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HAMILTONIAN STABILITY IN SOME OPEN SURFACES WITH SIMPLE SINGULARITIES

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Abstract. We complement to some open surfaces with singularities the basic characterization theorem obtained by Jarque–Nitecki [Ergod. Th. & Dynam Sys. **20** (2000) 775–799] for Hamiltonian flows in the Euclidean plane which are structurally stable among Hamiltonian flows. We also describe the Hamiltonian stability on $C = \{x^2 + y^2 = z^2\}$ (Cone) and $D = \{xy = 0\}$ (Double Crossing) by presenting some characterization theorems for Hamiltonian stability and some natural consequences.

1. Introduction and statement of the results

We examine in detail some important dynamical properties of the Hamiltonian flows whose phase space is an open surface embedded in the euclidian space. This surface, o more specifically, the phase space, might admit the following *simple singularities* [7]: $C = \{x^2 + y^2 = z^2\}$ (Cone) and $D = \{xy = 0\}$ (Double Crossing). In this context, we extend to these surfaces with simple singularities the basic results obtained by Jarque and Nitecki [14] for planar Hamiltonian dynamics (Theorem 4.6). They prove that the planar Hamiltonian flow $(X_f)_t$ is Hamiltonian C^r–stable if and only if:

(1*) every equilibrium of $(X_f)_t$ is either a hyperbolic saddle or a non-degenerate center,

(2*) a separatrix of a finite saddle is isolated from the separatrices of all other finite of infinite saddles for $(X_f)_t$. This means that for each separatrix ℓ_q of a saddle q there exists V_q, an open neighborhood of ℓ_q where q is the unique equilibrium point such that

$$V_q \setminus \{\ell : \ell \text{ is a separatrix of } q\}$$

is disjoint from all the separatrices.

Consequently, $(X_f)_t$ satisfies the Definition 3.5 with $\mathcal{M}$ equal to the plane P.

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We research the Hamiltonian flows on open surfaces with simple singularities. These kind of degeneracy were studied by Gutiérrez and Sotomayor in [7, 8], since the simple singularities appear when the regularity conditions in the definition of smooth submanifolds of $\mathbb{R}^3$ in terms of implicit functions and immersions are broken in a stable manner. However, we have to present an exact definition of Hamiltonian vector field on this type of surfaces, and also we have to describe the precise topology on the space of the Hamiltonian vector fields. To this end, we restrict our study to the dynamics induced by Hamiltonian vector fields tangent to $\mathcal{M} \in \{C, D, P, A\}$, where $A = \{x^2 + y^2 = 1\}$ (Cylinder).

1.1. **Sufficient conditions for Hamiltonian stability.** The first results of this paper are Theorem 5.1 and Theorem 6.9, where we establish a dynamic characterization for Hamiltonian stable flows on $\mathcal{M} \in \{C, D, A\}$. These Hamiltonian stable flows are characterized by having topologically simple singular points (Definition 3.12), isolated unbounded separatrices (Definition 4.10) and some dynamical properties on the complement set of $\mathcal{S}^2(\mathcal{M})$, the smooth part of $\mathcal{M}$ defined just before start §2.1.

Theorem 5.1 *Consider a Hamiltonian vector field $X_f \in \mathcal{H}^r(\mathcal{M})$, with $r \geq 1$ and $\mathcal{M} \in \{C, A\}$. The induced Hamiltonian flow $(X_f)_t$ is* Hamiltonian C^r–stable *if the following conditions hold.*

(a) *The flow $(X_f)_t$ only has topologically simple singular point on $\mathcal{S}^2(\mathcal{M})$.*
(b) *For each separatrix ℓ_q, with $q \in \mathcal{M}$ it satisfies one of the following conditions:*
  - *ℓ_q is isolated from the separatrices of all other finite of infinite saddles.*
  - *ℓ_q is unbounded and admits an open isolated neighborhood $\ell_q \subset V_q \subset C$.*

Similarly, the Hamiltonian stability in $D = \{xy = 0\}$ is characterized in the following theorem, where the notations of Lemma 6.6 is needed.

Theorem 6.9 *Consider $X_f \in \mathcal{H}^r(D)$, with $r \geq 1$. The induced Hamiltonian flow $(X_f)_t$ is Hamiltonian C^r–stable if both flows $(Y_g)_t$ with $Y_g \in \{X_x, X_y\}$ (as in Lemma 6.6) satisfy the following conditions hold.*

(a) *The flow $(Y_g)_t$ only has topologically simple singular point on $P \backslash (z-axis)$.*
(b) *For each separatrix ℓ_q, with $q \in P$ it satisfies one of the following conditions:*
  - *ℓ_q is isolated from the separatrices of all other finite of infinite saddles.*
  - *ℓ_q is unbounded and admits an open isolated neighborhood $\ell_q \subset V_q \subset P$ with $q \in z - axis$.*

(c) *If either the $z - axis$ has no isolated singularities or the $z - axis$ is free of singuarities, then this vertical axis admits a open neighborhood obtained as the saturation of some small transversal open segment.*

In Theorem 5.1 and Theorem 6.9 the condition (a) is closed related to the assumption (1*) of Jarque–Nitecki. In the case of smooth surfaces $\mathcal{M} = \mathcal{S}^2(\mathcal{M})$, both conditions (a) and (b) are reduced to (1*) and (2*), respectively. However, in the case of open surfaces with simple singularities it is absolutely necessary to take into account some properties on the singular part, as either the vertex in the cone or the vertical axis on D.

1.2. **Natural consequences of Hamiltonian stability.** In the next theorem are presented the necessary conditions for the Hamiltonian stability on $\mathcal{M} \in \{C, D, P, A\}$. We also describe the boundary of the hyperbolic sector associated to the saddle at infinity, Definition 4.1.

Theorem 4.11 *Suppose that the Hamiltonian flow $(X_f)_t$ with $X_f \in \mathcal{H}^r(\mathcal{M})$ is* Hamiltonian C^r–stable *then,*

(a) *The flow $(X_f)_t$ only has topologically simple equilibrium points on the respective smooth part, $\mathcal{S}^1(\mathcal{M}) \sqcup \mathcal{S}^2(\mathcal{M})$.*
(b) *For each separatrix of a saddle $q \in \mathcal{S}^2(\mathcal{M})$ there exists an open isolated neighborhood $V_q \subset \mathcal{S}^2(\mathcal{M})$. Moreover, the vertex cone does not admit a saddle connection.*
(c) *The boundary of a hyperbolic sector associated to a saddle at infinity is either connected (homoclinic contour at infinity) or it has infinitely many components.*

Theorem 4.11 with $\mathcal{M} = P$ coincides with the results of [14], since (1*) and (2*) hold. Moreover, the conclusion (c) is related with the properties of a saddle at infinity in the sense of [10, 11]. Therefore, as far as we know, our theorems improve all the previous results related with the subject.

Motivated by the interesting results of [12, 13, 15, 20], we also present a few new types of stability, even in the plane. For instance, we study the Hamiltonian flows with an invariant line, and we show that some set of no isolated singularities is stable in the sense of Theorem 4.9. In addition, the necessary conditions of that kind of stability are presented in Proposition 4.7. Similarly, in Theorem 6.4 we characterize the so called Weak Hamiltonian stability on D, where we do not assume that the topologically equivalence is near to the identity [2, 15].

