XVA Analysis

and the Embedded Probabilistic, Risk Measure, and Machine Learning Issues

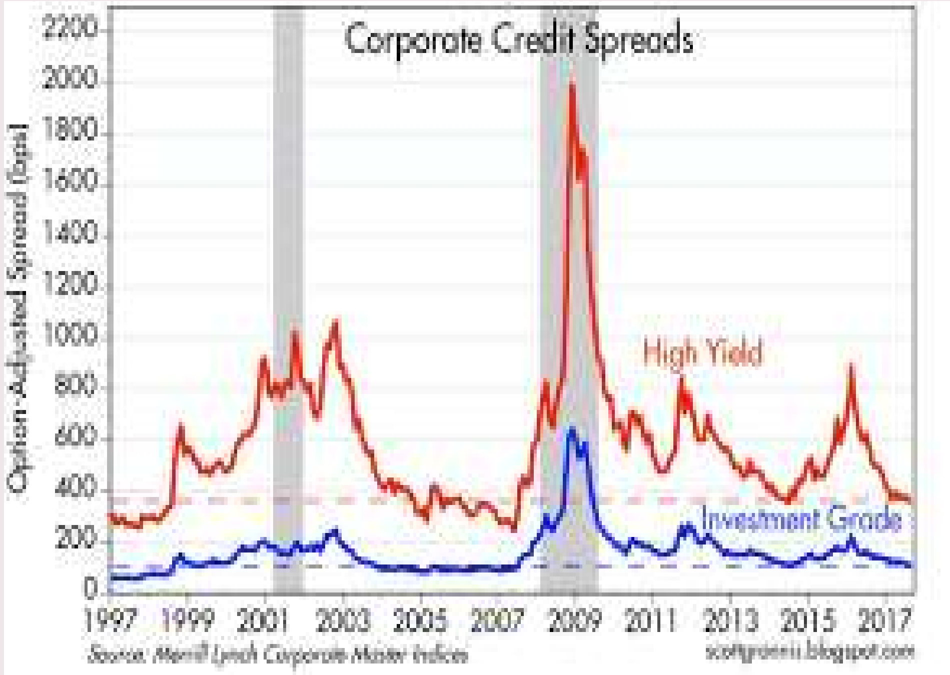
Stéphane Crépey
Université de Paris / LPSM

https://www.lpsm.paris/pageperso/crepey

First Banco de Santander Financial Engineering School, organised by the Centre de Recerca Matemàtica (CRM), Barcelona

March 4 and 11, 2021

(left) corporate and (right) bank credit spreads across the last financial crises; (top) until 2017 and (bottom) covid-19 crisis



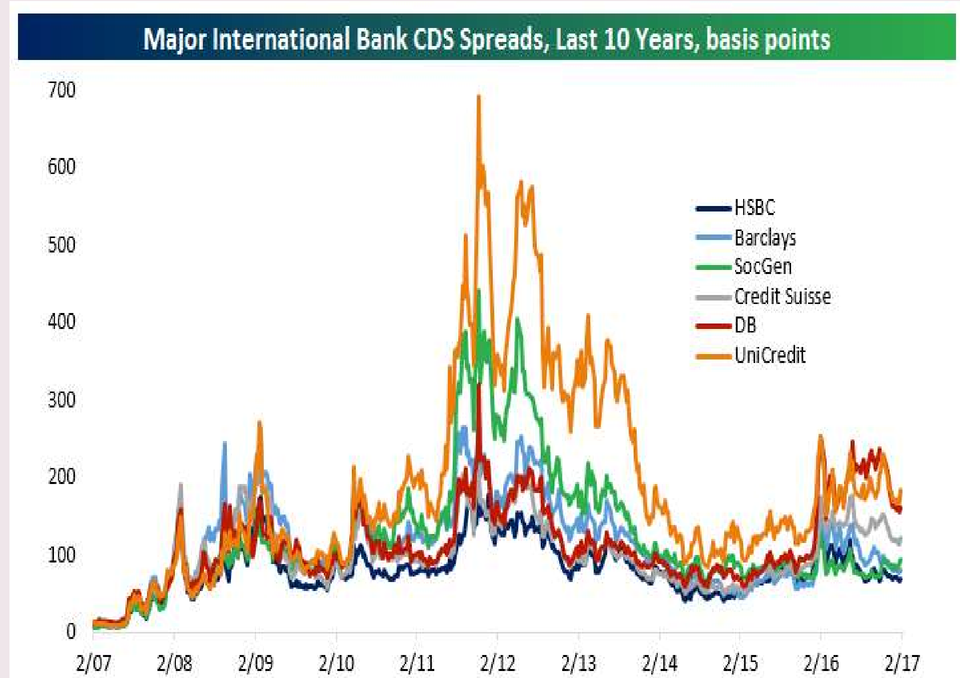
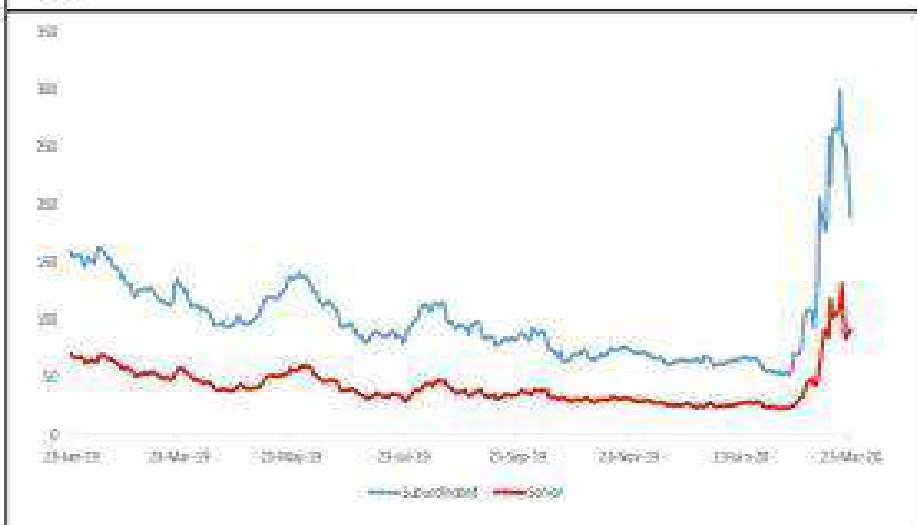
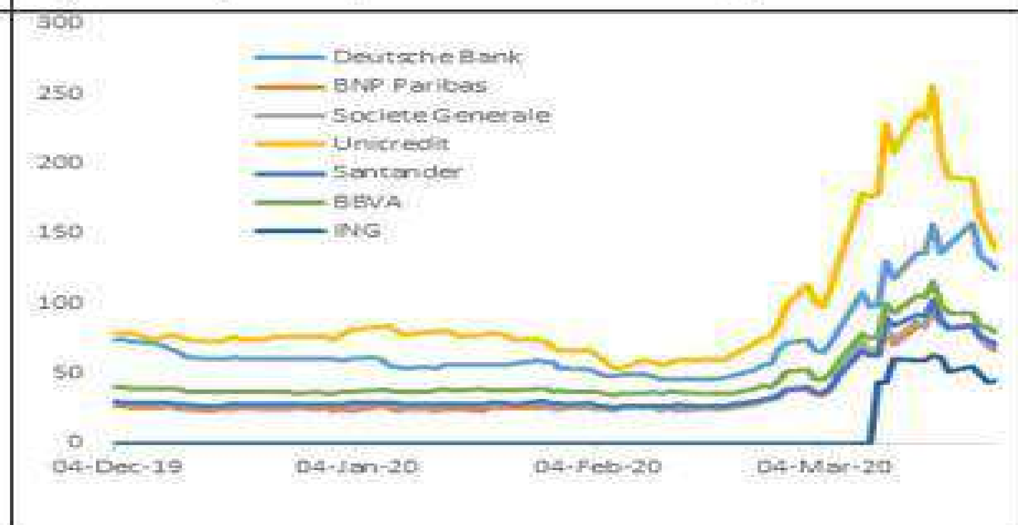


Figure 10: Itraxx indexes for senior and subordinated debt



Source: Bloomberg and ECON calculations. Note: last observation 30 March 2020

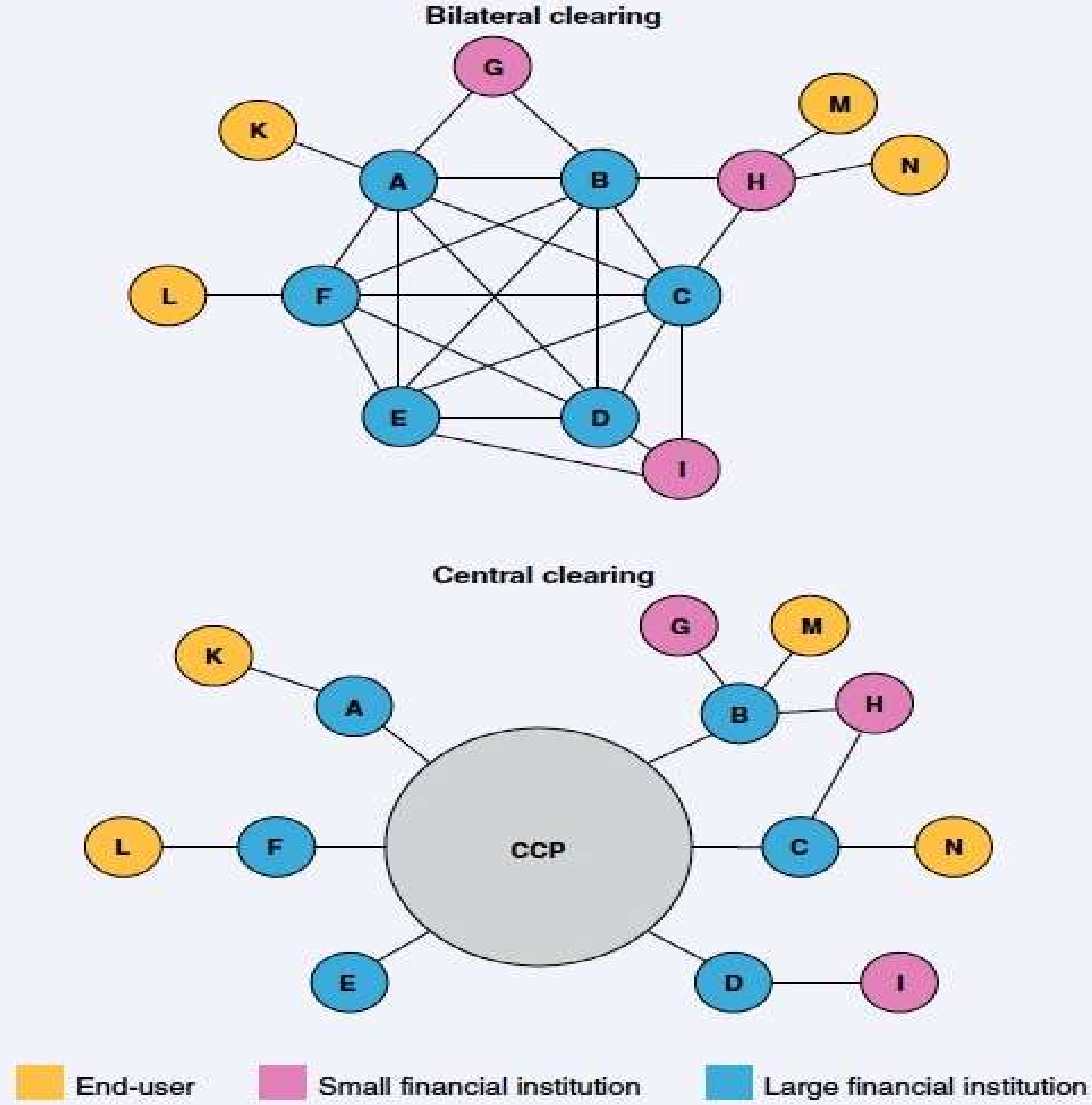
Figure 11: 5yr CDS spreads for selected European banks



Source: Bloomberg and ECON calculations. Note: last observation 30 March

- In the aftermath of the 2008-09 financial crisis, derivative markets regulators launched a major banking reform effort aimed at securing the financial system by raising collateral and capital requirements.
- Clearing of standardized derivatives through central counterparties (CCPs) was progressively enforced or strongly incentivized

1. OTC Derivatives Counterparty Relationships



Source: Reserve Bank of Australia, Central Clearing of OTC Derivatives in Australia (June 2011), available at: http://www.rba.gov.au/publications/consultations/201106-otc-derivatives/central-clearing-otc-derivatives.html

- In (counterparty credit risk) complete markets, collateral and capital requirements would be indifferent to banks.
- The quantification by banks of market incompleteness, based on various XVA metrics, emerged as the unintended consequence of the banking reform.
- XVAs: Pricing add-ons (or rebates) with respect to the counterparty-risk-free value of financial derivatives, meant to account for counterparty risk and its capital and funding implications.
- VA stands for valuation adjustment and X is a catch-all letter to be replaced by C for credit, D for debt, F for funding, M for (initial) margin, and K for capital.

