVA} = \gamma\sum_{ccp}(\mathrm{IM}^\mathrm{ccp} + \mathrm{DF}^\mathrm{ccp} + \overline{\mathrm{IM}}^\mathrm{ccp} + \overline{\mathrm{DF}}^\mathrm{ccp}),$$

$$J(\mathcal{C} - \mathrm{CVA}) = J\Big(\sum_{c,ccp}(1-J_c)(\mathcal{P}_c^{ccp} - \mathrm{MtM}_c^{ccp} - \mathrm{IM}_c^{ccp})^+ + \sum_{ccp}\mu^{ccp}\mathcal{L}^{ccp} - \mathrm{CCVA} + \sum_b(1-J_b)(\mathcal{P}_b - \mathrm{VM}_b - \mathrm{IM}_\mathrm{b})^+ - \mathrm{BCVA}\Big).$$

Continuous-Time Model

- Minimal continuous-time setup
  - Single CCP services, no bilateral or proprietary trading
  - Risky clearing members $i = 0, \ldots, n$ with default time τ_i and survival indicator processes J^i
  - Risk-free "buffer"
    - The CCP itself is default-free
  - The external clients of the banks are default-free.
- CCP portfolio with final horizon (margin period of risk δ included) T
  - notation $t^\delta = t + \delta$, in particular clearing member i portfolio is liquidated at time τ_i^δ
  - joint default and liquidation times τ_Z and τ_Z^δ, $Z \subseteq \{0, 1, 2, \ldots, n\}$

- Cash Flows ingredients:
  - mark-to-market processes MtM^i, unpaid cash flows accumulated during the margin period of risk Δ^i,
  - margins VM^i and IM^i,
  - IM funding spread $\lambda^i \leq$ unsecured funding spread $\bar{\lambda}^i$,
  - No funded default fund but unfunded default fund without cap, i.e. the survivors pay for the losses triggered by defaultors above their margins.
- Pricing Setup:
  - Pricing stochastic basis $(\mathbb{G}, \mathbb{Q}^*)$, with model filtration $\mathbb{G}$ and risk-neutral pricing measure $\mathbb{Q}^*$, corresponding expectation and conditional expectation denoted by $\mathbb{E}^*$ and $\mathbb{E}_t^*$
  - We also introduce the $\mathbb{Q}^*$ value-at-risk and expected shortfall of level a ($\geq 50\%$), $\mathbb{V}\mathrm{a}\mathbb{R}^{*,a}$ and $\mathbb{ES}^{*,a}$, and their conditional versions $\mathbb{V}\mathrm{a}\mathbb{R}_t^{*,a}$ and $\mathbb{ES}_t^{*,a}$.
  - Clearing member i XVA metrics computed with respect to its survival measure $\mathbb{Q}^i$, with expectation denoted by $\mathbb{E}^i$
  - $r = 0$

By application of the results of Crépey and Song (2018) (cf. the condition (A) there):

Lemma 8

For every $\mathbb{Q}^i$ *(resp. sub-, resp. resp. super-) martingale* Y*, the process* Y *stopped before* τ_i*, i.e.* $J^i Y + (1 - J^i) Y_{\tau_i -}$*, is a* $\mathbb{Q}^*$ *(resp. sub-, resp. resp. super-) martingale.* ∎

Remark 6

This survival measure formulation is a light presentation of an underlying reduction of filtration setup similar to the one of Chapter 4 (for each clearing member with regard to the CCP). Regarding Lemma 8, see also Collin-Dufresne, Goldstein, and Hugonnier (2004, Lemma 1). ∎

Member CVA, MVA, and KVA

- Let $\epsilon^i_{\tau^\delta_Z} = \mu^i_{\tau^\delta_Z} \sum_{j\in Z} \left(\mathrm{MtM}^j_{\tau^\delta_Z} + \Delta^j_{\tau^\delta_Z} - (\mathrm{VM}^j_{\tau_Z -} + \mathrm{IM}^j_{\tau_Z -})\right)^+$
- We have $L^i_0 = 0$ and, for $t < \tau_i$,

$$dL^i_t = J^i_t \sum_Z \epsilon^i_{\tau^\delta_Z} \boldsymbol{\delta}_{\tau^\delta_Z}(dt) + \lambda^i_t \mathrm{IM}^i_t dt + d\mathrm{CVA}^i_t + d\mathrm{MVA}^i_t, \quad (51)$$

where for $t < \tau_i$,

$$\mathrm{CVA}^i_t = \mathbb{E}^i_t \sum_{t<\tau^\delta_Z \le T} \mu^i_{\tau^\delta_Z} \sum_{j\in Z} \left(\mathrm{MtM}^j_{\tau^\delta_Z} + \Delta^j_{\tau^\delta_Z} - (\mathrm{VM}^j_{\tau_Z -} + \mathrm{IM}^j_{\tau_Z -})\right)^+,$$

$$\mathrm{MVA}^i_t = \mathbb{E}^i_t \int_t^T \lambda^i_s \mathrm{IM}^i_s ds$$

- Member i shareholder capital at risk and capital valuation adjustment, SCR^i and KVA^i, defined as usual based on the above loss process L^i.