This paper is organized as follows. In Section 2, we give some notation and preliminary definitions. By using the smooth part of $\mathcal{M}$, we characterize, in (2.3), the Vector fields of class C^0 on $\mathcal{M} \in \{C, D, P, A\}$, where $P = \{y = 0\}$ (Plane). We also describe the strong topology in some space of functions defined on $\mathcal{M}$. This § 2 concludes by proving that those spaces are independent of the System of Directions (Definition 2.3). In Section 3, we present the basic properties of the Hamiltonian dynamics on $\mathcal{M}$, as in Proposition 3.3 and Proposition 3.6. Moreover, in Proposition 3.10 we characterize the simple singularities of the Hamiltonian vector fields. In Section 4, we prove the consequences of the Hamiltonian stability and new type of stability on the plane. In Section 5 is proved Theorem 5.1 and characterize the Hamiltonian stability in the Cone and the Open Infinity Cylinder. Finally, in Section 6 we prove Theorem 6.9 and obtain some sufficient conditions of the Hamiltonian stability in the Double Crossing Plane, by using Theorem 4.9.

2. Preliminaries

Take the Plane $P = \{(x, y, z) \in \mathbb{R}^3 : y = 0\}$ and the Infinity Open Cylinder $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$, two smooth submanifolds of $\mathbb{R}^3$. Their Tangent Bundle can be seen as a subset in $\mathbb{R}^3$ where each Tangent Space is tangent to the submanifold in the usual geometric sense. We also consider the Cone $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\}$ and the Double Crossing Plane $D = \{(x, y, z) \in \mathbb{R}^3 : xy = 0\}$, so that a *continuous* vector field $\widehat{X} : \mathcal{M} \to \mathbb{R}^3$ is said to be **tangent** to $\mathcal{M} \in \{C, D, P, A\}$ when the respective restriction of $\widehat{X}$ defines a section of the Tangent Bundle of the **smooth part of** $\mathcal{M}$. It is defined as the disjoint union $\mathcal{S}^1(\mathcal{M}) \sqcup \mathcal{S}^2(\mathcal{M})$ where $\mathcal{S}^2(C) = C \setminus \{\text{origin}\}$, $\mathcal{S}^1(D) = z-\text{axis}$, $\mathcal{S}^2(D) = D \setminus (z - \text{axis})$, $\mathcal{S}^2(P) = P$, $\mathcal{S}^2(A) = A$ and $\mathcal{S}^i(\mathcal{M})$ is the empty set on the rest cases. These Tangent Vector Fields on $\mathcal{M}$ define

$$\mathfrak{X}^0(\mathcal{M}) = \left\{\widehat{X} \colon \mathcal{M} \longrightarrow \mathbb{R}^3 : \widehat{X} \text{ is continuos on } \mathcal{M} \text{ and tangent to } \mathcal{M}\right\} \tag{2.1}$$

and then $\mathfrak{X}^0(\mathcal{S}^2(\mathcal{M}))$ is the set of the continuous section respect to the Tangent Bundle on the sub–manifold $\mathcal{S}^2(\mathcal{M}) \subset \mathbb{R}^3$. The general definition of Tangent Vector Fields on a *two–dimensional manifold with simple singularities* appears in [8].

2.1. Vector fields of class C^0 on $\mathcal{M}$.

A vector field $X : \mathcal{S}^2(\mathcal{M}) \to \mathbb{R}^3$ is said to be of **class C^0 on $\mathcal{M}$** if it admits a continuous extension to all $\mathcal{M}$. That is, there exists a continuous $\widehat{X} \colon \mathcal{M} \to \mathbb{R}^3$, in $\mathfrak{X}^0(\mathcal{M})$ such that the restriction $\widehat{X}| \colon \mathcal{S}^2(\mathcal{M}) \to \mathbb{R}^3$ equals $X \colon \mathcal{S}^2(\mathcal{M}) \to \mathbb{R}^3$. In this context,

$$\mathcal{X}^0(\mathcal{S}^2(\mathcal{M}) = \left\{X \in \mathfrak{X}^0(\mathcal{S}^2(\mathcal{M})) : X \text{ is of class } C^0 \text{ on } \mathcal{M}\right\}.$$

The Tangent Spaces on $\mathcal{S}^2(C)$ intersect the Cone on some straight line passing through the origin $0 \in C$. They are pairwise transversal planes in $\mathbb{R}^3$ which contain $X(0)$, for any $X \in \mathfrak{X}^0(C)$. By using the continuity of the vector fields along three of these intersection lines, it is not difficult to prove that: any Tangent Vector Field on the Cone has a **singularity** in the origin (i.e., $X(0) = 0$).

The definition of class C^0 on the Cone implies that any vector field in $\mathcal{X}^0(\mathcal{S}^2(C))$ must be continuously extended to the origin. This extension is unique in the sense that it sends the origin to zero. Therefore, the mapping $\widehat{X} \mapsto X$, where X is the restriction $\widehat{X}| : \mathcal{S}^2(C) \to \mathbb{R}^3$ defines a bijection from $\mathfrak{X}^0(C)$ onto $\mathcal{X}^0(\mathcal{S}^2(C))$:

$$\mathcal{X}^0(\mathcal{S}^2(C)) \approx \mathfrak{X}^0(C) = \left\{X \in \mathfrak{X}^0(C) : X(0) = 0\right\}. \tag{2.2}$$

Consequently, the unitary vector fields, tangent to the smooth part of the Cone are not continuous on the whole Cone.

To prove the bijective result: $\mathcal{X}^0(\mathcal{S}^2(D)) \approx \mathfrak{X}^0(D)$, we consider a *continuous* extension $\widehat{X}: D \to \mathbb{R}^3$ of a vector field $X \in \mathcal{X}^0(\mathcal{S}^2(D))$ to all the surface D. Since $\mathcal{S}^1(D) = \{x = 0\} \cap \{y = 0\}$, $\widehat{X}(\{x = 0\}) \subset \{x = 0\}$ and $\widehat{X}(\{y = 0\}) \subset \{y = 0\}$, the restriction $\widehat{X}| : \mathcal{S}^1(D) \to \mathbb{R}^3$ is a section of the Tangent Bundle of $\mathcal{S}^1(D)$. The continuity of $\widehat{X}$ on D proves that $\widehat{X}|: \mathcal{S}^1(D) \to \mathbb{R}^3$ is unique in the sense that it does not depend on the extension $\widehat{X}$ to D. Therefore, the mapping $X \mapsto \widehat{X}$ is a bijection from $\mathcal{X}^0(\mathcal{S}^2(D))$ onto $\mathfrak{X}^0(D)$.

In both cases: the plane P and the cylinder A, the bijection is the identity map. Therefore, we obtain that

$$\mathfrak{X}^0(\mathcal{M}) \approx \mathcal{X}^0(\mathcal{S}^2(\mathcal{M})). \tag{2.3}$$

This result (2.3) fails when the codimension on $\mathcal{M}$ is different from zero. Take for instance $Z(x, y, z) = (x, -y, 0)$ and $Z^*(x, y, z) = (x, 0, 0)$. It is not difficult to see that $Z, Z^* \in \mathfrak{X}^0(D)$, but the restrictions $Z| : \mathcal{S}^1(D) \to \mathbb{R}^3$ and $Z^*| : \mathcal{S}^1(D) \to \mathbb{R}^3$ are the zero section on $\mathcal{S}^1(D)$. Note that, $Z(q) \neq 0$, for all $q \in \mathcal{S}^2(D)$.

Remark 2.1. The set $\mathcal{X}^0(\mathcal{S}^2(D)$ is properly contained in $\mathfrak{X}^0(\mathcal{S}^2(D)$, the set of all the continuous sections of $\mathcal{S}^2(D)$. It is shown by the following continuous section $Y: \mathcal{S}^2(D) \to \mathbb{R}^3$ defined on $\mathcal{S}^2(D) = \{(x, y, z) \in D :$ either $x \neq 0$ or $y \neq 0\}$ as

$$Y(x, y, z) = \begin{cases} (0,\ 0, -x\ ) & x \neq 0, y = 0; \\ \left(0, 0, \dfrac{y}{||y||}\right) & x = 0, y \neq 0. \end{cases}$$

Since Y can not be continuously extended to all D, we have $Y \notin \mathcal{X}^0(\mathcal{S}^2(\mathcal{M})$.