- Pricing XVA add-ons at trade level
  - funds transfer price (FTP)
- But also accounting XVA entries at the aggregate portfolio level
  - In June 2011 the Basel Committee reported that

    During the financial crisis, roughly two-thirds of losses attributed counterparty credit risk were due to CVA losses and only about one-third were due to actual defaults
  - In January 2014 JP Morgan has recorded a $1.5 billion FVA loss
  - https://www.risk.net/derivatives/7526696/fva-losses-back-in-spotlight-after-coronavirus-stress

    Banks face a new round of losses after two key inputs for calculati funding costs for uncollateralised derivatives—interest rates and fundi spreads—saw wild moves last month, contributing to a combined loss almost $2 billion at Bank of America, Goldman Sachs and JP Morga
- Individual FTP of a trade actually defined as trade portfolio incremental XVAs of the trade

Objectives of the course

- Deriving sound, principle based XVA metrics, for both bilateral and centrally cleared transactions
- Addressing the related computational challenges

References

Mainly based on:

- S. Crépey. Positive XVAs, *Working Paper* 2021.
- C. Albanese, S. Crépey, Rodney Hoskinson, and Bouazza Saadeddine. XVA Analysis From the Balance Sheet, *Quantitative Finance* 2021.
- C. Albanese, S. Crépey, and Yannick Armenti. XVA metrics for CCP optimization, *Statistics & Risk Modeling* 2020.

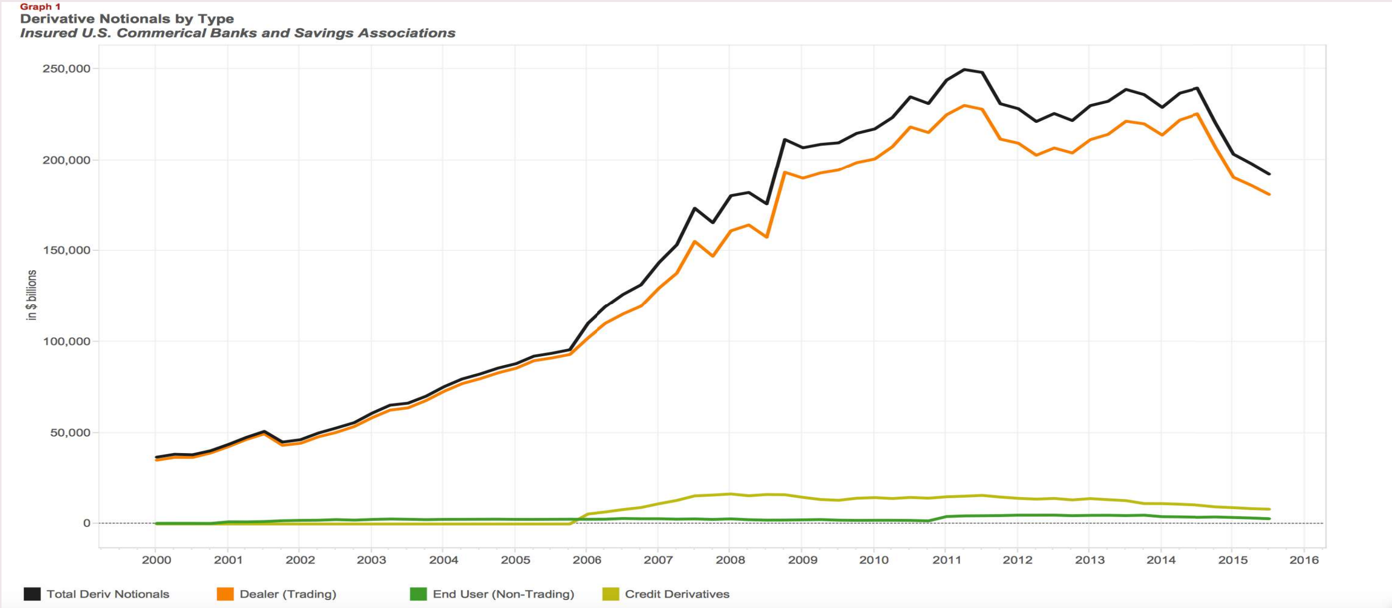
Outline

- Before coming to the technical (computational) implications, the fundamental points are to
  - understand what deserves to be priced and what does not
    - $\supseteq$ "double counting" (overlap) issues
  - by establishing, not only a pricing, but also the corresponding collateralization, accounting, and dividend policy of the bank

The Sustainable Pricing and Dividends Problem

- We want to devise a pricing, collateralization, accounting, and dividend policy for a dealer bank, sustainable in the sense of ensuring to its shareholders a constant instantaneous return rate h on their capital at risk, even in the limiting case of a portfolio held on a run-off basis, i.e. without future deals.

Ponzi scheme in the 2008–09 global financial crisis



- Moreover, the corresponding policy of the bank should satisfy several regulatory constraints.

- Firstly, the market risk of the bank should be hedged as much as possible.
  - As a result, mainly counterparty risk remains.

Secondly, reserve capital should be maintained by the bank at the level of its *expected* counterparty credit losses, along two lines:

- the credit valuation adjustment (CVA) of the bank, meant to cope with the counterparty risk of the bank clients
  - i.e. with the expected losses of the bank due to client defaults;
- the funding valuation adjustment (FVA) of the bank, meant to cope with the counterparty risk of the bank itself,
  - i.e. with its expected risky funding expenses.

- Thirdly, capital should be set at risk by the bank to deal with its *exceptional* (above expected) *losses*.
    - The above return rate h is then meant at a hurdle rate for the bank shareholders, i.e. a risk premium for their capital at risk within the bank.

- Reserve capital (RC), like capital at risk (CR) , should obviously be nonnegative.
- Furthermore, it should not decrease simply because the credit risk of the bank itself has worsened, a property which we refer to as monotonicity
  - see Section 3.1 in Albanese and Andersen (2014) for the relevant wordings from Basel Committee on Banking Supervision (2012) and Federal Register (2014)

- Further requirements on a solution to the above sustainable pricing and dividend release policy problem are
  - economic interpretability and logical consistency
    - for intellectual adhesion by market participants
  - numerical feasibility and robustness at the level of a realistic banking portfolio
    - for practicality
  - minimality in the sense of being, all things equal, as cheap as possible
    - for competitiveness

Solution Setup

The starting point of the cost-of-capital XVA solution to the sustainable pricing and dividends problem is an organizational and accounting separation between three kinds of business units within the bank: the CA (contra-assets) desks, the clean desks, and the management of the bank.

The CA desks

- are themselves split between the CVA desk and the FVA desk (or Treasury, or ALM) of the bank,
- respectively in charge of the default risk of the clients and of the risky funding expenses of the bank.
- The corresponding cash flows are collectively called the contra-assets (CA).
- The CA desks fully guarantee the trading of the clean desks against client and bank defaults, through a clean margin (CM) account of re-hypothecable collateral , which also funds the trading of the clean desks at the risk-free rate.

The clean desks

- Thanks to this work accomplished by the CA desks, the clean desks can focus on the market risk of the contracts in their respective business lines, as if there was no counterparty risk
  - even if some of their positions are liquidated, this will occur at no loss from their perspective

The management

- The management of the bank is in charge of its dividend release policy.
- We consider a level of capital at risk (CR) sufficient to make the bank resilient to a forty-year adverse event, i.e. at least as large as an economic capital (EC) defined as the expected shortfall of the losses of the bank in the next year at the confidence level $\alpha = 97.5\% = 1 - \frac{1}{40}$.
- The implementation of a sustainable dividend remuneration policy requires a dedicated risk margin (RM) account, on which bank profits are initially retained so that they can then be gradually released as dividends at a hurdle rate h on shareholder capital at risk
    - as opposed to being readily distributed as day-one profit

- Counterparty default losses, as also funding payments, are materialities for default if not paid.
- By contrast, risk margin payments, i.e. dividends, are at the discretion of the bank management, hence they do not represent an actual liability to the bank.
- As a consequence, the amount on the risk margin account (RM) is also loss-absorbing, i.e. part of capital at risk (CR).
- With minimality in view, we thus set

$$\mathrm{CR} = \max(\mathrm{EC}, \mathrm{RM}). \tag{1}$$

Physical or Risk-Neutral?

- Let there be given a physical probability measure on a σ algebra $\mathfrak{A}$ and a risk-neutral pricing measure on a financial σ algebra $\subseteq \mathfrak{A}$
  a. The risk-neutral measure is calibrated to the market (prices of fully collateralized transaction for which counterparty risk is immaterial)
  b. The physical probability measure expresses user views on the unhedgeable risk factors
  c. The risk-neutral and physical measures are assumed equivalent on the financial σ algebra
- One can think of our reference probability measure $\mathbb{Q}^*$ as the unique probability measure[1] on $\mathfrak{A}$ that coincides
  i. with the risk-neutral pricing measure *on* the financial σ algebra (and is then is calibrated to the market via a. above)
  ii. with the physical measure *conditional* on the financial σ algebra.
- Risk-free asset used as numéraire (except in the numérics)

[1] See Proposition 2.1 in Artzner, Eisele, and Schmidt (2020), building on Dybvig (1992), for a proof.

Rules of the Game

- In line with the first-edicted sustainability requirement, the portfolio is supposed to be held on a run-off basis between inception time 0 and its final maturity.
  - The bank locks its portfolio at time 0 and lets it amortize in the future,
- All bank accounts are marked-to-model, i.e. continuously and instantaneously readjusted to theoretical target levels, specifically the following balance conditions hold:

$$\mathrm{CM} = \mathrm{MtM}\,,\quad \mathrm{RC} = \mathrm{CA} = \mathrm{CVA} + \mathrm{FVA}\,,\quad \mathrm{RM} = \mathrm{KVA},$$

  for some theoretical target levels MtM, CVA, FVA, and KVA, which will be defined later in view of yielding a solution to the sustainable pricing and dividends problem.