Funds Transfer Pricing Policy

- Let the trade incremental

$$\mathrm{FTP}^i = \Delta\mathrm{CVA}^i + \Delta\mathrm{MVA}^i + \Delta\mathrm{KVA}^i.$$

- The all-inclusive XVA add-on passed to the client of a new deal is

$$\mathrm{FTP} = \sum_i J^i \mathrm{FTP}^i, \tag{52}$$

used for resetting the reserve capital and risk margin accounts of each alive member i by $(\Delta\mathrm{CVA}^i + \Delta\mathrm{MVA}^i)$ and $\Delta\mathrm{KVA}^i$.

Hurdle rate

Proposition 11

Under the above dynamic and trade incremental cost-of-capital XVA strategy, assuming that the model filtration $\mathbb{G}$ is quasi-left continuous and that deal times are $\mathbb{G}$ predictable, then the cumulative dividend processes of the shareholders of each clearing member i are $\mathbb{Q}^*$ submartingales on $\mathbb{R}_+$ with drift coefficients $h\mathrm{SCR}^i$ killed at τ_i.

Proof

- Between time 0 and before the next deal, the dividends to clearing member i shareholders correspond to the process

$$-(dL_t^i + d\mathrm{KVA}_t^i), \tag{53}$$

stopped before time τ_i.

- Moreover, if the next deal time θ_1 is finite, the funds transfer policy defined by the above FTP allows the CCP to reset the reserve capital and risk margin accounts of each clearing member to their theoretical target values corresponding to the new CCP portfolio (including the new deal), without contribution of the clearing members themselves
  - No dividend at θ_1, as all the money required for these resets is sourced from the clients of the deals.

- In addition, in a quasi-left continuous filtration, a trading loss L^i and a process KVA^i as per (51) and (**??**) cannot not jump at a predictable time θ_1, so that the equality between clearing member i dividends and the process (53) stopped before τ_i holds until θ_1 included.

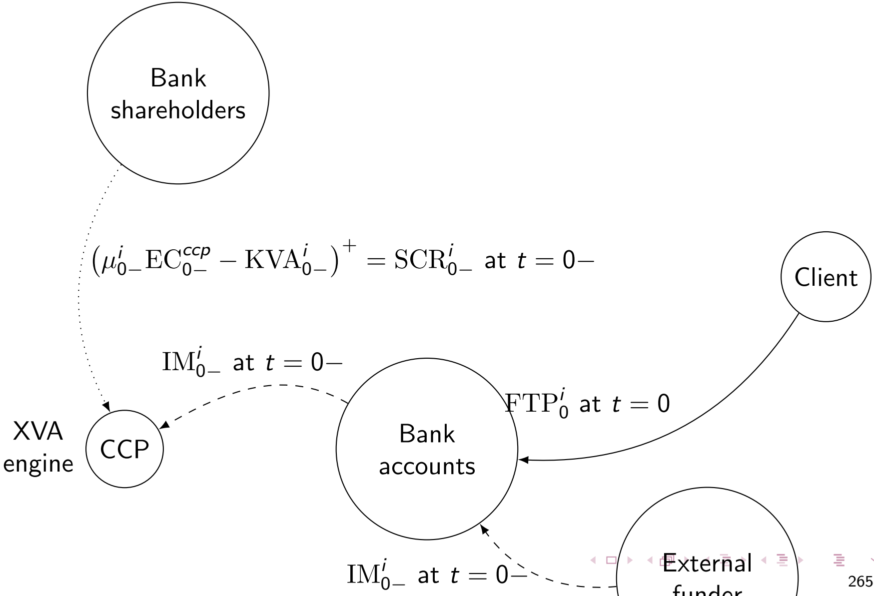
- By definition of L^i and KVA^i, the process (53) (or this process stopped before τ_i) is a $\mathbb{Q}^i$ submartingale with drift coefficient

$$h(\mu^i \mathcal{EC} - \mathrm{KVA}^i)^+ = h\mathrm{SCR}^i$$

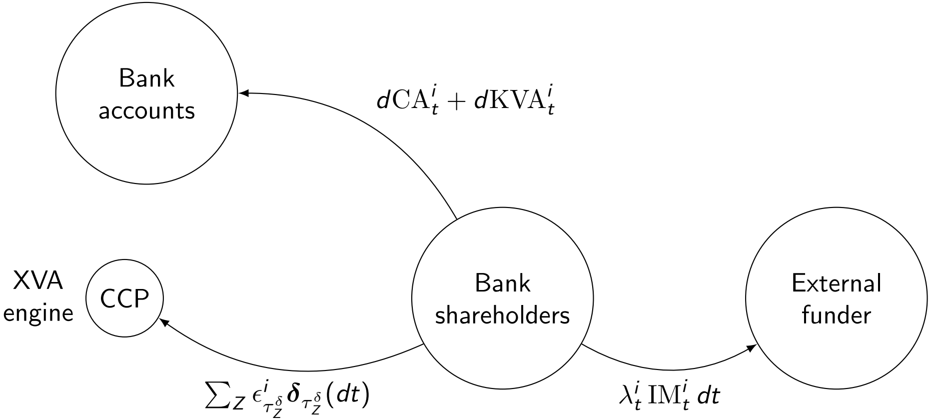
- As a consequence, the process (53) stopped before τ_i is a $\mathbb{Q}^i$ (hence $\mathbb{Q}^*$, by Lemma 8) submartingale with drift coefficient $h\mathrm{SCR}$ killed at τ_i.
- In view if the above, so is therefore the clearing member i dividend process on $[0, \theta_1]$.