2.2. **The Strong topology.** A trajectory of $X \in \mathfrak{X}^0(\mathcal{M})$ starting at $q \in \mathcal{M}$ is defined by a curve $t \to \gamma_q(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$ such that **(a)** t varies on some open real interval which contain the zero whose image satisfies $\gamma_q(0) = q$; **(b)** $\gamma_q(t) \in \mathcal{M}$, there exist the real derivatives $\frac{dx}{dt}(t)$, $\frac{dy}{dt}(t)$, $\frac{dz}{dt}(t)$, and; **(c)** $\dot{\gamma}_q(t) = \left(\frac{dx}{dt}(t), \frac{dy}{dt}(t), \frac{dz}{dt}(t)\right)$ the velocity vector field of γ_q at $\gamma_q(t)$ equals $X(\gamma_q(t))$. We assume the uniqueness of the trajectories of X and we denote by X_t the **induced flow on** $\mathcal{M}$. This local flow is well–defined in the sense that: for each $q \in \mathcal{M}$, even in $0 \in C$, the curve $t \mapsto X_t(q) \in \mathcal{M}$ gives the trajectory starting at q whose domain $I_q \subset \mathbb{R}$ is the maximal interval of definition and $X_s(X_t(q)) = X_{s+t}(q)$, for all s and t such that $t, s + t \in I_q$. Such X is called the Velocity Vector Field of X_t.

Remark 2.2. Any connected component of the smooth part of $\mathcal{M}$, say S, is *invariant* under the induced flow; that is, $X_t(q) \in S$ for all t as long as $q \in S$. Moreover, if $q \in \mathcal{M}$ does not belong to the smooth part of $\mathcal{M}$, (2.2) proves that q is a *singularity*, or equivalently it is an equilibrium point of the flow (i.e., $X_t(q) = q, \forall t$).

We recall [16] that a function $g : M \to \mathbb{R}$, is said to be smooth on $M \subset \mathbb{R}^3$ if around each point $q \in M$ there exist a function $g_q : U \to \mathbb{R}$ smooth on some open set $U \subset \mathbb{R}^3$ such that g_q coincides with g throughout $U \cap M$. Since these functions are continuous, we consider the set $\mathcal{F}^0(M)$ of all the continuous functions $f : M \to \mathbb{R}$ and we define the set

$$\mathcal{N}^0_\varepsilon(f, M) = \Big\{ g \in \mathcal{F}^0(M) : \big|g(q) - f(q)\big| < \varepsilon(q),\ \forall q \in M \Big\},$$

where $\varepsilon : M \to (0, +\infty) \subset \mathbb{R}$ is a positive valued function. These sets characterize the **Strong C^0–Topology of Whitney** in $\mathcal{F}^0(M)$, and it has all possible sets of the form $\mathcal{N}^0_\varepsilon(f, M)$ for a base.

Definition 2.3 (System of Directions)**.** A *System of Directions* on $\mathcal{M}$ is an ordered pair $\mathfrak{D} = (U, V)$ of continuous vector fields $U, V : \mathcal{M} \to \mathbb{R}^3$ such that:

(1) U and V are tangents to $\mathcal{M}$;
(2) the restrictions $U| : S \to \mathbb{R}^3$ and $V| : S \to \mathbb{R}^3$ are smooth functions on manifolds as long as S is a connected component in $\mathcal{S}^1(\mathcal{M}) \sqcup \mathcal{S}^2(\mathcal{M})$;
(3) the set $\big\{U(p), V(p), U(p) \times V(p)\big\}$ is an orthogonal positive basis of $\mathbb{R}^3$, for each $p \in \mathcal{S}^2(\mathcal{M})$; and
(4) $V(p) = (0, 0, 1)$, for all $p \in \mathcal{S}^1(\mathcal{M})$.

Given $f \in \mathcal{F}^0(\mathcal{M})$, a continuous function $f : \mathcal{M} \to \mathbb{R}$, we define $f_u(p)$ and $f_v(p)$ the **$\mathfrak{D}$–Partial Derivatives** of f at p, as the limits:

$$f_u(p) = \lim_{t\to 0} \frac{f(\widetilde{U}_t(p)) - f(p)}{t} \quad \text{and} \quad f_v(p) = \lim_{t\to 0} \frac{f(\widetilde{V}_t(p)) - f(p)}{t},$$

where $\widetilde{U}_t$ and $\widetilde{V}_t$ are the induced flows given by the unitary vector fields $\widetilde{U} = \frac{U}{||U||}$ and $\widetilde{V} = \frac{V}{||V||}$, respectively. In the points where these limits are real numbers, the vectors

$$\frac{\partial^* f}{\partial u}|p = f_v(p)\widetilde{U}(p) \quad \text{and} \quad \frac{\partial^* f}{\partial v}|p = f_u(p)\widetilde{V}(p),$$

are tangents. In particular, the vector field $\frac{\partial^* f}{\partial u}| : \mathcal{S}^2(\mathcal{M}) \to \mathbb{R}^3$ given by $p \mapsto \frac{\partial^* f}{\partial u}|p$ might be of class $\mathcal{C}^0$ on $\mathcal{M}$, as in § 2.1. Thus, we define the space

$$\mathcal{F}^1(\mathcal{M}; \mathfrak{D}) = \Big\{ f \in \mathcal{F}^0(\mathcal{M}) : \frac{\partial^* f}{\partial u}|, \frac{\partial^* f}{\partial v}| \in \mathcal{X}^0(\mathcal{S}^2(\mathcal{M})) \Big\}.$$

Remark 2.4. When $f \in \mathcal{F}^1(\mathcal{M}; \mathfrak{D})$, the $\mathfrak{D}$–Partial Derivatives define the continuous functions $f_u : \mathcal{S}^2(\mathcal{M}) \to \mathbb{R}$ and $f_v : \mathcal{S}^1(\mathcal{M}) \sqcup \mathcal{S}^2(\mathcal{M}) \to \mathbb{R}$. Thus, the number of $\mathfrak{D}$–partial derivatives coincides with the dimension of the manifold.

As a consequence of these properties we obtain.

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Proposition 2.5. *Suppose that $f \in \mathcal{F}^0(\mathcal{M};\mathfrak{D})$ and its $\mathfrak{D}$–Partial Derivatives define the vector fields $\frac{\partial f}{\partial u}| : \mathcal{S}^2(\mathcal{M}) \to \mathbb{R}^3$ and $\frac{\partial f}{\partial u}| : \mathcal{S}^2(\mathcal{M}) \to \mathbb{R}^3$ by the following formulaes*

$$\frac{\partial f}{\partial u}|p = f_u(p)\widetilde{U}(p) \quad \textit{and} \quad \frac{\partial f}{\partial v}|p = f_v(p)\widetilde{V}(p).$$

These vector fields, $\frac{\partial f}{\partial u}|$ and $\frac{\partial f}{\partial v}|$ are of class $\mathcal{C}^0$ on $\mathcal{M}$ if and only if $f \in \mathcal{F}^1(\mathcal{M};\mathfrak{D})$.