- At time 0:
  - The clean desks pay MtM_0 to the clients and the CA desks put an amount MtM_0 on the clean margin account if $\mathrm{MtM}_0 > 0$, whereas the clean desks put an amount $(-\mathrm{MtM}_0)$ on the clean margin account if $\mathrm{MtM}_0 < 0$;
  - The CA desks charge to the clients an amount CA_0 and add it on the reserve capital account;
  - The management of the bank charges the amount KVA_0 to the clients and adds it on the risk margin account.
- Between time 0 and the bank default time τ (both excluded), mark-to-model readjustments of all bank accounts are on bank shareholders.

The broad rule regarding the settlement of contracts following defaults is that, at the liquidation time t_c of a netting set c between two counterparties:

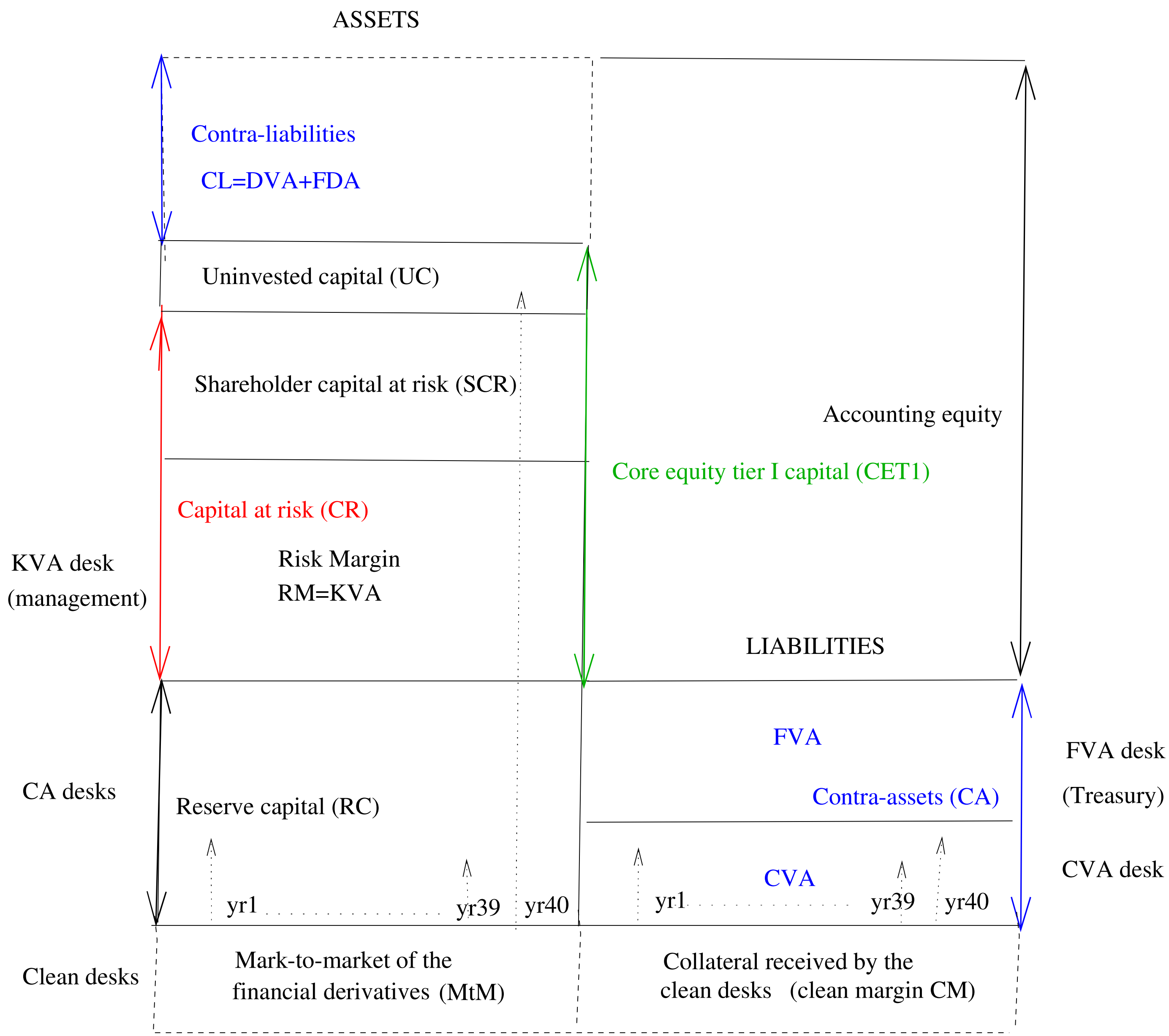
- In the case where only one of the two involved counterparties is in default at t_c, then:
  - If the debt of the counterparty in default toward the other does not exceed its posted margin, then this debt is reimbursed in totality to the other party;
  - Otherwise, this debt is only reimbursed at the level of this posted margin plus a fraction (recovery rate of the defaulted party) times the residual debt beyond the margin;

- In the case where both involved counterparties are in default at t_c, then:
  - If one is indebted to the other beyond its posted margin (as we will detail later this cannot occur for both jointly), then this counterparty transfers to the other the property of its posted margin plus its recovery rate times its residual debt beyond the margin;
  - Otherwise the debt between the two parties is fully settled.

Here debt is understood on a counterparty-risk-free basis and gross of the promised contractual cash flows unpaid during the liquidation period. Within the bank, the CVA desk is in charge of the liquidation close-out cash flows at t_c.

- If the bank itself defaults, then any residual amount on the reserve capital and risk margin accounts, as well as any remaining trading cash flows, are transferred to the creditor of the bank, who also needs to address the liquidation costs of the bank.
- These are outside the scope of the model, as is also the primary business of the clients of the bank, which motivates their deals with the bank.

The Balance Sheet Invites Itself Into Pricing



Hedging Assumptions

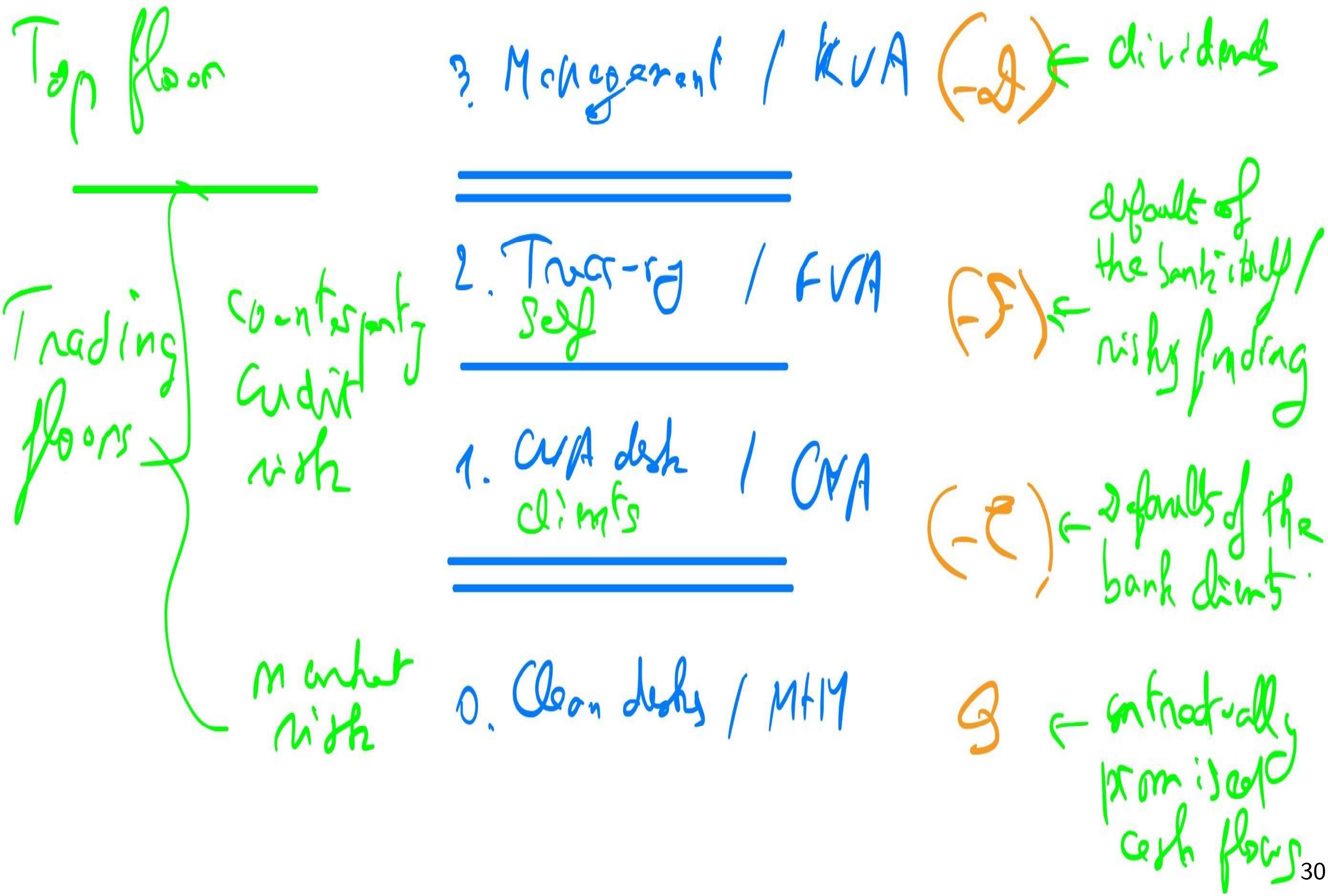
- For simplicity and in line with the no prop. trading Volcker rule, we assume perfect hedging by the clean desks, in the sense that their trading loss, inclusive of their hedging loss, vanishes.
- But, from a conservative XVA perspective, we assume that the CA desks do no hedge.

$\rightarrow$ The derivative portfolio and its hedge reduces to its counterparty risk related cash flows

Remark 1

- One could include further a (partial) XVA hedge
  - of the embedded market risk, as opposed to jump-to-default risk
- Conversely, one could relax the perfect clean hedge assumption
- The related extensions of the setup would change nothing to the qualitative conclusions of the paper, only implying additional terms in the trading loss L of the bank and accordingly modified economic capital and KVA figures.

Now: What are the Cash Flows??



Outline

- In this section we present the main ideas of the cost-of-capital XVA approach in an elementary static one-year setup
- Assume that at time 0 a bank enters a derivative position (or portfolio) with a client.

- Deal with promised cash flows (from the client to the bank) $\mathcal{P}$
- But the bank and its client are both default prone with zero recovery.
- We denote by J and J_1 the survival indicators of the bank and its client at time 1
  - Both being assumed alive at time 0
  - With default probability of the bank $\mathbb{Q}^*(J=0)=\gamma\in(0,1)$

- We assume that unsecured borrowing is fairly priced as $\gamma \times$ the amount borrowed by the bank for funding its trading, which is assumed paid at time 1 by the bank, irrespective of its default status.
- We assume further that a fully collateralized back-to-back market hedge is set up by the bank in the form of a deal with a third party, with no entrance cost and a payoff to the bank $(\mathrm{MtM} - \mathcal{P})$ at time 1, irrespective of the default status of the bank and the third party at time 1.

- For simplicity in a first stage, we will ignore the possibility of using capital at risk for funding purposes, only considering in this respect reserve capital $\mathrm{RC} = \mathrm{CA}$.
- The additional free funding source provided by capital at risk will be introduced later, as well as collateral between bank and clients.