- The reset at θ_1 puts us in a position to repeat the above reasoning relative to the time interval $[\theta_1, \theta_2]$, where θ_2 is the following deal time (i.e. on $[\theta_1, +\infty)$ if no next deal happens and θ_2 is infinite), and so on iteratively, so that the stated submartingale property holds on $\mathbb{R}_+$. ∎

Cash-flows affecting the clearing member bank at $t = 0 = \theta_0$



Cash-flows affecting the clearing member bank in $[t; t + \mathrm{d}t], 0 < t < \theta_1 \wedge \tau^{\cdot}$



Comments

- As XVA computations are delegated to the CCP, the vexing modeling situation mentioned above is solved.
- The CCP is even in a position to decide to which clearing member a new deal should be allocated, optimally in XVA terms at the level of the system as a whole
  - i.e. to the clearing member for which the ensuing FTP (52) is minimal.

The proposed XVA setup offers a risk analysis environment that can be used for versatile purposes, including

- optimal trade allocation
- XVA compression and collateral optimization,
- detection of XVA cross-selling opportunities,
- credit limits monitoring at the trade or counterparty level
    - with sensitivities to credit market (wrong way-risk) and credit credit correlations, which are missing with the usual (purely market risk based) metrics such as the potential future exposure
- reverse stress testing in the context of CCAR exercises
    - capital analysis and review US regulatory framework

Setting the refill allocation weights

- A CCP trading loss process is written as (recall our CCP is default-free):

$$
\begin{aligned}
&\mathcal{L}_0 = 0 \text{ and, for } t \in (0, T],\\
&d\mathcal{L}_t = \sum_i \left((\mathrm{MtM}^i_{\tau_i^\delta} + \Delta^i_{\tau_i^\delta}) - (\mathrm{VM}^i_{\tau_i} + \mathrm{IM}^i_{\tau_i}) \right)^+ \boldsymbol{\delta}_{\tau_i^\delta}(dt)\\
&\qquad + \sum_{i=0}^{n} J^i_t \lambda^i_t \mathrm{IM}^i_t dt\\
&\qquad + d\mathcal{CVA}_t + d\mathcal{MVA}_t
\end{aligned}
\tag{54}
$$

(and $\mathcal{L}$ constant from time T onward),

- where, for $0 \leq t \leq T$,

$$\mathcal{CVA}_t = \mathbb{E}_t^* \sum_{i=0,\ldots,n;\, t<\tau_i^\delta<T} \left((\mathrm{MtM}^i_{\tau_i^\delta} + \Delta^i_{\tau_i^\delta}) - (\mathrm{VM}^i_{\tau_i} + \mathrm{IM}^i_{\tau_i}) \right)^+$$

$$\mathcal{MVA}_t = \mathbb{E}_t^* \int_t^T \sum_{i=0}^n J_s^i \lambda_s^i \mathrm{IM}_s^i ds.$$

- The ensuing economic capital process of the CCP is

$$\mathcal{E}\mathcal{C}_t = \mathbb{ES}_t^{*,a_{df}} \left(\int_t^{t+1} d\mathcal{L}_s \right),$$

where $\mathbb{ES}^{*,a_{df}}$ represents the $\mathbb{Q}^*$ expected shortfall of level a_{df},

- The corresponding member decremental economic capital is

$$\Delta_i \mathcal{E}\mathcal{C} = \mathcal{E}\mathcal{C} - \mathcal{E}\mathcal{C}^{ccp(-i)},$$

where $ccp(-i)$ refers to the CCP deprived from its i^{th} member, i.e. with the i^{th} member replaced by a risk-free "buffer" in all its CCP transactions.

- We can set

$$\mu^i = \frac{\Delta_i \mathcal{E}\mathcal{C}}{\sum_j J^j \Delta_j \mathcal{E}\mathcal{C}}, \text{ for every (alive) member } i.$$

  - reflects the exposure of the CCP to the clearing members, as should be
  - instead of the opposite in the case of an IM proportional allocation.