Proof. The unitary vector fields $\widetilde{U}$ and $\widetilde{V}$ are continuous on $\mathcal{M}$. Therefore, this propositon is directly obtaned form the definitions of the spaces $\mathcal{F}^1(\mathcal{M};\mathfrak{D})$ and $\mathcal{X}^0(\mathcal{S}^2(\mathcal{M}))$, given in § 2.1. □

Given $f \in \mathcal{F}^1(\mathcal{M};\mathfrak{D})$, by Remark 2.4 f_u and f_v are continuous functions. Thus, we can apply the same constructions to f_u and f_v. These definitions give the $\mathfrak{D}-$*Partial Derivatives of second order:* f_{uu}, f_{uv}, f_{vu} and f_{vv}. On a similar way, there exist the vector fields $\frac{\partial^* f_u}{\partial u}|$, $\frac{\partial^* f_u}{\partial v}|$, $\frac{\partial^* f_v}{\partial u}|$ and $\frac{\partial^* f_v}{\partial v}|$, which might be of class C^0 on $\mathcal{M}$. Therefore, we define:

$$\mathcal{F}^2(\mathcal{M};\mathfrak{D}) = \Big\{ f \in \mathcal{F}^1(\mathcal{M};\mathfrak{D}) : \frac{\partial^* f_u}{\partial u}|, \frac{\partial^* f_u}{\partial v}|, \frac{\partial^* f_v}{\partial u}|, \frac{\partial^* f_v}{\partial v}| \in \mathcal{X}^0(\mathcal{S}^2(\mathcal{M})) \Big\}.$$

For each $r \geq 1$ a positive integer we can proceed inductively and characterize the $\mathfrak{D}-$*Partial Derivatives of order* r and the space $\mathcal{F}^r(\mathcal{M};\mathfrak{D})$. These spaces satisfy

$$\mathcal{F}^r(\mathcal{M};\mathfrak{D}) \subset \mathcal{F}^{r-1}(\mathcal{M};\mathfrak{D}) \subset \cdots \subset \mathcal{F}^1(\mathcal{M};\mathfrak{D}) \subset \mathcal{F}^0(\mathcal{M};\mathfrak{D}).$$

In this context, if $f \in \mathcal{F}^r(\mathcal{M};\mathfrak{D})$ and $p \in \mathcal{S}^1(\mathcal{M}) \sqcup \mathcal{S}^2(\mathcal{M})$, let $||f(p)||_{C^r}$ be the maximum among the $\mathfrak{D}$–Partial derivatives up to and including order r, all evaluated at p. For every $\varepsilon : \mathcal{M} \to (0, +\infty)$ consider

$$\mathcal{N}^r_\varepsilon(f,\mathcal{S}) = \Big\{ g \in \mathcal{F}^r(\mathcal{M};\mathfrak{D}) : \big|\big|g(p) - f(p)\big|\big|_{C^r} < \varepsilon(p), \forall p \in \mathcal{S}^1(\mathcal{M}) \sqcup \mathcal{S}^2(\mathcal{M}) \Big\}.$$

Therefore, a basis for the neighborhoods of $f \in \mathcal{F}^r(\mathcal{M};\mathfrak{D})$ in the **Strong** C^r– **Topology of Whitney** is given by the sets

$$\mathcal{N}^r_\varepsilon(f,\mathcal{M}) = \mathcal{N}^0_\varepsilon(f,\mathcal{M}) \cap \mathcal{N}^r_\varepsilon(f,\mathcal{S}), \tag{2.4}$$

when $\varepsilon : \mathcal{M} \to (0, +\infty)$ varies over the positive functions on $\mathcal{M}$.

We conclude this § 2, by proving that the spaces $\mathcal{F}^r(\mathcal{M};\mathfrak{D})$ are independent of the system of directions.

Proposition 2.6. *Suppose that $\overline{\mathfrak{D}} = (\overline{U}, \overline{V})$ is a System of Directions on $\mathcal{M}$.*

(a) *The $\overline{\mathfrak{D}}$–Partial Derivatives $f_{\overline{u}}$ and $f_{\overline{v}}$ are equal to f_u and f_v, respectively.*
(b) *A function $f \in \mathcal{F}^1(\mathcal{M};\mathfrak{D})$ if and only if $\frac{\partial^* f}{\partial \overline{u}}|, \frac{\partial^* f}{\partial \overline{v}}| \in \mathcal{X}^0(\mathcal{S}^2(\mathcal{M}))$.*
(c) *The basic neighborhood in (2.4) is independent of the System of Directions. Therefore, $\mathcal{F}^r(\mathcal{M};\overline{\mathfrak{D}}) = \mathcal{F}^r(\mathcal{M};\mathfrak{D})$.*

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Proof. The unitary vector fields satisfy $\frac{\overline{U}}{||\overline{U}||} = \frac{U}{||U||}$ and $\frac{\overline{V}}{||\overline{V}||} = \frac{V}{||V||}$. Thus (a) holds. Statement (b) is obtained directly from (a) and the definition of such topological space. Moreover, since (a) and (b) are true, a direct application of the definitions gives (c) and concludes this proof. □

The space $\mathcal{F}^r(\mathcal{M};\mathfrak{D})$ will be denoted by $\mathcal{F}^r(\mathcal{M})$, on clear context.

3. Dynamic of Tangent Vector Fields on $\mathcal{M}$

Let $X = (f_1, f_2, f_3)$ denote a Tangent Vector Field on $\mathcal{M}$. Given $r \geq 1$ a positive integer, we define the set of the Tangent Vector Fields of **class** C^r **on** $\mathcal{M}$ as

$$\mathfrak{X}^r(\mathcal{M}) = \Big\{(f_1, f_2, f_3) \in \mathfrak{X}^0(\mathcal{M}) : f_1, f_2, f_3 \in \mathcal{F}^r(\mathcal{M})\Big\}.$$

Moreover, $\mathfrak{X}^r(\mathcal{M})$ is endowed with the Strong C^r–Topology of Whitney, this is obtained by applying the estimates in (2.4) to the components which always belong to $\mathcal{F}^r(\mathcal{M})$. Therefore, if $\varepsilon : \mathcal{M} \to (0, +\infty)$ and $X = (f_1, f_2, f_3)$, the set

$$\mathcal{N}^r_\varepsilon(X, \mathcal{M}) = \Big\{(g_1, g_2, g_3) \in \mathfrak{X}^r(\mathcal{M}) : g_i \in \mathcal{N}^r_\varepsilon(f_i, \mathcal{M}), \forall i = 1, 2, 3\Big\}$$

describe a basic neighborhood of X in $\mathfrak{X}^r(\mathcal{M})$.

By Proposition 2.6, the topological space $\mathfrak{X}^r(\mathcal{M})$ is independent of the System of Directions.

3.1. The space $\mathcal{X}^r(\mathcal{S}^2(\mathcal{M}))$.

The space $\mathcal{X}^0(\mathcal{S}^2(\mathcal{M}))$ is defined in § 2.1, where we saw that every element $X \in \mathcal{X}^0(\mathcal{S}^2(\mathcal{M}))$ admits a unique continuous extension $\widehat{X} : \mathcal{M} \to \mathbb{R}^3$ which belongs to $\mathfrak{X}^0(\mathcal{M})$ and then $X = \widehat{X}| : \mathcal{S}^2(\mathcal{M}) \to \mathbb{R}^3$. Similarly if $r \geq 1$ is a positive integer, we define the space

$$\mathcal{X}^r(\mathcal{S}^2(\mathcal{M})) = \Big\{X \in \mathcal{X}^0(\mathcal{S}^2(\mathcal{M})) : \widehat{X} \in \mathfrak{X}^r(\mathcal{M})\Big\},$$

endowed with the Strong C^r–Topology of Whitney, defined as the weakest topology under which the following mapping

$$\mathcal{X}^r(\mathcal{S}^2(\mathcal{M})) \ni X \mapsto \widehat{X} \in \mathfrak{X}^r(\mathcal{M})$$

is continuous.