Cash flows

Lemma 1

Given the to be specified MtM *and* CA *amounts, the credit and funding cash flows* $\mathcal{C}$ *and* $\mathcal{F}$ *of the bank and its trading loss (and profit)* L *satisfy* $L = \mathcal{C} + \mathcal{F} - \mathrm{CA}$, *with*

$$\begin{aligned}
\mathcal{C} &= (1 - J_1)\mathcal{P}^+ - (1 - J)\mathcal{P}^- = J(1 - J_1)\mathcal{P}^+ \\
&\quad - (1 - J)\big(\mathcal{P}^- - (1 - J_1)\mathcal{P}^+\big) \\
\mathcal{F} &= \gamma(\mathrm{MtM} - \mathrm{CA})^+ - (1 - J)(\mathrm{MtM} - \mathrm{CA})^+ = J\gamma(\mathrm{MtM} - \mathrm{CA})^+ \\
&\quad - (1 - J)\big((\mathrm{MtM} - \mathrm{CA})^+ - \gamma(\mathrm{MtM} - \mathrm{CA})^+\big).
\end{aligned}$$

Proof. The trading desks of the bank pay $\mathrm{MtM} - \mathrm{CA}$ for the deal, whereas they receives on the hedge and as portfolio settlement

$$
\begin{aligned}
&(\mathrm{MtM} - \mathcal{P}) + (1 - J_1)J(-\mathcal{P}^-) + J_1(1 - J)\mathcal{P}^+ + (1 - J_1)(1 - J)\,0 + J_1 J\mathcal{P} \\
&= \mathrm{MtM} + (1 - J_1)J(-\mathcal{P}^+) + J_1(1 - J)\mathcal{P}^- + (1 - J_1)(1 - J)(-\mathcal{P}) \\
&= \mathrm{MtM} + (1 - J_1)J(-\mathcal{P}^+) + J_1(1 - J)\mathcal{P}^- + (1 - J_1)(1 - J)(\mathcal{P}^- - \mathcal{P}^+) \\
&= \mathrm{MtM} + (1 - J_1)(-\mathcal{P}^+) + (1 - J)\mathcal{P}^-,
\end{aligned}
$$

i.e. the bank pays

$$(1 - J_1)\mathcal{P}^+ - (1 - J)\mathcal{P}^- - \mathrm{CA}$$

The funding side of the strategy yields to the bank a further cost $\gamma(\mathrm{MtM} - \mathrm{CA})^+$ and a windfall funding benefit $(1 - J)(\mathrm{MtM} - \mathrm{CA})^+$,

- money borrowed at time 0 and kept at time 1 in the case where the bank defaults
- if $\mathrm{MtM} - \mathrm{CA} < 0$ then the bank is actually lender at time 0 and reimbursed at time 1 whatever its default status at time 1 ∎

- Flipping the signs in the above, the result of the bank over the year (appreciation of its accounting equity) is rewritten as

$$
\begin{aligned}
&-\underbrace{J(1-J_1)\mathcal{P}^+}_{J\mathcal{C}}+J\mathrm{CVA}-\underbrace{J\gamma(\mathrm{MtM}-\mathrm{CA})^+}_{J\mathcal{F}}+J\mathrm{FVA}\\
&\qquad+\underbrace{(1-J)\big(\mathcal{P}^- -(1-J_1)\mathcal{P}^+\big)}_{(-(1-J)\mathcal{C})}+(1-J)\mathrm{CVA}\\
&\qquad+\underbrace{(1-J)\big((\mathrm{MtM}-\mathrm{CA})^+-\gamma(\mathrm{MtM}-\mathrm{CA})^+\big)}_{(-(1-J)\mathcal{F})}+(1-J)\mathrm{FVA}.
\end{aligned}
$$

- However, the cash flows in the last two lines are only received by the bank if it is in default at time 1, hence they go to the estate of the defaulted bank (liquidators of the bank, sometimes dubbed bank creditors below).
- Hence, the profit-and-loss of bank shareholders reduces to the first line, i.e. the bank shareholders' trading loss is

$$\mathcal{J}\mathcal{L} = \mathcal{J}\mathcal{C} - \mathrm{JCVA} + \mathcal{J}\mathcal{F} - \mathrm{JFVA}. \tag{2}$$

Remark 2

- The above derivation implicitly allows for negative equity (that arises whenever $JL > \mathrm{CET1}$) which is interpreted as recapitalization.
- In a variant of the model excluding recapitalization, the default of the bank would be modeled in a structural fashion as the event $\{L = \mathrm{CET1}\}$, where

$$L = \big((1 - J_1)\mathcal{P}^+ + \gamma(\mathrm{MtM} - \mathrm{CA})^+ - \mathrm{CA}\big) \wedge \mathrm{CET1},$$

and we would obtain, instead of the above, the bank trading loss

$$\mathbb{1}_{\{\mathrm{CET1} > L\}} L + \mathbb{1}_{\{\mathrm{CET1} = L\}} (\mathrm{CET1} - \mathcal{P}^- - (\mathrm{MtM} - \mathrm{CA})^+).$$

Shareholder valuation

- Let $\mathbb{E}^*$ and $\mathbb{E}$ denote the expectations with respect to the measure $\mathbb{Q}^*$ and the associated bank survival measure, $\mathbb{Q}$, i.e., for any random variable $\mathcal{Y}$,
$$\mathbb{E}\mathcal{Y} = (1-\gamma)^{-1}\mathbb{E}^*(J\mathcal{Y}) \qquad (3)$$
  - $= \mathbb{E}J\mathcal{Y}$
  - $= \mathbb{E}^*\mathcal{Y}$ if $\mathcal{Y}$ is independent from J.

Lemma 2

For any random variable $\mathcal{Y}$ and constant Y, we have

$$Y = \mathbb{E}^*(J\mathcal{Y} + (1-J)Y) \iff Y = \mathbb{E}\mathcal{Y}.$$

Proof. Indeed,

$$Y = \mathbb{E}^*(J\mathcal{Y} + (1-J)Y) \iff \mathbb{E}^*(J(\mathcal{Y} - Y)) = 0$$
$$\iff \mathbb{E}(\mathcal{Y} - Y) = 0 \iff Y = \mathbb{E}\mathcal{Y},$$

where the passage to the second line is justified by (3). ∎

MtM, CVA, and FVA

- Clean and CA desks make their shareholder trading losses $\mathbb{Q}^*$ centered
- The clean desks pay to the client MtM such that

$$\mathbb{E}^*(J\mathcal{P} - J\mathrm{MtM}) = 0, \text{ i.e. } \mathrm{MtM} = \mathbb{E}^*\big(J\mathcal{P} + (1-J)\mathrm{MtM}\big).$$

- CA desks charge to the client CVA and FVA add-ons such that

$$\mathbb{E}^*(J\mathcal{C} - J\mathrm{CVA}) = \mathbb{E}^*(J\mathcal{F} - J\mathrm{FVA}) = 0, \qquad (4)$$

i.e.

$$\mathrm{CVA} = \mathbb{E}^*\big(J\mathcal{C} + (1-J)\mathrm{CVA}\big)\,, \;\; \mathrm{FVA} = \mathbb{E}^*\big(J\mathcal{F} + (1-J)\mathrm{FVA}\big).$$

- These are MtM, CVA, and FVA equations.

- However, in terms of the bank survival expectation, Lemma 2 yields $\mathrm{MtM} = \mathbb{E}(J\mathcal{P})$ and

$$\mathrm{CVA} = \mathbb{E}(J\mathcal{C}) = \mathbb{E}\big((1 - J_1)\mathcal{P}^+\big)\,,\ \ \mathrm{FVA} = \mathbb{E}(J\mathcal{F}) = \gamma(\mathrm{MtM} - \mathrm{CA})^+$$

(as the latter is deterministic), hence by (2)

$$JL = J\mathcal{C} - J\mathrm{CVA}. \qquad (5)$$

- The possibility for the clean desks to find hedge counterparties at the price MtM leads to assume that $\mathrm{MtM} = \mathbb{E}^*\mathcal{P}$

$\rightarrow$

$$(\mathbb{E}^* J)(\mathbb{E}^*\mathcal{P}) = (1-\gamma)(\mathbb{E}\mathcal{P}) = \mathbb{E}^*(J\mathcal{P}), \qquad (6)$$

by (3).

Remark 3

Even if the clean desks were able to find (clients and) hedge counterparties accepting to deal with the bank on the basis of an MtM process that would be the bank shareholder value of $\mathcal{P}$ but not its value process, the corresponding discrepancy between valuation and shareholder valuation of $\mathcal{P}$ would be an indication of extreme dependence between the derivative portfolio and the default of the bank itself, such as the bank trading its own default risk,

- note that $\mathcal{P} = \pm J$ violates (6) (having assumed $\gamma \in (0, 1)$),

which should be considered with caution.

- We have the following semi-linear equation for $\mathrm{FVA} = \mathrm{CA} - \mathrm{CVA}$:

$$\mathrm{FVA} = \gamma(\mathrm{MtM} - \mathrm{CVA} - \mathrm{FVA})^+, \tag{7}$$

which has the unique solution

$$\mathrm{FVA} = \frac{\gamma}{1+\gamma}(\mathrm{MtM} - \mathrm{CVA})^+. \tag{8}$$

- The creditors of the bank get

$$(-(1-J)L) = (-(1-J)\mathcal{C}) + (1-J)\mathrm{CVA} + (-(1-J)\mathcal{F}) + (1-J)\mathrm{FVA}$$

- Let $\mathrm{CL} = \mathrm{DVA} + \mathrm{FDA}$, where

$$\mathrm{DVA} = \mathbb{E}^*\big((-(1-J)\mathcal{C}) + (1-J)\mathrm{CVA}\big)$$
$$\mathrm{FDA} = \mathbb{E}^*\big((-(1-J)\mathcal{F}) + (1-J)\mathrm{FVA}\big)$$

  - debt valuation adjustment and funding debt adjustment

- As $\mathbb{E}^*(J\mathcal{F}) = \mathbb{E}^*(-(1-J)\mathcal{F}) = \gamma(1-\gamma)(\mathrm{MtM} - \mathrm{CA})^+$, we have

$$\begin{aligned}\mathrm{FVA} =& \mathbb{E}^*\big(J\mathcal{F} + (1-J)\mathrm{FVA}\big) = \\ & \mathbb{E}^*\big((-(1-J)\mathcal{F}) + (1-J)\mathrm{FVA}\big) = \mathrm{FDA}.\end{aligned}$$

- Writing $\mathcal{C} = J\mathcal{C} - (-(1-J)\mathcal{C})$ and $\mathcal{F} = J\mathcal{F} - (-(1-J)\mathcal{F})$, also note that the fair valuation $\mathrm{FV} = \mathbb{E}^*(\mathcal{C} + \mathcal{F})$ of counterparty credit risk satisfies

$$\begin{aligned}\mathrm{FV} =& \mathbb{E}^*\mathcal{C} = \mathbb{E}^*J\mathcal{C} - \mathbb{E}^*(-(1-J)\mathcal{C}) \\ &= \mathbb{E}^*\big(J\mathcal{C} + (1-J)\mathrm{CVA}\big) - \mathbb{E}^*\big((-(1-J)\mathcal{C}) + (1-J)\mathrm{CVA}\big) \\ &= \mathrm{CVA} - \mathrm{DVA} = \mathrm{CA} - \mathrm{CL}.\end{aligned}$$

KVA and Funds Transfer Price

- Let EC denote economic capital, i.e. the theoretical target level of capital at risk that a regulator would like to see in the bank from a structural point of view.
- For simplicity we assess EC "on a going concern" as

$$\mathrm{EC} = \mathbb{ES}(JL)$$

  - 97.5% expected shortfall of the bank shareholder trading loss JL under the bank survival measure $\mathbb{Q}$
  - nonnegative, as JL is $\mathbb{Q}^*$ centered, hence $\mathbb{Q}$ centered by (3).