Specialist Lending of Initial Margin

- Let us consider a bank, say the clearing member 0 in the above
- Removing all indices 0 in the notation, let $\bar{\lambda} = \gamma(1 - R)$ denote the credit spread of the bank, where γ is its risk-neutral default intensity.
- The time-t bank MVA of the bank when its IM is funded through unsecured borrowing is defined by

$$\mathrm{MVA}_t^{ub} = \mathbb{E}_t\left[\int_t^T \bar{\lambda}_s \mathrm{IM}_s ds\right] \tag{55}$$

  - computed as should be with respect to the bank survival measure, in line with a risk-neutral martingale property of the ensuing bank shareholder trading loss process

- However, instead of assuming its initial margin borrowed by the bank on an unsecured basis at the spread $\bar{\lambda}$, one can consider an alternative scheme whereby IM is funded through a liquidity supplier, dubbed "specialist lender" (private equity fund,..), which lends IM and, in case of default, receives back the portion of IM unused to cover losses and a claim against the bank estate.
- Note that, as long as such margin lending is implemented off the balance sheet of the clearing member bank, it is not a violation of pari passu rules. It is just a form of collateralized lending.

- We assume that the specialist lender funds are kept at the segregated account for initial margin.
- In case of default, this account is depleted by the amount $(G^+_{\tau^\delta} \wedge \mathrm{IM}_\tau)$, where the time-$t$ gap G_t is given as

$$G_t = \mathrm{MtM}_t + \Delta_t - \mathrm{VM}_{t-\delta},$$

and the lender receives a claim against the bank estate.

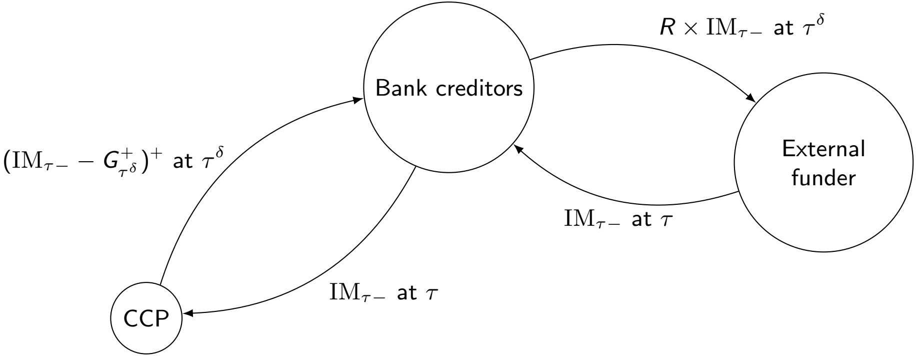
- We assume that the specialist lender funds are kept at the segregated account for initial margin.
- In case of default, this account is depleted by the amount $(G^+_{\tau^\delta} \wedge \mathrm{IM}_\tau)$, where the time-$t$ gap G_t is given as

$$G_t = \mathrm{MtM}_t + \Delta_t - \mathrm{VM}_{t-\delta},$$

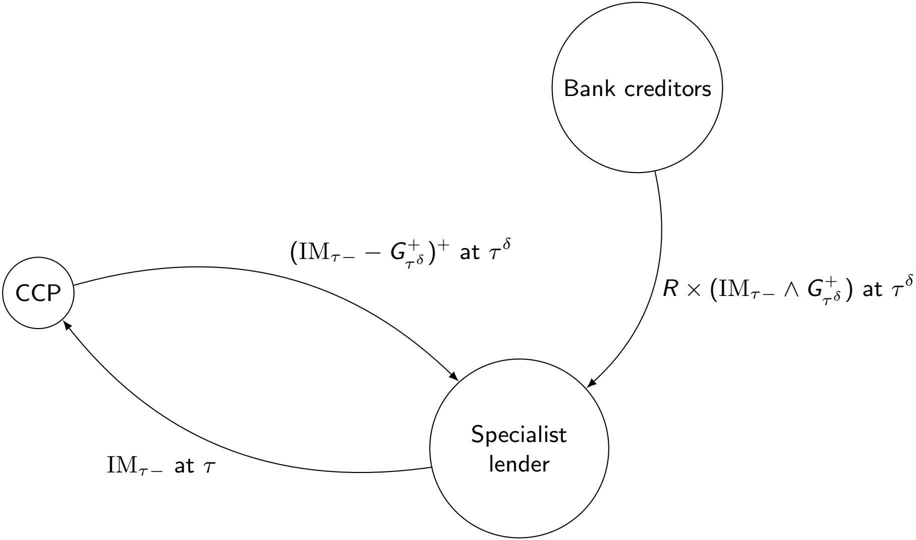
and the lender receives a claim against the bank estate.

- The specialist lender on the bottom panel is lending the same IM amount to the CCP (on behalf of the bank) than the external funder is lending to the bank (who is then lending it to the CCP) on top.
- But, in case the bank defaults, the specialist lender receives back from the CCP the portion of IM unused to cover losses. Hence it is reimbursed at a much higher effective recovery rate than the nominal recovery rate R embedded in the bank credit spread.