Proposition 3.1. *Consider $r \geq 1$ a positive integer.*

(a) *If $X \in \mathcal{X}^r(\mathcal{S}^2(\mathcal{M}))$ and $\varepsilon : \mathcal{M} \to (0, +\infty)$ is positive, then the set*

$$\mathcal{N}^r_\varepsilon(X, \mathcal{S}^2(\mathcal{M})) = \Big\{Y \in \mathcal{X}^r(\mathcal{S}^2(\mathcal{M})) : \widehat{Y} \in \mathcal{N}^r_\varepsilon(\widehat{X}, \mathcal{M})\Big\},$$

where $\widehat{Y}$ extends Y to all $\mathcal{M}$ is a basic neighborhood of X in $\mathcal{X}^r(\mathcal{S}^2(\mathcal{M}))$.

(b) *The map $X \mapsto \widehat{X}$ defines a homeomorphism between $\mathcal{X}^r(\mathcal{S}^2(\mathcal{M}))$ and $\mathfrak{X}^r(\mathcal{M})$.*

Proof. From the definition of the topology over $\mathcal{X}^r(\mathcal{S}^2(\mathcal{M}))$ it is not difficult to show that all the sets $\mathcal{N}^r_\varepsilon(X, \mathcal{S}^2(\mathcal{M}))$ are open. So, by using that $\mathcal{N}^r_\varepsilon(X, \mathcal{M})$ always are basic neighborhoods of X in $\mathfrak{X}^r(\mathcal{M})$ we conclude that (a) holds.

To prove (b) we recall (2.3) that

$$\mathfrak{X}^r(\mathcal{M}) \ni \widehat{X} \mapsto \widehat{X}| \in \mathfrak{X}^r(\mathcal{S}^2(\mathcal{M})),$$

where $\widehat{X}| : \mathcal{S}^2(\mathcal{M}) \to \mathbb{R}^3$ is the restriction to $\mathcal{S}^2(\mathcal{M})$ gives an injective mapping. Since $\mathcal{X}^r(\mathcal{S}^2(\mathcal{M}))$ is its image we obtain that $\widehat{X} \mapsto \widehat{X}|$ is a bijection from $\mathfrak{X}^r(\mathcal{M})$ onto $\mathcal{X}^r(\mathcal{S}^2(\mathcal{M}))$. Moreover, for every basic neighborhood $\mathcal{N}^r_\varepsilon(X, \mathcal{S}^2(\mathcal{M}))$ its inverse image under that map is open in $\mathfrak{X}^r(\mathcal{M})$. Thus $\widehat{X} \mapsto \widehat{X}|$ is a continuous bijection whose inverse is also continuous, by our construction. Therefore, this defines a homeomorphism and then (b) is true. □

3.2. **Hamiltonian dynamics on $\mathcal{M}$.** To present the Hamiltonian flows on $\mathcal{M}$ we consider the definition of $\mathcal{F}^1(\mathcal{M})$ and also the following.

Example 3.2. For every $f \in \mathcal{F}^1(\mathcal{M})$ we define X_f, a Tangent Vector Field on $\mathcal{M}$, as the unique continuous extension of $\frac{\partial^* f}{\partial u}| - \frac{\partial^* f}{\partial v}|$ to all $\mathcal{M}$; that is,

$$X_f = \widehat{\frac{\partial^* f}{\partial u}}| - \widehat{\frac{\partial^* f}{\partial v}}|.$$

An inductive argument proves that $X_f \in \mathfrak{X}^r(\mathcal{M})$ as long as $f \in \mathcal{F}^{r+1}(\mathcal{M})$.

Two flows X_t and Y_t are **topologically equivalent** on $\mathcal{M}$ when there is a homeomorphism $h\colon \mathcal{M} \to \mathcal{M}$ which sends trajectories of X onto trajectories of Y preserving their orientations. It defines a equivalence relation whose equivalence classes give a natural partition, for instance, on the space of the C^r sections on a smooth manifold. If the manifold is compact the interior points (in the C^r-compact open topology) in such a class of C^r sections are called C^r-structurally stable [8, 9]. However, in the general case of an open surface we assume that the equivalence belongs to $\mathcal{N}_{co}(K, U; \mathcal{M})$: a *compact–open neighborhood of the identity on* $\mathcal{M}$ given by a pair (K, U) of a compact set K contained in an open subset U in $\mathcal{M}$. This neighborhood $\mathcal{N}_{co}(K, U; \mathcal{M})$ is the set of all the continuous functions $g\colon \mathcal{M} \to \mathcal{M}$ such that $g(K) \subset U$. Note that, $\mathcal{N}_{co}(K, U; \mathcal{M})$ contains the continuous functions for which

$$\big|g(p) - p\big| < \delta \quad \forall p \in K,$$

if the fixed $\delta > 0$ is less than the euclidian distance from K to $\mathcal{M} \setminus U$.

Proposition 3.3. *Suppose that $f \in \mathcal{F}^2(\mathcal{M}; \mathfrak{D})$ and $\mathfrak{D} = (U, V)$.*

(a) $X_f \in \mathfrak{X}^1(\mathcal{M})$ *and* $\frac{\partial^* f}{\partial u}|$ *has an extension* $\widehat{\frac{\partial^* f}{\partial u}}| \in \mathfrak{X}^0(\mathcal{M})$ *such that* $\widehat{\frac{\partial^* f}{\partial u}}|_p = 0$, *for all* $p \in \mathcal{S}^1(\mathcal{M})$.

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(b) *If* $\overline{\mathfrak{D}} = (\overline{U}, \overline{V})$ *is a System of Directions on* $\mathcal{M}$ *and* $\overline{V} = V$*, then* $f \in \mathcal{F}^2(\mathcal{M}; \overline{\mathfrak{D}})$, $\widehat{\frac{\partial^* f}{\partial \overline{v}}|} = \widehat{\frac{\partial^* f}{\partial v}|}$ *and* $(\widehat{\frac{\partial^* f}{\partial u}|})_t$ *is topologically equivalent to* $\widehat{Z}_t$*, where* $Z(p) = f_v(p)\overline{U}(p), \forall p \in \mathcal{S}^2(\mathcal{M})$.

Proof. It is not difficult to see that $\frac{\partial^* f}{\partial u}|$ and $\frac{\partial^* f}{\partial v}|$ are of class C^0 on $\mathcal{M}$, and $X_f \in \mathfrak{X}^1(\mathcal{M})$ as long as $f \in \mathcal{F}^2(\mathcal{M})$. In particular, $\frac{\partial^* f}{\partial u}| \in \mathcal{X}^0(\mathcal{S}^2(D))$ admits a tangent extension $\widehat{\frac{\partial^* f}{\partial u}|} \in \mathfrak{X}^0(D)$ to D and the respective restriction $(\widehat{\frac{\partial^* f}{\partial u}|})| : \mathcal{S}^1(D) \to \mathbb{R}^3$ is tangent to $\mathcal{S}^1(D)$. The induced flow $(\widehat{\frac{\partial^* f}{\partial u}|})_t$ on D is well-defined, and the nontrivial curves $t \mapsto (\widehat{\frac{\partial^* f}{\partial u}|})_t(q)$ with $q \in \mathcal{S}^2(D)$ are horizontal. Therefore, we can proceed by contradiction and by using the trajectory of $\widehat{\frac{\partial^* f}{\partial u}|}$ at $p \in \mathcal{S}^1(D)$ it is not difficult to conclude that $\widehat{\frac{\partial^* f}{\partial u}}|_p = 0$, for any point $p \in \mathcal{S}^1(D)$. This is (a).