- Under the cost of capital XVA approach, the bank charges to its client an additional amount (retained margin, which is loss absorbing) such that

$$\mathrm{KVA} = \mathbb{E}^*\big(Jh(\mathrm{EC} - \mathrm{KVA})^+ + (1 - J)\mathrm{KVA}\big),$$

  for some so called hurdle rate parameter h (e.g. 10%),
- i.e.

$$\mathrm{KVA} = \mathbb{E}h(\mathrm{EC} - \mathrm{KVA})^+ = h(\mathrm{EC} - \mathrm{KVA})^+, \tag{9}$$

  i.e.

$$\mathrm{KVA} = \frac{h}{1+h}\mathrm{EC}. \tag{10}$$

Funds Transfer Price

The all-inclusive XVA add-on aligning the entry price of the deal to shareholder interest , which we call funds transfer price (FTP), is

$$\begin{aligned}\mathrm{FTP} &= \underbrace{\mathrm{CVA} + \mathrm{FVA}}_{\text{Expected costs } \mathrm{CA}} + \underbrace{\mathrm{KVA}}_{\text{Risk premium}} \\ &= \underbrace{\mathrm{CVA} - \mathrm{DVA}}_{\text{Fair valuation } \mathrm{FV}} + \underbrace{\mathrm{DVA} + \mathrm{FDA}}_{\text{Wealth transfer } \mathrm{CL}} + \underbrace{\mathrm{KVA}}_{\text{Risk premium}} ,\end{aligned}$$

where the random variable used to size the economic capital EC in the KVA formula (9) is the bank shareholders loss-and-profit JL as per (5).

Monetizing the Contra-Liabilities?

- Let us now assume, for the sake of the argument, that the bank would be able to hedge its own jump-to-default risk through a new deal, whereby the bank would deliver a payment $(-(1-J)L)$ at time 1 in exchange of a premium fairly valued as

$$\mathrm{CL} = \mathbb{E}^*(-(1-J)L) = \mathrm{DVA} + \mathrm{FDA},$$

and deposited in the reserve capital account at time 0.

- Accounting for the new deal and assuming the client provides $\mathrm{FV} = \mathrm{CA} - \mathrm{CL}$ (instead of CA before) in the reserve capital account of the bank, the amount that needs by borrowed by the CA desk for implementing the strategy is still $\gamma(\mathrm{MtM} - \mathrm{CA})^+$ as before and the bank trading loss is now given by

$$\begin{aligned}\mathcal{C} + \mathcal{F} - \mathrm{FV} + (-(1-J)L) - \mathrm{CL} = \\ \mathcal{C} + \mathcal{F} - \mathrm{CA} + (-(1-J)L) = L + (-(1-J)L) = JL,\end{aligned}$$

for JL as in (5).

- Hence, because of the new deal:
  - The client is better off by the amount $\mathrm{CA} - \mathrm{FV} = \mathrm{CL}$;
  - The creditors are left without any ressources to address the liquidation costs of the bank;
  - The shareholders are indifferent as always to the (duly priced) deal.
- Summing up, the CL originating cash flow $(-(1 - J)L)$ has been hedged out and monetized by the shareholders, which have passed the corresponding benefit to the client.

- In this situation, the bank would still charge to its client a KVA add-on $\frac{h}{1+h}\mathrm{EC}$, where EC is the same as before (as the random variable JL is the same as before).
- If the bank could also hedge its client default, then the bank trading loss and the KVA would vanish and the FTP would reduce to

$$\mathrm{FTP} = \mathrm{FV} = \mathrm{CVA} - \mathrm{DVA} = \mathrm{CA} - \mathrm{CL}.$$

Initial Margin

- In case of variation margin (VM) that would be exchanged between the bank and its client, and of initial margin that would be received (RIM) and posted (PIM) by the bank, at the height of, say for simplicity, some $\mathbb{Q}$ value-at-risk of $\pm(\mathcal{P} - \mathrm{MtM})$, then
- $\mathcal{P}^+$ needs be replaced by $(\mathcal{P} - \mathrm{VM} - \mathrm{RIM})^+$ in $\mathcal{JC}$, whence an accordingly modified (in principle: diminished) CVA.

- There would be an additional initial margin related cash flow in $J\mathcal{F}$ given as $J\gamma\mathrm{PIM}$, triggering an additional adjustment MVA in CA, where

$$\mathrm{MVA} = \mathbb{E}^*\big(J\gamma\mathrm{PIM} + (1-J)\mathrm{MVA}\big) = \gamma\mathrm{PIM};$$

- There would be additional initial margin related cash flows in $(-(1-J)\mathcal{F})$, triggering an additional adjustment $\mathrm{MDA} = \mathrm{MVA}$ in CL;
- Because of this additional MVA, the FVA formula (8) would become $\mathrm{FVA} = \frac{\gamma}{1+\gamma}(\mathrm{MtM} - \mathrm{VM} - \mathrm{CVA} - \mathrm{MVA})^+$.

Fungibility of Capital at Risk as a Funding Source

- In order to account for the additional free funding source provided by capital at risk, one would need to replace $(\mathrm{MtM} - \mathrm{CA})^{\pm}$ by $(\mathrm{MtM} - \mathrm{CA} - \max(\mathrm{EC}, \mathrm{KVA}))^{\pm}$ everywhere in the above.
  - Note that the marginal cost of capital for using capital as a funding source for variation margin is nil, because when one posts cash as variation margin, the valuation of the collateralized hedge is reset to zero and the total capital amount does not change.
  - If, instead, the bank were to post capital as initial margin, then the bank would record a "margin receivable" entry on its balance sheet, which however cannot contribute to capital since this asset is too illiquid and impossible to unwind without unwinding all underlying derivatives.
  - Hence, capital can only be used as VM, while IM must be borrowed entirely.

This would end-up in (the same modified CVA formula as above and) the following *system* for the random variable JL and the FVA and the KVA numbers (cf. (2), (7), and (10)):

$$
\begin{aligned}
JL &= J(1 - J_1)(\mathcal{P} - \mathrm{VM} - \mathrm{RIM})^+ - \mathrm{JCVA} \\
\mathrm{KVA} &= \frac{h}{1+h}\mathbb{ES}(JL) \\
\mathrm{FVA} &= \gamma(\mathrm{MtM} - \mathrm{VM} - \mathrm{CA} - \mathrm{EC})^+ \\
&= \frac{\gamma}{1+\gamma}(\mathrm{MtM} - \mathrm{VM} - \mathrm{CVA} - \mathrm{MVA} - \mathbb{ES}(\mathrm{JL}))^+.
\end{aligned}
$$

Outline

Probabilistic Pricing setup

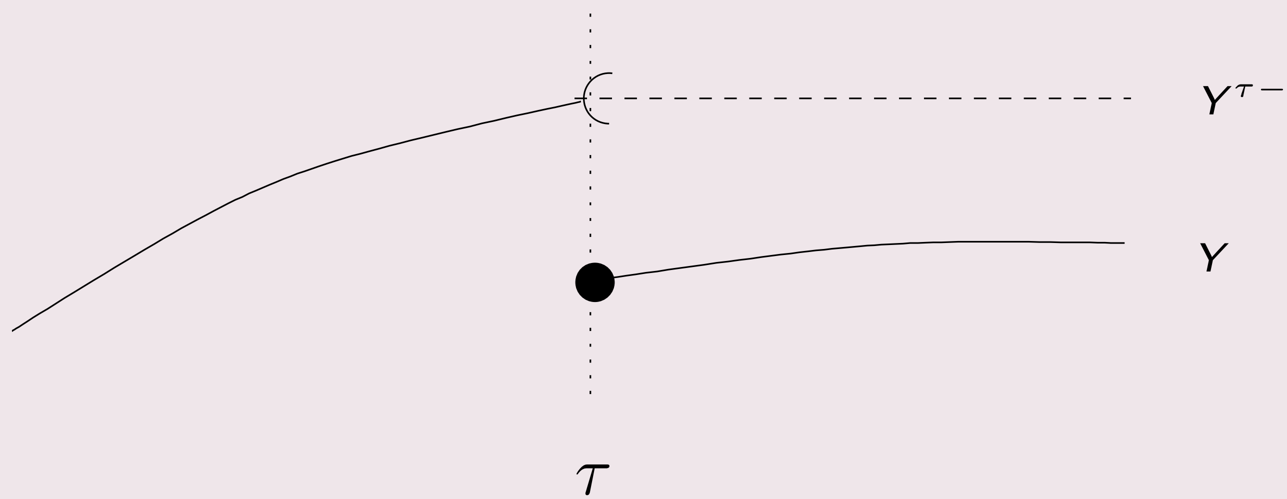
- Stochastic basis $(\mathbb{G}, \mathbb{Q}^*)$, with $\mathbb{G} = (\mathfrak{G}_t)$
- (Risk-neutral) value process of a financial cash flow stream: $(\mathbb{G}, \mathbb{Q}^*)$ conditional expectation process its of future cash flows.
  - (implicitely) discounted through our choice of the risk-free asset as a numéraire
- Portfolio first assumed held on a run-off basis, with final maturity T
  - also including the time (assumed bounded, in practice of the order of one or two weeks) of liquidating defaulted positions

- Bank default time τ with survival indicator process $J = \mathbb{1}_{[\![0,\tau[\![}$ and intensity γ, i.e. the process $dJ_t + \gamma_t dt$ is a martingale
  - additive martingale vs. multiplicative martingale $Je^{\int_0^\cdot \gamma_s ds}$
- For any left-limited process Y, we denote by

$$Y^{\tau-} = JY + (1 - J)Y_{\tau-}$$

and ${}^{\tau-}Y = Y - Y^{\tau-}$ the processes Y stopped before and starting before the bank default time τ.

Stopping before τ



- Reduced filtration $\mathbb{F} = (\mathfrak{F}_t) \subseteq \mathbb{G}$ with $\mathbb{Q}^*(\tau > T \,|\, \mathfrak{F}_T) > 0$.
- *Reduction* of any $\mathbb{G}$ predictable (resp. optional) process
  - the unique $\mathbb{F}$ predictable (resp. optional) process on $[0, T]$ coinciding with it until (resp. before) τ
- Invariance probability measure $\mathbb{P} \sim \mathbb{Q}^*$ on $\mathfrak{F}_T$ such that
  - $(\mathbb{F}, \mathbb{P})$ local martingales on $[0, T]$ stopped before τ are $(\mathbb{G}, \mathbb{Q})$ local martingales;
  - $\mathbb{F}$ reductions of $(\mathbb{G}, \mathbb{Q})$ local martingales on $[0, \tau \wedge T]$ without jump at τ are $(\mathbb{F}, \mathbb{P})$ local martingales on $[0, T]$.
- *Clean valuation* of an $\mathbb{F}$ adapted cash flow stream, with respect to $(\mathbb{F}, \mathbb{P})$
- $(\mathfrak{G}_t, \mathbb{Q}^*)$ and $(\mathfrak{F}_t, \mathbb{P})$ conditional expectations denoted by $\mathbb{E}_t^*$ and $\mathbb{E}_t$.