Reference clearing member bank own-default related funding cash-flows



Cash-flows affecting the clearing member bank in $[t; t + \mathrm{d}t], 0 < t < \theta_1 \wedge \tau$



- Let ξ be a $\mathbb{G}$ predictable process, which exists by Corollary 3.23 2) in He, Wang, and Yan (1992), such that

$$\xi_\tau = \mathbb{E}^*_{\tau-}\left[(G^+_{\tau^\delta} \wedge \mathrm{IM}_{\tau-})\right] \leq \mathrm{IM}_{\tau-}. \tag{56}$$

- We assume that the clearing member pays continuous-time service fees $\bar{\lambda}_t \xi_t dt = \lambda_t \mathrm{IM}_t dt$ to the specialist lender

- Such arrangement can be deemed fair to the specialist lender, in view of the identities

$$\begin{aligned}\mathbb{E}^*_t\Big[\mathbb{1}_{t<\tau<T}(1-R)\big(G^+_{\tau^\delta}\wedge \mathrm{IM}_{\tau-}\big)\Big]\\ &= \mathbb{E}^*_t\Big[\mathbb{1}_{t<\tau<T}(1-R)\mathbb{E}^*_{\tau-}\big[\big(G^+_{\tau^\delta}\wedge \mathrm{IM}_{\tau-}\big)\big]\Big]\\ &= \mathbb{E}^*_t\Big[\mathbb{1}_{t<\tau<T}(1-R)\xi_\tau\Big] = \mathbb{E}^*_t\left(\int_t^{\tau\wedge T}\bar{\lambda}_s\xi_s ds\right),\ \ 0\le t\le T.\end{aligned} \tag{57}$$

- The corresponding time-t bank MVA of the bank is defined by

$$\mathrm{MVA}_t^{sl} = \mathbb{E}_t[\int_t^T \bar{\lambda}_s \xi_s ds] \tag{58}$$

  - in line with a risk-neutral martingale property of the ensuing bank shareholder trading loss process

Proposition 12

The respective IM funding spreads $\lambda = \lambda^{ub}$ *and* $\lambda = \lambda^{sl}$ *of the bank corresponding to unsecured borrowing and margin lending are* $\lambda^{ub} = \bar{\lambda}$ *and* λ^{sl} *such that*

$$\frac{\lambda^{sl}}{\bar{\lambda}} = \frac{\xi}{\mathrm{IM}_-} \leq 1.$$

In particular, $\mathrm{MVA}^{sl} \leq \mathrm{MVA}^{ub}$. ∎

Proof. In view of (55) and (58), this follows from (56) . ∎

- Under a basic specification where

$$\mathrm{IM}_t = \mathbb{V}\mathrm{a}\mathbb{R}_t^{*,a_{im}}\left(\mathrm{MtM}_{t^\delta} + \Delta_{t^\delta} - \mathrm{MtM}_t\right), \quad t \le \tau \wedge T, \qquad (59)$$

  assuming for simplicity continuous-time variation margining $\mathrm{VM}_t = \mathrm{MtM}_t$ until time τ, then the blending factor is typically significantly less than one.
- Hence λ is significantly less than $\bar{\lambda}$ and MVA_0^{sl} should in turn be significantly less than MVA_0^{ub}.

- But margin lending practicalities are an active investor (such as private equity fund) business, whereas the available liquidity is mostly with passive investors (insurance, pension funds,...).
- Hence, the implementation of margin lending naturally calls for a two-layered structure, whereby an active investor bridges to a passive one.

- The above developments (whether they regard margin lending or unsecured borrowing) are only on the credit side of the problem, with short-term funding assumed continuously rolled over in time.
- As always with credit, there is another, liquidity side to it.

- This is the fact that the lender (the so called external lender in the case of unsecured borrowing and specialist lender in the case of margin lending) may want to cease to roll-over its loan, not because of the credit risk of the borrower, but just because of liquidity squeeze in the market
  - The lender may be short of cash (or liquid assets), or want to keep the latter for other (own) purposes.

- It may then be argued that the liquidity issue is more stringent for margin lending, with its two-layered structuring, than for unsecured borrowing.
    - This is also why margin lending is more difficult to implement for VM than for IM

- In order for an IM margin lending business to obtain the blessing of a regulator, it would be better to address this liquidity issue in the legal structure of the setup.
- This could for instance take the form of an option for the clearing member bank to force one roll-over by the specialist lender (once in the life of the contractual relation, say, and in exchange of a then higher interest rate).
    - Such a "liquidity option" would have to be priced into the structure.

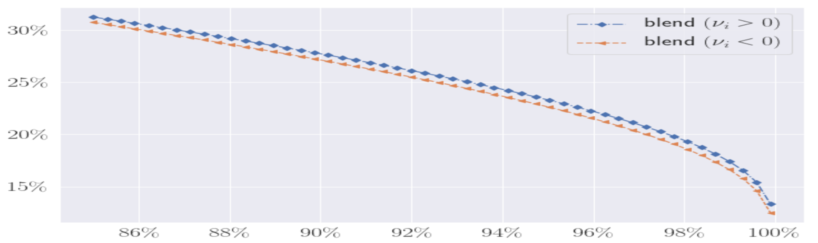
- Unsecured borrowing is not exempt of liquidity issues either
- A detailed comparison of unsecured borrowing and margin lending also accounting for these liquidity issues could be an interesting topic of further research.