To obtain (b) note that $\widehat{\frac{\partial^* f}{\partial \overline{v}}|} = \widehat{\frac{\partial^* f}{\partial v}|}$ on $\mathcal{M}$. Moreover, by Proposition 2.6, $f \in \mathcal{F}^2(\mathcal{M}; \overline{\mathfrak{D}})$. Thus, (2.2) and (a) imply that

$$\widehat{\frac{\partial^* f}{\partial u}}|_p = \widehat{\frac{\partial^* f}{\partial \overline{u}}}|_p = 0, \quad \forall p \notin \mathcal{S}^2(\mathcal{M}).$$

Since, $f_v = f_{\overline{v}}$ and $V = \overline{V}$ the part (c) in Definition 2.3 gives the existence of a continuous and nonnegative function $\alpha : \mathcal{M} \to [0, +\infty)$ such that

$$\alpha(p) > 0 \quad \text{and} \quad \widehat{\frac{\partial^* f}{\partial u}}|_p = \alpha(p) Z(p), \quad \forall p \in \mathcal{S}^2(\mathcal{M}).$$

By using these properties it is not difficult to show that $(\widehat{\frac{\partial^* f}{\partial u}|})_t$ is topologically equivalent to $\widehat{Z}_t$. This concludes the proof. □

Given $X \in \mathfrak{X}^r(\mathcal{M})$, the flow X_t is *globally* C^r*–structurally stable* if given a set $\mathcal{N}_{co}(K, U; \mathcal{M})$, there is a neighborhood $\mathcal{N}^r_\varepsilon(X, \mathcal{M})$ such that every flow Y_t with $Y \in \mathcal{N}^r_\varepsilon(X, \mathcal{M})$ is topologically equivalent to X_t and its equivalence $h \in \mathcal{N}_{co}(K, U; \mathcal{M})$. A similar definition has previously been used in [14, 15] in the case $\mathcal{M} = P$.

Definition 3.4. A flow X_t with $X \in \mathfrak{X}^r(\mathcal{M})$ is called *Hamiltonian* if there exists a function $f \in \mathcal{F}^{r+1}(\mathcal{M})$ such that X equals X_f, where $X_f \in \mathfrak{X}^r(\mathcal{M})$ is defined in Example 3.2. We denote by $\mathcal{H}^r(\mathcal{M})$ the subspace of $\mathfrak{X}^r(\mathcal{M})$ given by the Velocity Vector Fields of these Hamiltonian flows with the induced topology from $\mathfrak{X}^r(\mathcal{M})$.

Observe that, for each $f \in \mathcal{F}^{r+1}(\mathcal{M})$ the flow $(X_g)_t$, where $g = -f$ is also Hamiltonian, and the reverse curve $t \mapsto (X_g)_{-t}$ defines the trajectory of X_f at q.

Definition 3.5 (Hamiltonian stable)**.** A Hamiltonian flow $(X_f)_t$ with $X_f \in \mathcal{H}^r(\mathcal{M})$ is *Hamiltonian* C^r*–stable* if given a compact–open neighborhood $\mathcal{N}_{co}(K, U; \mathcal{M})$ of the identity on $\mathcal{M}$, there is a basic neighborhood $\mathcal{N}^r_\varepsilon(X_f, \mathcal{M})$

of X_f such that every Hamiltonian flow Y_t with $Y \in \mathcal{N}^r_\varepsilon(X_f, \mathcal{M}) \cap \mathcal{H}^r(\mathcal{M})$ is topologically equivalent to $(X_f)_t$ and its equivalence $h \in \mathcal{N}_{co}(K, U; \mathcal{M})$.

The space $\mathcal{H}^r(\mathcal{M})$ is the image of the mapping

$$\mathcal{F}^{r+1}(\mathcal{M}) \ni f \longmapsto X_f \in \mathfrak{X}^r(\mathcal{M}). \tag{3.1}$$

Moreover, $\mathcal{N}^r_\varepsilon(X_f, \mathcal{M}) \cap \mathcal{H}^r(\mathcal{M})$ is a basic neighborhood of X_f in $\mathcal{H}^r(\mathcal{M})$ and there is some $\tilde{\varepsilon} : \mathcal{M} \to (0, +\infty)$ such that

$$g \in \mathcal{N}^{r+1}_{\tilde{\varepsilon}}(f, \mathcal{M}) \Longrightarrow X_g \in \mathcal{N}^r_\varepsilon(X_f, \mathcal{M}) \cap \mathcal{H}^r(\mathcal{M}).$$

Therefore, the map in (3.1) is continuous and then the Hamiltonian stability can be formulated in terms of $\mathcal{F}^{r+1}(\mathcal{M})$: "$f \in \mathcal{F}^{r+1}(\mathcal{M})$ is *Hamiltonian* C^r*–stable* if given $\mathcal{N}_{co}(K, U; \mathcal{M})$, there is $\mathcal{N}^{r+1}_{\tilde{\varepsilon}}(f, \mathcal{M})$ such that every flow $(X_g)_t$ with $g \in \mathcal{N}^{r+1}_{\tilde{\varepsilon}}(f, \mathcal{M})$ is topologically equivalent to $(X_f)_t$ and $h \in \mathcal{N}_{co}(K, U; \mathcal{M})$". See [14].

Proposition 3.6. *If $f \in \mathcal{F}^2(\mathcal{M})$, the following hold:*

(a) *This function is constant along the trajectories of $X_f \in \mathcal{H}^1(\mathcal{M})$; that is, $f((X_f)_t(q)) = f(q), \forall t$ as long as $q \in \mathcal{M}$. A nonconstant $g \in \mathcal{F}^0(\mathcal{M})$ with this property is called: (global)* ***first integral of*** *X_f.*

(b) *$X_f(p) = 0$ if and only if $f_u(p) = f_v(p) = 0$, for all $p \in \mathcal{S}^2(\mathcal{M})$.*

(c) *Such f is nonconstant on any open subset in $\mathcal{M}$ if and only if the set of the* ***critical points*** *of f, given by*

$$\mathcal{C}r(f, \mathcal{M}) = \{q \in \mathcal{M} : X_f(q) = 0\}$$

has no interior points on $\mathcal{M}$; that is, $\mathcal{M} \setminus \mathcal{C}r(f, \mathcal{M})$ is dense on $\mathcal{M}$. A first integral with this property is named: ***strong first integral.***

Proof. To prove (a) apply Remark 2.2 to $(X_f)_t$. Thus, either $(X_f)_t(0) = 0 \in C, \forall t$ or $(X_f)_t(q) \in S, \forall t$, where $S \subset \mathcal{S}^1(\mathcal{M}) \sqcup \mathcal{S}^2(\mathcal{M})$ is the connected component containing $q \in \mathcal{M}$. The first case trivially implies that f is constant along the trajectory of $X_f \in \mathfrak{X}^1(C)$ at 0. The second one gives two different cases: either $S \subset \mathcal{S}^2(\mathcal{M})$ or $S \subset \mathcal{S}^1(\mathcal{M})$. If $S \subset \mathcal{S}^2(\mathcal{M})$, the definitions of X_f and $\mathfrak{D}-$Partial Derivatives imply that f is constant along the X_f-trajectories on $\mathcal{S}^2(\mathcal{M})$. If $S \subset \mathcal{S}^1(\mathcal{M})$, Propositon 3.3 proves that $\widehat{\frac{\partial^* f}{\partial u}}|_p = 0$ for any $p \in \mathcal{S}^1(D)$. Thus, both planes $\{x = 0\} \subset D$ and $\{y = 0\} \subset D$ are invariants under $(X_f)_t$ and we obtain a similar result with the X_f-trajectories on $\mathcal{S}^1(D)$. Therefore (a) holds.