Lemma 3

Given an optional, integrable process $\mathcal{Y}$ stopped at T (cumulative cash flow stream in the financial interpretation), the shareholder valuation equation of $\mathcal{Y}$: $Y_T = 0$ on $\{T < \tau\}$ and

$$Y_t = \mathbb{E}^*_t\big(\mathcal{Y}_{\tau-} - \mathcal{Y}_t + Y_{\tau-}\big)\,,\;\; t < \tau,$$

is equivalent, "within suitable spaces of square integrable solutions", to the clean valuation equation of $\mathcal{Y}'$

$$Y'_t = \mathbb{E}_t(\mathcal{Y}'_T - \mathcal{Y}'_t), t \leq T.$$

Proof. *(Sketched, i.e. martingale square integrability considerations aside)* Differential variations on the equations for Y and Y':

$$\begin{aligned} & Y_T^{\tau-} = 0 \text{ on } \{T < \tau\} \text{ and, for } t \le \tau \wedge T, \\ & dY_t^{\tau-} = -d\mathcal{Y}_t^{\tau-} + d\nu_t, \\ & \text{for some } (\mathbb{G}, \mathbb{Q}^*) \text{ square integrable martingale } \nu, \end{aligned} \tag{11}$$

respectively

$$\begin{aligned} & Y'_T = 0 \text{ and, for } t \le T, \\ & dY'_t = -d\mathcal{Y}'_t + d\mu_t, \\ & \text{for some } (\mathbb{F}, \mathbb{P}) \text{ square integrable martingale } \mu. \end{aligned} \tag{12}$$

By definition of $\mathbb{F}$ optional reductions, the terminal condition in (12) obviously implies the one in (11). Conversely, taking the $\mathfrak{F}_T$ conditional expectation of the terminal condition in (11) yields

$$0 = \mathbb{E}[Y_T^{\tau-} \mathbb{1}_{\{T<\tau\}} | \mathfrak{F}_T] = \mathbb{E}[Y'_T \mathbb{1}_{\{T<\tau\}} | \mathfrak{F}_T] = Y'_T \mathbb{Q}^*(\tau > T \,|\, \mathfrak{F}_T),$$

hence $Y'_T = 0$ (as by assumption $\mathbb{Q}^*(\tau > T \,|\, \mathfrak{F}_T) > 0$), which is the terminal condition in (12).

The martingale condition in (12) implies the one in (11), by stopping before τ and application to $\nu = \mu^{\tau-}$ of the invariance probability measure direct condition.
Conversely, the martingale condition in (11) implies that $(Y', \mu = \nu')$ satisfies the second line in (12) on $[\![0, \tau \wedge T]\!]$, hence on $[0, T]$ (by uniqueness of the reduction of the null process). Moreover, by application of the invariance probability measure converse condition, $\mu = \nu'$ is an $(\mathbb{F}, \mathbb{P})$ martingale. ∎

Assuming τ endowed with a $(\mathbb{G}, \mathbb{Q}^*)$ intensity process $\gamma = \gamma J_-$ such that $e^{\int_0^T \gamma_s ds}$ is $\mathbb{Q}^*$ integrable

- Bank survival probability measure $\mathbb{Q}$ associated with $\mathbb{Q}^*$:
  - Probability measure $\mathbb{Q}$ on $(\Omega, \mathfrak{A})$ with $(\mathbb{G}, \mathbb{Q}^*)$ density process $Je^{\int_0^\cdot \gamma_s ds}$
  - cf. Schönbucher (2004) and Collin-Dufresne, Goldstein, and Hugonnier (2004)
- Crépey and Song (2017b):
  - Clean valuation $\sim$ valuation with respect to $(\mathbb{G}, \mathbb{Q})$
  - $\mathbb{P} = \mathbb{Q}_{|\mathfrak{F}_T}$
- Reduction of filtration into $(\mathbb{F}, \mathbb{P})$ is the systematic way to address "computations under the (singular) survival probability measure $\mathbb{Q}$"
- Mainstream immersion setup where

$$\mathbb{P} = \mathbb{Q}^*(= \mathbb{Q}) \text{ on } \mathfrak{F}_T$$

.

Remark 4

- For $A \in \mathfrak{A}$,

$$\mathbb{Q}^*(A \,|\, \tau > T) = \frac{\mathbb{Q}^*(A \cap \{\tau > T\})}{\mathbb{Q}^*(\{\tau > T\})} = \mathbb{E}^{\mathbb{Q}}\Big[\frac{\mathbb{1}_A e^{-\int_0^T \gamma'_s ds}}{\mathbb{E}^{\mathbb{Q}}(e^{-\int_0^T \gamma'_s ds})}\Big]$$

where the first equality follows from Bayes' rule and the second follows from the definition of the probability measure $\mathbb{Q}$:

$$\mathbb{Q}^*(A \cap \{\tau > T\}) = \int_\Omega \mathbb{1}_{A \cap \{\tau > T\}}(\omega)\mathbb{Q}^*(d\omega)$$
$$= \int_\Omega \mathbb{1}_{A \cap \{\tau > T\}}(\omega)\frac{d\mathbb{Q}^*}{d\mathbb{Q}}(\omega)\mathbb{Q}(d\omega) = \int_\Omega \mathbb{1}_{A \cap \{\tau > T\}}(\omega)e^{-\int_0^{\tau \wedge T} \gamma_s ds}\mathbb{Q}(d\omega)$$
$$= \mathbb{E}^{\mathbb{Q}}\big[\mathbb{1}_{A \cap \{\tau > T\}} e^{-\int_0^{\tau \wedge T} \gamma'_s ds}\big] = \mathbb{E}^{\mathbb{Q}}\big[\mathbb{1}_A e^{-\int_0^T \gamma'_s ds}\big].$$

- Hence $\mathbb{Q}$ coincides with the conditional probability $\mathbb{Q}^*(\cdot \,|\, \tau > T)$ if and only if γ' is a deterministic (measurable and Lebesgue integrable) function of time.

Trading Cash-Flows

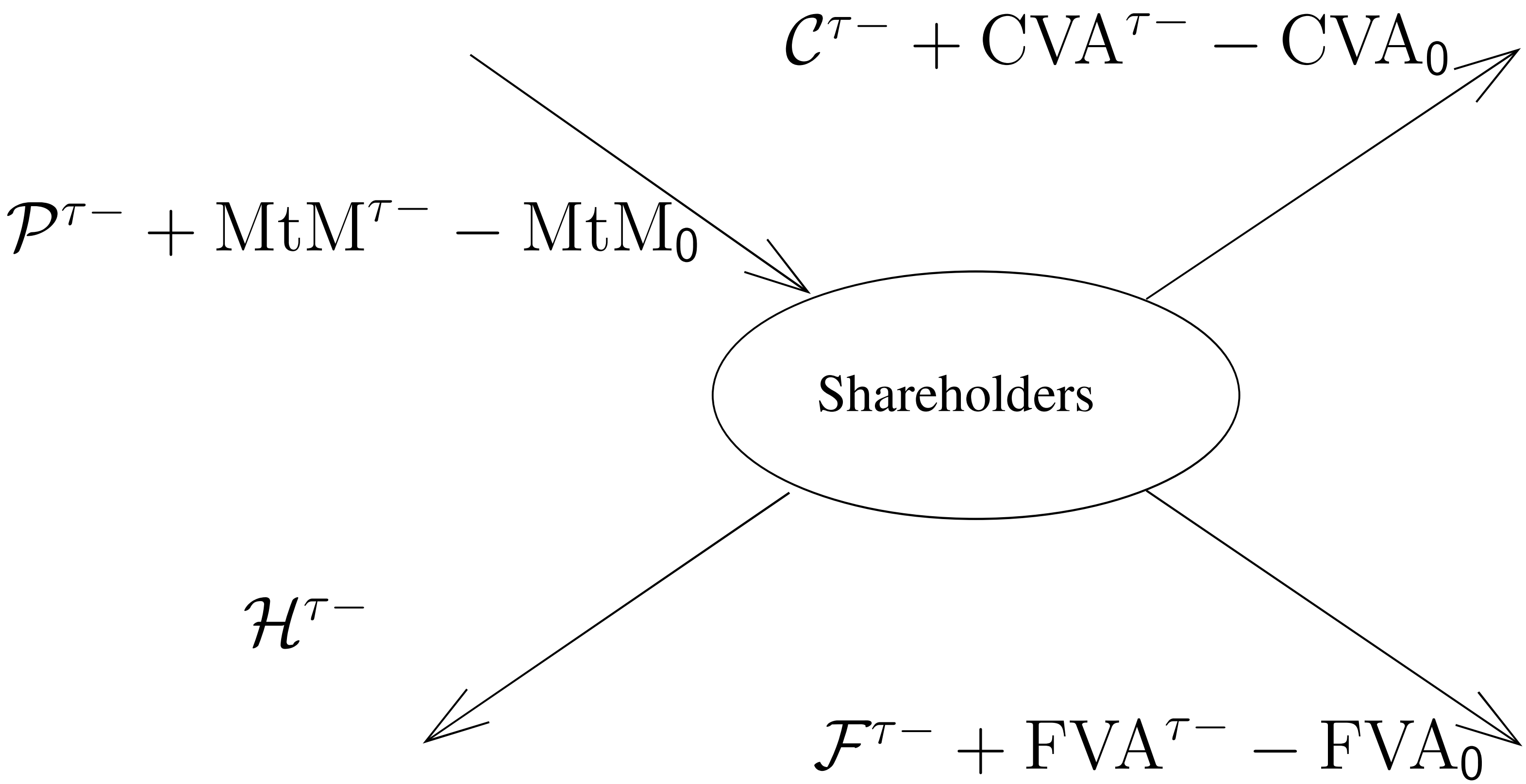
- Cash flows to the clean desks $\mathcal{P}$ and from the CVA and FVA desks $\mathcal{C}$ and $\mathcal{F}$
- $\sim$ contractually promised cash flows $\mathcal{P}$
- Counterparty credit cash flows $\mathcal{C}$
  - a finite variation process with nondecreasing component $\mathcal{C}^{\tau-}$
- Risky funding cash flows $\mathcal{F}$
  - a (zero valued) martingale with nondecreasing $\mathcal{F}^{\tau-}$ and ${}^{\tau-}(-\mathcal{F})$ components, stopped at $\tau \wedge T$.
- Hedging (of market risk) cash flows $\mathcal{H}$
  - a (zero valued) martingale with martingale $(\cdot)^{\tau-}$ component

Remark 5

Martingales with martingale $(\cdot)^{\tau-}$ component include

- martingales without jump at τ
  - in particular, continuous martingales,
- all the $\mathbb{F}$ (càdlàg) martingales in a standard progressive enlargement of filtration setup with the immersion property, provided the $\mathbb{F}$ Azéma supermartingale of τ is continuous and nonincreasing
  - see Lemma 2.1(ii) in Crépey (2015b)