Numerics

- We use $m = 10^5$ simulated paths of an underlying Black–Scholes swap rate S and default scenarios, in a CCP toy model consisting of nine clearing members trading the corresponding swap with each other.
- Semi-explicit formulas for all the quantities of interest, except for a term structure of the economical capital of the CCP, which is obtained by Monte Carlo simulation.
- All the reported numbers are in basis points. The nominal of the swap is fixed so that each leg equals $1 = 10^4$ bps at time 0.
- Unless stated otherwise we use $a_{im} = 85\%$ and $a_{df} = 97.5\%$.

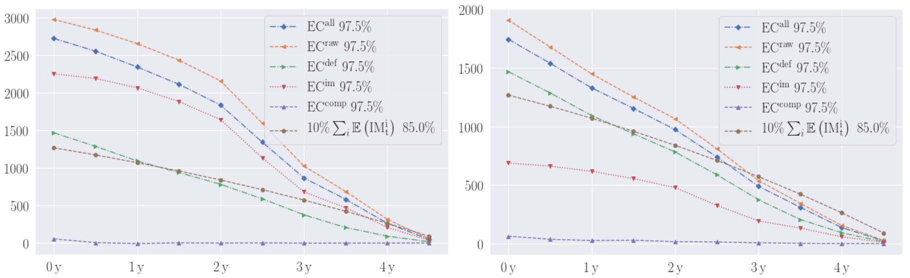
IM specialist lender funding spread blending factor,...

...as a function of the IM quantile level a_{im}, for clearing members short (orange) vs. long (blue) in the swap.



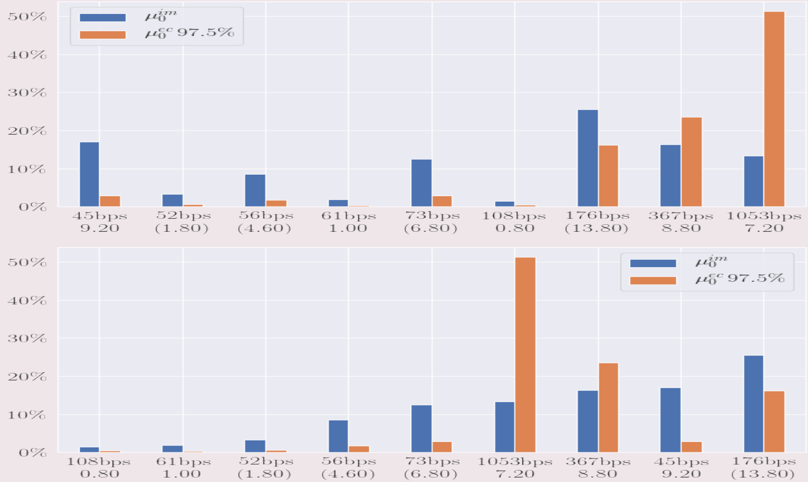
Economic capital based default fund of the CCP, as a function of time *(blue)*,...

...and analogous term structures of economic capital obtained by only considering the terms in the first line in (54) *(orange)*, the terms in the second line *(purple)*, the default and CVA terms *(green)*, and the IM and MVA terms *(red)*. The 10%×IM term structure is also shown *(brown)* as a proxy of what would be the level of a Cover 2 funded default fund. *Left*: Case of unsecured borrowing IM. *Right*: Case of specialist lender IM.



Time-0 default fund allocation based on member initial margin vs. member decremental $\mathcal{EC}$

Bottom: Members ordered by increasing position $|\nu_i|$. *Top*: Members ordered by increasing credit spread Σ_i



XVAs and FTP

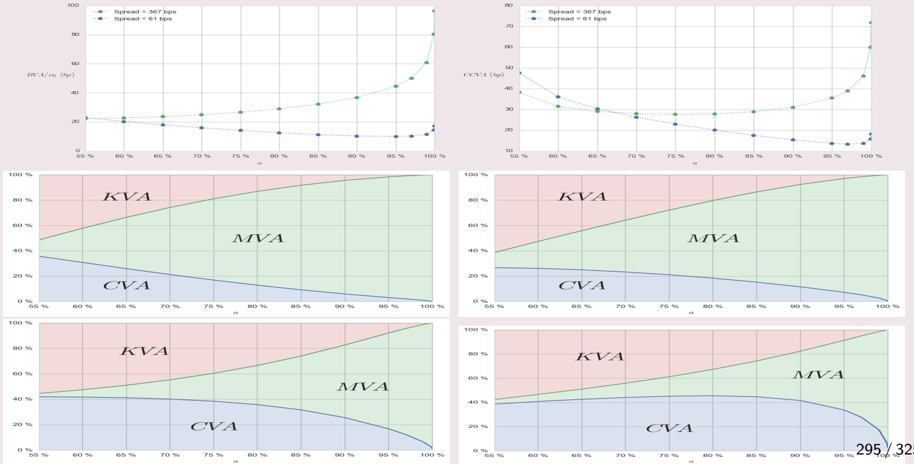
...under unsecured borrowing *(left)* vs. specialist lender *(right)* initial margin

	ub	*sl*
$\sum_{i=0}^{n} \mathrm{CVA}_0^i$	102.23	102.23
$\sum_{i=0}^{n} \mathrm{MVA}_0^i$	804.25	248.28
$\sum_{i=0}^{n} \mathrm{KVA}_0^i$	610.90	353.87
$\mathrm{FTP}_0 =$	1517.38	704.38

Table: CCP portfolio-wide XVAs and FTP under unsecured borrowing *(left)* vs. margin lending *(right)* IM raising policies. ∎

- Unless efficient (such as specialist lender) IM raising strategies are implemented, the IM funding expenses are in fact the main contributor to the economical capital of the CCP.
- This is an illustration of the transfer of counterparty risk into liquidity risk triggered by extensive collateralization.