To obtain (b), observe that

$$X_f(p) = \frac{\partial^* f}{\partial u}|_p - \frac{\partial^* f}{\partial v}|_p, \quad \forall p \in \mathcal{S}^2(\mathcal{M}).$$

Thus, $X_f(p) = 0$ when both $\mathfrak{D}$–Partial Derivatives of f at p are zero. Reciprocally, if $\frac{\partial^* f}{\partial u}|_p - \frac{\partial^* f}{\partial v}|_p = 0$, by using the inner product with each vector $\widetilde{U}(p)$ and $\widetilde{V}(p)$, and the assumption (3) in Definition 2.3 we have that $f_u(p) = f_v(p) = 0$. This concludes the proof of (b)

Since any nonempty open subset in $\mathcal{M}$ has interior points in $\mathcal{S}^2(\mathcal{M})$, we obtain (c) by using (b). □

Remark 3.7. There are Hamiltonian flows with the same strong first integral, but they are not topologically equivalent. This is shown by the function $f(x,0,z) = z$, and the Hamiltonian flows on the Plane $P = \{y = 0\}$ induced by $X_f(x,0,z) = (1,0,0)$ and $X_h(x,0,z) = (3z^2,0,0)$ con $h(x,0,z) = z^3$.

Since the generating function $f \in \mathcal{F}^{r+1}(\mathcal{M})$ of a Hamiltonian flow $(X_f)_t$ is constant along trajectories, the structure of the flow is closely related to the behavior of the **level sets** of the function $f \in \mathcal{F}^{r+1}(\mathcal{M})$, defined as

$$\mathcal{L}_c(f,\mathcal{M}) = \mathcal{M}\{f = c\} = \big\{p \in \mathcal{M} : f(p) = c\big\}. \tag{3.2}$$

In particular, if this function is free of critical points (the set $\mathcal{C}r(f,\mathcal{M})$ in Proposition 3.6 is empty) the connected components of these level sets $\mathcal{L}_c(f,\mathcal{M})$ naturally induces the leaves of a foliation on $\mathcal{S}^2(\mathcal{M})$.

Example 3.8. For every $f \in \mathcal{F}^1(\mathcal{M})$ we define a tangent vector field on $\mathcal{M}$ as the unique continuous extension of $\frac{\partial f}{\partial u}| + \frac{\partial f}{\partial v}|$ to all $\mathcal{M}$; (see Proposition 2.5) that is,

$$X_f^* = \widehat{\frac{\partial f}{\partial u}}| + \widehat{\frac{\partial f}{\partial v}}|.$$

In particular, if $f \in \mathcal{F}^{r+1}(\mathcal{M})$, the following properties hold:

- $X_f^* \in \mathfrak{X}^r(\mathcal{M})$, and it induces a local flow $(X_f^*)_t$.
- A local maximun (minimum) of f is a sink (source) for $(X_f^*)_t$.
- $(X_f^*)_t$ induces a foliation on $\mathcal{S}^2(\mathcal{M}) \setminus \mathcal{C}r(f,\mathcal{M})$, transversal to (3.2).

3.2.1. *Some generic properties.* To describe some dynamical properties of $X_f \in \mathfrak{X}^1(\mathcal{M})$, in Proposition 3.10, we need some preliminary notations and concepts.

Given a System of Directions on $\mathcal{M}$, the restrictions $U|: \mathcal{S}^2(\mathcal{M}) \to \mathbb{R}^3$ and $V|: \mathcal{S}^2(\mathcal{M}) \to \mathbb{R}^3$ are smooth sections on the sub–manifold $\mathcal{S}^2(\mathcal{M}) \subset \mathbb{R}^3$, which is locally diffeomorphic to $\mathbb{R}^2$. Thus, if $p \in \mathcal{S}^2(\mathcal{M})$, there exists an homeomorphism $\phi_p : U_0 \subseteq \mathbb{R}^2 \to V_p \subset \mathcal{S}^2(\mathcal{M})$ between open sets such that:

(1) $0 \in U_0$, $p \in V_p$, $\phi_p(0) = p$ and both homeomorphisms ϕ_p and its inverse $(\phi_p)^{-1}$ are smooth maps of manifolds.
(2) The derivative of ϕ_p at $q \in U_0$ sends

$$\mathbb{R}^2 \ni (1,0) \longmapsto U(q) \quad \text{and} \quad \mathbb{R}^2 \ni (0,1) \longmapsto V(q).$$

These maps $\phi_p : U_0 \to V_p$ are called **local $\mathfrak{D}$–parametrizations** of $\mathcal{S}^2(\mathcal{M})$.

The $\mathfrak{D}$–parametrizations let us to give a local description of $(X_f)_t$ around many points $p \in \mathcal{S}^2(\mathcal{M})$ where $X_f(p) = 0$. In particular, the linearization of X_f at p is defined by the following matrix of the $\mathfrak{D}$–partial derivatives of second order:

$$\begin{pmatrix} f_{vu}(p) & f_{vv}(p) \\ -f_{uu}(p) & -f_{uv}(p) \end{pmatrix}. \tag{3.3}$$

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This is called the **$\mathfrak{D}$–linearization matrix** at p of X_f.

A point $p \in \mathcal{S}^2(\mathcal{M})$ with $X_f(p) = 0$ is said to be **simple** [17] when the respective $\mathfrak{D}$–linearization matrix does not have zero as an eigenvalue. The characteristic polynomial of (3.3) is $\lambda^2 - \mathcal{D}_s f(p)$, where

$$\mathcal{D}_s f(p) = (f_{uv}(p))^2 - f_{uu}(p) f_{vv}(p)$$

is the discriminant of f at p, we then have the following duality on simple zeroes: $p \in \mathcal{S}^2(\mathcal{M})$ is an *hyperbolic saddle* (resp. a *non–degenerate center*) of X_f provided that all the eigenvalues λ of (3.3) have non–zero (resp. zero) real parts. Therefore, on simple zeroes

- hyperbolic saddle of $X_f \Leftrightarrow \mathcal{D}_s f(p) > 0$, and
- non–degenerate center of $X_f \Leftrightarrow \mathcal{D}_s f(p) < 0$.

If $p \in \mathcal{S}^2(\mathcal{M})$ is an hyperbolic saddle of X_f, the eigenvalues satisfy $\lambda_s < 0 < \lambda_u$, and so one tangent direction contracts and the other one expands. Thus, p is a critical point of f which is not a local extremum. Moreover, the flow $(X_f)_t$ induces two immersed submanifolds ℓ^s and ℓ^u with $p \in \ell^s \cap \ell^u$ and $\ell^s \setminus \{p\}$ (resp. $\ell^u \setminus \{p\}$) is make up by regular points whose $\omega-$limit set (resp. $\alpha-$limit) is $\{p\}$. Similarly, when $p \in \mathcal{S}^2(\mathcal{M})$ is a non–degenerate center of X_f, both eigenvalues are non–zero purely imaginary numbers. So it is a local extremum of f, where $(X_f)_t$ induces a small open neighborhood given by periodic orbits enclosing the fixed point.

Definition 3.9. If $f \in \mathcal{F}^2(\mathcal{M})$, $H_f : \mathcal{M} \to \mathbb{R}^3$ is given by the product $X_{f_u} \times X_{f_v}$; that is,

$$H_f(p) = \left(\widehat{\frac{\partial^* f_u}{\partial u}}|_p - \widehat{\frac{\partial^* f_u}{\partial v}}|_p \right) \times \left(\widehat{\frac{\partial^* f_v}{\partial u}}|_p - \widehat{\frac{\partial^* f_v}{\partial v}}|_p \right),$$

for all $p \in \mathcal{M}$.

Observe that, $H_f \notin \mathfrak{X}^0(\mathcal{M})$, but $H_f(q) = 0$ for all $q \notin \mathcal{S}^2(\mathcal{M})$.