Pre-bank default trading cash flows



Valuation compensates shareholder trading cash flows,

i.e. $\mathrm{MtM}_T = \mathrm{CVA}_T = \mathrm{FVA}_T = 0$ on $\{T < \tau\}$ and, for $t \le T$,

$$0 = \mathbb{E}_t^* \int_t^T d(\mathcal{P}_s^{\tau-} + \mathrm{MtM}_s^{\tau-} - \mathcal{H}_s^{\tau-}) = \mathbb{E}_t^* \int_t^T d(\mathcal{P}_s^{\tau-} + \mathrm{MtM}_s^{\tau-})$$
$$= \mathbb{E}_t^* \int_t^T d(\mathcal{C}_s^{\tau-} + \mathrm{CVA}_s^{\tau-}) = \mathbb{E}_t^* \int_t^T d(\mathcal{F}_s^{\tau-} + \mathrm{FVA}_s^{\tau-}),$$

i.e., for $t \le \tau \wedge T$,

$$\mathrm{MtM}_t^{\tau-} = \mathbb{E}_t^* \big(\mathcal{P}_{\tau\wedge T}^{\tau-} - \mathcal{P}_t^{\tau-} + \mathbb{1}_{\{\tau\le T\}} \mathrm{MtM}_{\tau-}\big), \quad (13)$$
$$\mathrm{CVA}_t^{\tau-} = \mathbb{E}_t^* \big(\mathcal{C}_{\tau\wedge T}^{\tau-} - \mathcal{C}_t^{\tau-} + \mathbb{1}_{\{\tau\le T\}} \mathrm{CVA}_{\tau-}\big), \quad (14)$$
$$\mathrm{FVA}_t^{\tau-} = \mathbb{E}_t^* \big(\mathcal{F}_{\tau\wedge T}^{\tau-} - \mathcal{F}_t^{\tau-} + \mathbb{1}_{\{\tau\le T\}} \mathrm{FVA}_{\tau-}\big), \quad (15)$$

i.e. $\mathrm{MtM}_T = \mathrm{CVA}_T = \mathrm{FVA}_T = 0$ on $\{T < \tau\}$ and, for $t < \tau$,

$$\mathrm{MtM}_t = \mathbb{E}_t^*\big(\mathcal{P}_{\tau-} - \mathcal{P}_t + \mathrm{MtM}_{\tau-}\big), \tag{16}$$
$$\mathrm{CVA}_t = \mathbb{E}_t^*\big(\mathcal{C}_{\tau-} - \mathcal{C}_t + \mathrm{CVA}_{\tau-}\big), \tag{17}$$
$$\mathrm{FVA}_t = \mathbb{E}_t^*\big(\mathcal{F}_{\tau-} - \mathcal{F}_t + \mathrm{FVA}_{\tau-}\big), \tag{18}$$

i.e. by Lemma 3, for $t \leq T$,

$$\mathrm{MtM}'_t = \mathbb{E}_t\big(\mathcal{P}'_T - \mathcal{P}'_t\big), \tag{19}$$
$$\mathrm{CVA}'_t = \mathbb{E}_t\big(\mathcal{C}'_T - \mathcal{C}'_t\big), \tag{20}$$
$$\mathrm{FVA}'_t = \mathbb{E}_t\big(\mathcal{F}'_T - \mathcal{F}'_t\big). \tag{21}$$

Core Equity Tier I Capital

Proposition 1

The core equity of the bank satisfies

$$\mathrm{CET1} = \mathrm{CET1}_0 - L, \tag{22}$$

where L is the trading loss of the bank (i.e. of the CA desks), such that

$$L^{\tau-} = \mathcal{C}^{\tau-} + \mathcal{F}^{\tau-} + \mathrm{CA}^{\tau-} - \mathrm{CA}_0$$

is a local martingale on $[0, \tau \wedge T]$ *without jump at* τ*; The* $\mathbb{F}$ *reduction* L' *of* L *is an* $(\mathbb{F}, \mathbb{P})$ *local martingale on* $[0, T]$.

Economic capital

- Since contra-assets (not even talking about contra-liabilities) cannot be replicated, the regulator requires that capital be set at risk by the shareholders.
- The capital at risk (CR) of the bank is its resource devoted to cope with losses beyond their expected levels that are already taken care of by reserve capital $\mathrm{RC} = \mathrm{CA} = \mathrm{CVA} + \mathrm{FVA}$.
- Economic capital (EC) is the level of capital at risk that a regulator would like to see on an economic, structural basis, based on $\mathrm{CET1}$ depletions
- Recall from Proposition 1 that $\mathrm{CET1}$ depletions correspond to $L^{\tau-}$ in the present setup.
- For simplicity we assess EC on the following 'going concern" basis:

Definition 1

EC_t is the $(\mathfrak{F}_t, \mathbb{P})$ conditional 97.5% expected shortfall of $(L'_{t+1} - L'_t)$, killed at τ. ∎

- Let $\mathbb{ES}_t(\ell)$ denote the $(\mathfrak{F}_t,\mathbb{P})$ conditional expected shortfall, at some level α (e.g. $\alpha = 97.5\%$), of an $\mathfrak{F}_T$ measurable, $\mathbb{P}$ integrable random variable ℓ. That is, denoting by $q_t^a(\ell)$ the $(\mathfrak{F}_t,\mathbb{P})$ conditional value at risk (left quantile) of level a of ℓ (cf. Artzner, Delbean, Eber, and Heath (1999)):

$$\begin{aligned}
\mathbb{ES}_t(\ell) &= \mathbb{E}_t(\ell \,|\, \ell \geq q_t^a) \\
&= (1-\alpha)^{-1}\int_\alpha^1 q_t^a(\ell)da \\
&= \inf_{x\in\mathbb{R}}\Big((1-\alpha)^{-1}\mathbb{E}_t\big[(\ell-x)^+ + x\big]\Big) \\
&= \sup\big\{\mathbb{E}_t[\ell\chi]\,;\ \chi \text{ is } \mathfrak{F}_T \text{ measurable, } 0\leq\chi\leq(1-\alpha)^{-1}, \mathbb{E}_t[\chi]=1\big\}.
\end{aligned}$$

- For any integrable random variables ℓ_1 and ℓ_2, we have (cf. Lemma 6.10, Eq. (6.20) in Barrera et al. (2019) and its proof):

$$|\mathbb{ES}_t(\ell_1) - \mathbb{ES}_t(\ell_2)| \;\leq\; (1-\alpha)^{-1}\mathbb{E}_t[|\ell_1 - \ell_2|]\,,\;\; 0 \leq t \leq T.$$

- Note incidentally that we will only deal with martingale loss and profit processes $L^{\tau-}$ and therefore centered loss variables ℓ, for which $\mathbb{ES}_t(\ell) \geq 0$ holds in view of its third formulation above.

Capital at risk, KVA, and dividends

Assumption 1

The risk margin is loss-absorbing, hence part of capital at risk. ∎

As a consequence, shareholder capital at risk (SCR) is only the difference between the capital at risk (CR) of the bank and the risk margin $(\mathrm{RM} = \mathrm{KVA})$, i.e.

$$\mathrm{SCR} = \mathrm{CR} - \mathrm{KVA}. \quad (23)$$

Given a positive target hurdle rate h:

Definition 2

We set

$$\mathrm{CR} = \max(\mathrm{EC}, \mathrm{KVA}), \tag{24}$$

for a KVA process such that $\mathrm{KVA}_T = 0$ on $\{T < \tau\}$ and

$$(-\mathrm{KVA}^{\tau-}) \text{ has for drift coefficient } h\mathrm{SCR} \text{ killed at } \tau, \blacksquare$$

i.e. $\mathrm{KVA}_T = 0$ on $\{T < \tau\}$ and

$$\mathrm{KVA}_t = \mathbb{E}_t^*\Big[\int_t^{\tau\wedge T} h\mathrm{SCR}_s ds + \mathrm{KVA}_{\tau-}\Big], \quad t < \tau, \blacksquare$$

i.e.

$$\mathrm{KVA}'_t = \mathbb{E}_t\Big[\int_t^T h\mathrm{SCR}'_s ds\Big],\ \ 0 \le t \le T. \tag{25}$$

Note that, in view of (23) and (24), (25) is in fact a KVA' equation, namely

$$\begin{aligned}\mathrm{KVA}'_t =& \mathbb{E}_t\Big[\int_t^T h\big(\mathrm{CR}'_s - \mathrm{KVA}'_s\big)ds\Big]\\ =& \mathbb{E}_t\Big[\int_t^T he^{-h(s-t)}\max(\mathrm{EC}'_s, \mathrm{KVA}'_s)ds\Big],\ \ 0 \le t \le T.\end{aligned} \tag{26}$$

- Continuous-time analog of the risk margin formula under the Swiss solvency test cost of capital methodology: See Swiss Federal Office of Private Insurance (2006, Section 6, middle of page 86 and top of page 88).
- Can be used either in the direct mode, for computing the KVA corresponding to a given h, or in the reverse-engineering mode, for defining the "implied hurdle rate" associated with the actual RM level on the risk margin account of the bank.

Proposition 2

Shareholder capital (i.e. equity) $\mathrm{SHC} = \mathrm{SCR} + \mathrm{UC}$ *satisfies* $\mathrm{SHC} = \mathrm{SHC}_0 + \mathcal{D}$, *where*

$$\mathcal{D} = -(L^{\tau-} + \mathrm{KVA}^{\tau-} - \mathrm{KVA}_0),$$

is a submartingale with drift coefficient $h\mathrm{SCR}$ *on* $[0, \tau \wedge T]$, *without jump at* τ.

- Cost of capital proxies have always been used to estimate return on equity (ROE). The KVA is a refinement, dynamic and fine-tuned for derivative portfolios, but the base ROE concept itself is far older than even the CVA.
- In particular, the KVA is very useful in the context of collateral and capital optimization.

Portfolio-Wide XVAs are nonnegative

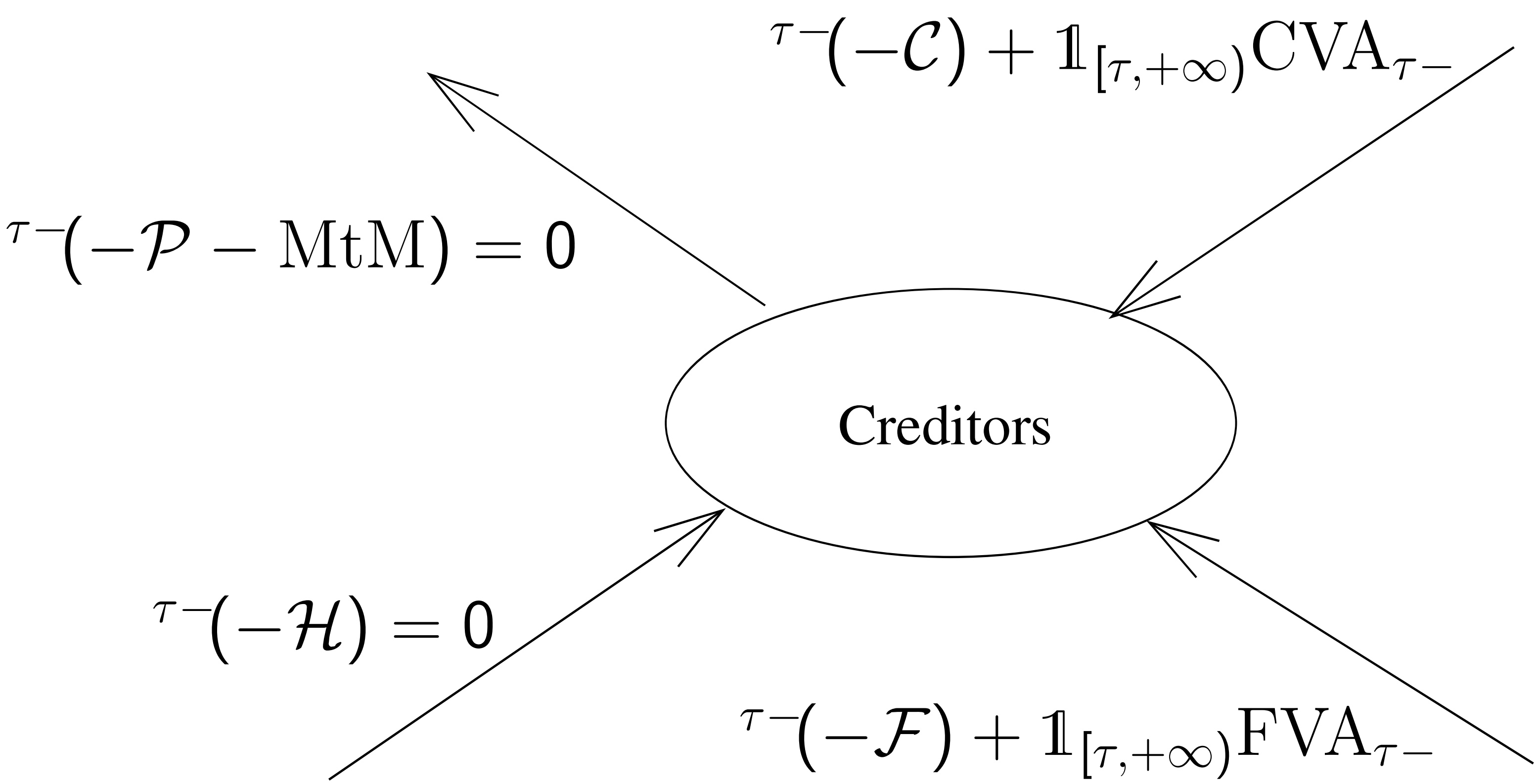
- and, even though we do crucially include the default of the bank itself in our modeling, unilateral
  - computed "under the bank survival probability measure"
- This makes them naturally in line with the regulatory requirement that capital should not diminish as an effect of the sole deterioration of the bank credit spread