Varying the quantile level used for setting the IM. *Left*: SIMM setup. *Right*: CCP setup. *Top*: FTP (scaled for netting, in bps for a swap with fixed leg equal to one). *Bottom*: XVA relative contributions (low credit name) *Middle*: XVA relative contributions (high credit name).



Outline

References

- On the funding issue,
  - Burgard and Kjaer (2011, 2013a, 2017) (BK) FVA papers,
  - Andersen, Duffie, and Song (2019).
- On the capital issue,
  - Green, Kenyon, and Dennis (2014) (GK) KVA paper,
  - The Solvency II actuarial literature.

Finance vs. Insurance

- The cost-of-capital XVA approach is a continuous-time and banking extension of the Swiss Solvency Test insurance methodology

finance	insurance
contra-assets $\mathrm{CA} = \mathrm{CVA} + (\mathrm{FVA} + \mathrm{MVA})$	liabilities best estimate or market consistent valuation
priced by conditional expectation of related future cash flows	priced by expectation of related future cash flows
economic capital EC	solvency capital requirement
sized as a conditional ES of future losses over one year	sized as ES of future losses over one year
capital valuation adjustment KVA	market value margin or risk margin
sized as a supermartingale with drift coefficient $h\mathrm{SCR}$ and zero terminal condition	sized as $h\mathrm{SCR}$

Table: finance vs. insurance $(\mathrm{SCR} = (\mathrm{EC} - \mathrm{KVA})^+)$

CVA and FVA: Comparison with the BK Approach

- BK advocate a replication XVA approach and blame risk-neutral approaches outside the realm of replication
  - see the first paragraph in their 2013 paper
- But BK end-up doing what they call semi-replication, which is nothing but a form of risk-neutral pricing without replication.

- BK rightfully claim that only bank pre-default cash-flows matter to share-
  holders. For instance, quoting the first paragraph in their second paper:

  > *"Some authors have considered cases where the post-default cash flows on the funding leg are disregarded but not the ones on the derivative. But it is not clear why some post default cashflows should be disregarded but not others"*

- However being rigorous with the above principle implies that the valuation jump of the portfolio at the own default of the bank should be disregarded in the shareholder cash flow stream. But their computations do not exclude this cash flow.

KVA: Comparison with the GK Approach

- In Green et al. (2014) and as also discussed in some actuarial literature (see Section 4.4 in Salzmann and Wüthrich (2010)), the KVA is treated as a liability.

- Viewing the KVA as a liability, hence part of the trading loss-and-profit of the bank (process L in our notation), implies to view RM as non loss-absorbing, i.e. $\mathrm{SCR} = \mathrm{CR}$ (as opposed to $\mathrm{SCR} = \mathrm{CR} - \mathrm{KVA}$ in our setup),
- which leads to $h\mathrm{CR}'$ instead of $h(\mathrm{CR}' - \mathrm{KVA}')$ in the first line of (26),
- or, equivalently, to no discounting at the hurdle rate h in the KVA formula in the second line (cf. also (40)).

- Moreover, if the KVA is viewed as a liability, forward starting one-year-ahead fluctuations of the KVA must be simulated for economic capital calculation.
- This makes it intractable numerically, which leads to switch from economic capital to regulatory capital in the XVA equations.

- Using (scriptural) regulatory instead of economic capital is less self-consistent.
- It loses the connection whereby the KVA input is the bank shareholder loss process $L^{\tau-}$.

FCA/FBA vs. FVA/FDA Accounting and FTP framework

- If a corporate holds a bank payable, it typically has a desire to close it, receive cash, and restructure the hedge with a par contract
  - The bank would agree to close the deal as a market maker, charging fees for the new trade
- Because of this natural selection, a bank is mostly in the receivables in its derivative business with corporates.
- If, by exception, the derivative portfolio of a bank is mostly in the payables, then FVA numbers are small and matter much less anyway

- This argument is sometimes used to defend a symmetric FVA (SFVA) such as, instead of the above FVA:

$$\mathrm{SFVA}_t = \mathbb{E}_t \int_t^T \widetilde{\lambda}_s(\mathrm{MtM}_s - \mathrm{VM}_s)ds\,, \quad t < \tau,$$

  for some VM blended funding spread $\widetilde{\lambda}_t$

  - cf. Piterbarg (2010), Burgard and Kjaer (2013b), and the discussion in Andersen, Duffie, and Song (2019).