Trading cash flows from bank default onward

- MtM,CVA, FVA, and KVA are so far unconstrained on $[\![\tau, +\infty[\![\bigcap \big(\{\tau \leq T\} \times \mathbb{R}_+\big)$.
- We define the three XVA processes as zero there.
- As they already vanish on $[T, +\infty)$ if $T < \tau$, either of them, say Y, is in fact killed at $\tau \wedge T$, hence such that

$$^{\tau-}Y = \mathbb{1}_{[\![\tau,+\infty[\![}(Y_\tau - Y_{\tau-}) = -\mathbb{1}_{[\![\tau,+\infty[\![}Y_{\tau-} = -\mathbb{1}_{[\![\tau,+\infty[\![}Y_\tau^{\tau-}.$$

- As for MtM, we suppose, for clean valuation consistency across different banks (with hedging in mind), that it is not only the shareholder value process of $\mathcal{P}$, but also its value process in the first place.
- This determines MtM on $[0, T]$ and implies that $\mathcal{P} + \mathrm{MtM}$ is a martingale with martingale $(\cdot)^{\tau-}$ component.


$^{\tau-}(-\mathcal{C}) + \mathbb{1}_{[\tau,+\infty)}\mathrm{CVA}_{\tau-}$
$^{\tau-}(-\mathcal{P} - \mathrm{MtM}) = 0$
Creditors
$^{\tau-}(-\mathcal{H}) = 0$
$^{\tau-}(-\mathcal{F}) + \mathbb{1}_{[\tau,+\infty)}\mathrm{FVA}_{\tau-}$

Contra-liabilities

Definition 3

By contra-liability value process CL, we mean $\mathrm{CL} = \mathrm{DVA} + \mathrm{FDA}$, where

- DVA (debt valuation adjustment) is the value process of ${}^{\tau-}(-\mathcal{C}) + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{CVA}_{\tau-}$;
- FDA (funding debt adjustment) is the value process of ${}^{\tau-}(-\mathcal{F}) + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{FVA}_{\tau-}$.

By fair valuation of counterparty credit risk, we mean the value process FV of $\mathcal{C} + \mathcal{F}$.

As is then immediate by the different martingale assumptions involved:

Lemma 4

We have $\mathrm{CL} = \mathrm{DVA} + \mathrm{FDA}$, *which is the value process of both* ${}^{\tau-}(-L)$ *and* $(-L)$. *Moreover, before* τ,

$$\mathrm{FVA} = \mathrm{FDA}\,,\;\; \mathrm{FV} = \mathrm{CA} - \mathrm{CL} = \mathrm{CVA} - \mathrm{DVA}. \blacksquare \tag{27}$$

Wealth Transfer Analysis

- We assume that the shareholders have no other business than their involvement within the bank.
- Like the bank clients, whose business with firms other than the bank (which provides their motivation for the deals) is not present in the model, creditors have to face the liquidation costs of the bank, which are outside the scope of the model.

Definition 4

We call wealth of the bank shareholders, $\mathcal{W}^{sh}$, the sum between their accumulated cash flows and the valuation of their future cash flows. ∎

- The wealth of the shareholders before entering the portfolio ("at time 0−") is implicitly (and conventionally) taken as zero in this definition.
- So $\mathcal{W}^{sh}$ is in fact a wealth *transfer*, namely the wealth transferred to the shareholders by the derivative portfolio of the bank
  - without the portfolio, their wealth process in the sense of Definition 4 would vanish identically.

Definition 5

We call wealth transfer to the creditor, denoted by $\mathcal{W}^{cr}$, the sum between the cash flows that they receive from the bank and the valuation of the corresponding future cash flows. ∎

Let

$$\mathrm{KVA}^{sh}_t = \mathbb{1}_{\{t<\tau\}}\mathbb{E}^*_t \int_t^\tau h\big(\mathrm{EC}_s - \mathrm{KVA}^{\tau-}_s\big)^+ ds\,,\;\; \mathrm{KVA}^{cr}_t = \mathbb{1}_{\{t<\tau\}}\mathbb{E}^*_t \mathrm{KVA}_{\tau-}. \tag{28}$$

Proposition 3

The shareholder and creditor wealth transfer processes are

$$\mathcal{W}^{sh} = -(L^{\tau-} + \mathrm{KVA}^{\tau-} - \mathrm{KVA}_0) + \mathrm{KVA}^{sh}, \tag{29}$$

$$\mathcal{W}^{cr} = {}^{\tau-}(-L) + \mathrm{CL} + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{KVA}_{\tau-} + \mathrm{KVA}^{cr}. \tag{30}$$

Shareholder and creditor wealth transfers are martingales starting from KVA_0^{sh} and $\mathrm{CL}_0 + \mathrm{KVA}_0^{cr}$ at time 0.

Proof. The first part follows from Definition 4 by inspection of the related cash flows, namely $\mathcal{D}$ as per (27) for shareholders and ${}^{\tau-}(-L) + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{KVA}_{\tau-}$ for creditors (recalling for (29) that the $(-L^{\tau-})$ component of $\mathcal{D}$ is zero-valued, as a martingale).
We have

$$\mathcal{W}^{sh} + \mathcal{W}^{cr} = \mathrm{KVA}_0 + \mathrm{CL} - L. \tag{31}$$

As seen in Lemma 4, this is a martingale. So are also $\mathcal{W}^{cr}$ and, by difference, $\mathcal{W}^{sh}$. ∎

- Should the shareholders decide to put the bank in default at time 0 right after the portfolio has been set up, they should not make any profit or loss, otherwise this would be a form of shareholder arbitrage.
- The fact that the shareholder wealth transfer martingale $\mathcal{W}^{sh}$ starts from $\mathrm{KVA}_0^{sh} > 0$ (positive initial wealth transfer to shareholders, unless the KVA vanishes) might suggest that the derivative trading of the bank entails shareholder arbitrage.
- Yet, given the rules of default settlement, upon bank default, the residual value on the (reserve capital and) risk margin account of the bank goes to creditors. So the shareholders would not monetize KVA_0^{sh} by putting the bank in default at time 0 right after the portfolio has been set up.
- The positive initial wealth transfer to shareholders does not entail any shareholder arbitrage, at least not in this sense.

- Likewise, the fact that the creditor wealth transfer martingale $\mathcal{W}^{cr}$ starts from $\mathrm{CL}_0 + \mathrm{KVA}_0^{cr} > 0$ (unless both CL and the KVA vanish) might suggest that the derivative trading of the bank entails a riskless profit to creditors.
- However, the scope of the model does not include the liquidation costs.
- For the creditors to monetize the wealth transfer triggered to them by the derivative portfolio of the bank, the bank has to default and there is a substantial cost associated to that to the creditors.

What-if Analysis

- Assume, for the sake of the argument, that the bank would be able to hedge its own jump-to-default risk by selling a new deal delivering the cumulative cash flow stream (zero valued martingale) $\mathrm{CL} - \mathrm{CL}_0 - L$.
- Accounting for the new deal and assuming that the CA desks would pass to the client (at time 0) and shareholders (through resets later on) the modified add-on $\mathrm{CA} - \mathrm{CL} = \mathrm{FV}$ (instead of CA before without the hedge), then the amount that needs by borrowed by the CA desk for implementing the strategy is the same as before and the trading loss of the bank would become

$$\begin{aligned} &\mathcal{C} + \mathcal{F} + \mathrm{FV} - \mathrm{FV}_0 + \mathrm{CL} - \mathrm{CL}_0 - L \\ &= \mathcal{C} + \mathcal{F} + \mathrm{CA} - \mathrm{CA}_0 - L = L - L = 0. \end{aligned}$$

- So $\mathrm{FV} = \mathrm{CA} - \mathrm{CL} = \mathrm{CVA} - \mathrm{DVA}$ is the cost of replicating counterparty risk in a theoretical, complete counterparty risk market;
- However:
  - the hedge of ${}^{\tau-}L$ is impossible because a bank cannot (is not even allowed) to sell credit protection on itself;
  - Hedging out $L^{\tau-}$ is not practical either, even in the case of a theoretical default-free bank, by lack of sufficiently liquid CDS instruments on the clients.
- Hence the shareholder and creditor wealth transfers can be interpreted as the wealth transferred to them by the trading of the bank, due to the inability of the bank to hedge counterparty risk.

Trade Incremental Cost-of-Capital XVA Policy

In the (realistic) case of an incremental portfolio, at each new trade, the funds transfer price (all-inclusive XVA add-on) sourced from the client is

$$\begin{aligned}\mathrm{FTP} &= \Delta\mathrm{CA} + \Delta\mathrm{KVA} = \Delta\mathrm{CVA} + \Delta\mathrm{FVA} + \Delta\mathrm{KVA} \\ &= \Delta\mathrm{FV} + \Delta\mathrm{CL} + \Delta\mathrm{KVA},\end{aligned}$$

computed on a trade incremental run-off basis.

- Meant incrementally at every new deal, the above FTP can be interpreted dynamically as the cost of the possibility for the bank to go into run-off,
  - i.e. lock its portfolio and let it amortize in the future,

  while staying in line with shareholder interest, from any point in time onward if wished.
- A "soft landing" or "anti-Ponzi" corrective pricing scheme accounting for counterparty risk incompleteness

Theorem 1

Under a trade incremental cost-of-capital XVA approach, consistently between and throughout deals: shareholder equity SHC *is a submartingale on* $\mathbb{R}_+$*, with drift coefficient* $h\mathrm{SCR}$ *killed before* τ*.*

- Hence, the preservation of the balance conditions in between and throughout deals yields a sustainable strategy for profits retention, which is already the key principle behind the Eurozone Solvency II insurance regulation.