- Such linear SFVA formula can be implemented by integration of (the) mark-to-market cube(s) against funding spread curves
  - no Monte Carlo simulations, not to mention simulation/regression schemes or nested Monte Carlo computations, beyond the generation of the mark-to-market cube
- For a suitably chosen blended spread $\widetilde{\lambda}_t$, the equation yields reasonable results in the case of a typical bank portfolio dominated by unsecured receivables.
- However, in the case of a portfolio dominated by unsecured payables, this equation could yield a negative FVA, i.e. an FVA benefit, proportional to the own credit spread of the bank, which is not acceptable from a regulatory point of view.

$\rightarrow$ FCA/FBA accounting and funds transfer pricing industry standard

  - detailed in the next slides

- An asymmetric FVA/FDA accounting and pricing framework is more rigorous and has been considered in Albanese and Andersen (2014), Albanese, Andersen, and Iabichino (2015), Crépey (2015a), Brigo and Pallavicini (2014), Bielecki and Rutkowski (2015), and Bichuch, Capponi, and Sturm (2018).
- Crépey, Sabbagh, and Song (2020) improve upon such asymmetric FVA models by accounting for the funding source provided by economic capital (cf. (52)).

- Denote by $x_c(\omega)$ = the "debt" (if positive, credit otherwise) of client c to the bank in scenario ω (ω omitted hereafter).
- The economically and mathematically correct formula FVA formula, which should be used both for decision taking and as a capital deduction by banks, is the asymmetric funding set FVA, $\mathbb{E}[(\sum x_c)^+]$.
- Instead, banks are calculating their FVA numbers aggregating over netting sets (i.e. clients) c.

- Such choice does not reflect the economics of collateral management.
- It is only justified by the desire to arrive at the numbers by simply retrofitting CVA calculators, which are based on distributed computing and are performed netting set by netting set, often with netting set specific approximations.

Specifically:

- For all their decision taking purposes, such as hedging and executives compensation (i.e. bonuses), banks use, instead of $\mathbb{E}[(\sum x_c)^+]$ as they should, $\mathbb{E}\sum x_c = \mathbb{E}\sum(x_c^+) - \mathbb{E}\sum(x_c^-)$ =FCA-FBA.
- As a capital deduction, instead of $\mathbb{E}[(\sum x_c)^+]$ again, they just use the FCA number.
  - Indeed, regulators insist that only asymmetric FVA numbers be used for the purpose of calculating a capital deduction.
  - They do not specify the aggregation level which could be at the netting set or funding set level and they are indifferent since the smaller is the level of aggregation, the larger and more conservative is the size of capital deduction.

In normal times:

- equity capital buffers are large enough to absorb the conservative capital deduction.
- Moreover, banks' balance sheets are dominated by assets, i.e. $0 < \sum x_c = (\sum x_c)^+$ holds in most scenarios
  - $\rightarrow$ the symmetric netting set FVA used by banks for decision taking , $\mathbb{E} \sum x_c$, is numerically close to the "right", asymmetric funding set FVA, $\mathbb{E}[(\sum x_c)^+]$

However, in the 2020 Covid crisis:

- Mark-downs swinged bank balance sheets towards liabilities, invalidating the above approximation.
- We saw an (e.g. Goldman Sachs) 8-fold credit spreads widening,
- Increasing default rates put pressure on bank capital.

As a result:

- The number FCA-FBA used for decision taking by banks went further and further from the correct one, implying erroneous hedges and executive compensation;
- The FCA number exploded and the corresponding capital reduction became needlessly punitive for banks, at the precise bad time where capital was becoming a stringent issue for banks
- The discrepancy between the (both wrong) FCA-FBA and FCA numbers increased, enhancing the corresponding misalignment of interest between the executives and the shareholders of the bank.

- A perfect storm to weather, through which only a mathematically rigorous treatment of accounting numbers, capital models and funding strategies can be of guidance.
- Ultimately one has to maximize shareholder value. Even funding strategies with debt are targeted to this ultimate objective.
- Any optimization problem has a merit function which is just a number. A "multi-objective" optimization simply targets to optimize a combination of objective functions, like a weighted sum.
- The perfect optimization target is the present value of future dividend streams, i.e. the KVA.

The concept of funding set aggregation is not only of use in the context of FVA metrics but it has numerous other applications, including:

- Cost of capital metrics such as KVA;
- Reverse stress testing;
- Liquidity risk management.

- Since banks fund themselves by issuing both equity and debt, a holistic optimisation of shareholder value ought to span the entire extent of the capital structure, including the FVA but also the KVA.
- Reverse stress testing can be seen as a third stage of evolution for mathematical methods in finance: pricing is about the calculation of averages, risk measures calculate tail-conditional expectations and reverse stress testing focuses on individual scenarios and their impact on risk measures. A FVA-KVA treatment can provide the foundation of a rigorous approach to the subject.
- A forward looking, simulation based approach to reverse stress testing also provides a suitable environment for liquidity risk management, by identification of the future market scenarios that would lead to an unacceptable level of funding risk, and of ways of dynamically hedging them.

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