Proposition 3.10. *If $f \in \mathcal{F}^2(\mathcal{M})$, the following hold:*

(a) *The component of $H_f(q)$ in the direction of $U(q) \times V(q)$ is $-\mathcal{D}_s f(q)$. More precisely,*

$$H_f(q) = -\mathcal{D}_s f(q)\big[U(q) \times V(q)\big], \quad \forall q \in \mathcal{S}^2(\mathcal{M}),$$

where $\mathcal{D}_s f(q)$ is the discriminant.

(b) *A singularity $p \in \mathcal{S}^2(\mathcal{M})$ of X_f is simple if and only if $H_f(p) \neq 0$.*

(c) *The set of the* ***non–degenerate*** *critical points of f, defined by*

$$\big\{q \in \mathcal{C}r(f, \mathcal{M}) : H_f(q) \neq 0\big\}$$

is equal to the set of the simple singularities of X_f. Moreover, this is a discrete set on $\mathcal{S}^2(\mathcal{M})$ and its image under f is either finite or countable.

(d) *If $X_f(q) = 0$ and the set $\big\{U(q), H_f(q), V(q)\big\}$ is a positive (resp. negative) basis of $\mathbb{R}^3$, then q is an hyperbolic saddle (resp. a non–degenerate center).*

(e) *The hyperbolic saddles of X_f are also saddles of X^*_f*

Proof. By Definition 3.9, $H_f(q)$ is equal to $\left(\frac{\partial^* f_u}{\partial u}|_q - \frac{\partial^* f_u}{\partial v}|_q\right) \times \left(\frac{\partial^* f_v}{\partial u}|_q - \frac{\partial^* f_v}{\partial v}|_q\right)$, for every $q \in \mathcal{S}^2(\mathcal{M})$. A direct computation gives $H_f(q) = -\mathcal{D}_s f(q)\Big[U(q) \times V(q)\Big]$, where $\mathcal{D}_s f(q) = (f_{uv}(q))^2 - f_{uu}(q) f_{vv}(q)$. Therefore, (a) holds.

Given a singularity $p \in \mathcal{S}^2(\mathcal{M})$ of X_f the $\mathfrak{D}$–linearization matrix (3.3) is nonsingular if and only if $\mathcal{D}_s f(q)$ is different form zero. Therefore (a) implies (b).

The simple singularities are contained in $\mathcal{S}^2(\mathcal{M})$, this implies the first part of (c). In order to conclude, it is enough to show that a point point $p \in \mathcal{S}^2(\mathcal{M})$ with $X_f(p) = 0$ and $H_f(p) \neq 0$ is isolated in $\mathcal{S}^2(\mathcal{M})$. This is obtained by a direct application of Proposition 3.1 in the Chapter 2 of [17] which says that the simple singular points are isolated. Thus, (c) is true.

Claim that both conditions $X_f(q) = 0$ and $\big\{U(q), H_f(q), V(q)\big\}$ is a positive (resp. negative) basis are satisfied, imply that $q \in \mathcal{S}^2(\mathcal{M})$ is a simple zero and $\mathcal{D}_s f(p) > 0$ (resp. $\mathcal{D}_s f(p) < 0$). This is obtained from the definitions, because $H_f(q) \neq 0$ and $\mathcal{D}_s f(p)$ equals the determinant of the matrix representation of the change of basis from $\big\{U(q), H_f(q), V(q)\big\}$ to $\big\{U(q), V(q), U(q) \times V(q)\big\}$. Therefore, the duality on simples singularites gives (d).

The $\mathfrak{D}$–parametrization around a point $p \in \mathcal{S}^2(\mathcal{M})$, where $X^*_f(p) = 0$ let us obtain the $\mathfrak{D}$–linearization matrix of X^*_f at p :

$$\begin{pmatrix} f_{uu}(p) & f_{uv}(p) \\ f_{vu}(p) & f_{vv}(p) \end{pmatrix}, \tag{3.4}$$

whose polynomial characteristic is $\lambda^2 - (f_{uu}(p) + f_{vv}(p))\lambda - \mathcal{D}_s f(p)$. Thus, $p \in \mathcal{S}^2(\mathcal{M})$ is an hyperbolic saddle of X^*_f as long as $-\mathcal{D}_s f(p)$, the product of the eigenvalues of (3.4), is less than zero. Therefore, since the hyperbolic saddles of X_f satisfies $\mathcal{D}_s f(p > 0$, the statement (e) holds. □

Proposition 3.11. *If $r \geq 2$ is a positive integer and*

$$\mathfrak{M} = \big\{X_f \in \mathcal{H}^r(\mathcal{M}) : \textit{every zero of } X_f \textit{ is simple}\big\},$$

then $\mathfrak{M}$ is an open and dense subset of $\mathcal{H}^r(\mathcal{M})$.

Proof. If $X_f \in \mathfrak{M}$ we consider the restriction $X_f| \in \mathcal{X}^r(\mathcal{S}^2(\mathcal{M}))$ whose singularities are isolated as shown Proposition 3.10. By the proof of [14, Lemma 1] there is an open neighborhood of this restriction $X_f|$ on $\mathcal{X}^r(\mathcal{S}^2(\mathcal{M}))$ whose elements only have simple singular points. Therefore, we obtain the openness of $\mathfrak{M}$ by using Proposition 3.1.

The homeomorphism in Proposition 3.1 sends the following set of restrictions

$$\big\{X_f| \in \mathcal{X}^r(\mathcal{S}^2(\mathcal{M})) : X_f \in \mathfrak{M}\big\} \tag{3.5}$$

on $\mathfrak{M} \subset \mathcal{H}^r(\mathcal{M})$. For every $X_f| : \mathcal{S}^2(\mathcal{M}) \to \mathbb{R}^3$, which is a C^r–section on $\mathcal{S}^2(\mathcal{M})$, we can apply Proposition 3.6 so $X_f|$ is Hamiltonian. As a consequence, the proof of Lemma 1 in [14] can be used and then the set in (3.5) is dense on $\mathcal{X}^r(\mathcal{S}^2(\mathcal{M}))$. Therefore, Proposition 3.1 shown that $\mathfrak{M}$ is dense on $\mathcal{H}^r(\mathcal{M})$, and conclude. □

The following definition will be needed.

Definition 3.12 (Topologically Simple). Given a flow X_t and an equilibrium point $p \in \mathcal{M}$, we will say that p is *topologically simple* if there exists some Hamiltonian vector field X_g which has a simple singularity at $q \in \mathcal{M}$ joint to two neighborhoods $U \subset \mathcal{M}$ of p, and $V \subset \mathcal{M}$ of q with $U \backslash \{p\}$ free of singularities such that the restriction flows $X_t|_U$ and $(X_g)_t|_V$ are *topologically equivalent*: there is some homeomorphism $h\colon U \to V$ taking directed X_t-trajectories in U to directed $(X_g)_t-$trajectories in V.

The topologically simple fixed–points are isolated and they remain topologically simple under any topological equivalence. Certainly, any degenerate center in the plane is a topologically simple fixed-point, but it is not simple [1, 4].

4. Stability in $\mathcal{H}^r(\mathcal{M})$.

We start with the study of the behavior at infinity of $(X_f)_t$ with $X_f(p) = f_v(p)U(p) - f_u(p)V(p)$, where $U(x,0,z) = (1,0,0)$ joint to $V(x,0,z) = (0,0,1)$ define a System of Directions on the plane. To this end, we take the orthogonal vector field $X^* = f_u(p)U(p) + f_v(p)V(p) \in \mathfrak{X}^r(P)$ and consider the region $S_h = S(p_1,p_2;q_1,q_2,\{\sigma_i\}) \subset P$ whose boundary made up of two unbounded semi–trajectories $[q_1,\infty)$ and $(\infty,q_2]$ of X_f, a compact arc of trajectory $[p_1,p_2]$ of X_f, two arcs of trajectory $[p_1,q_1]^*$, $[p_2,q_2]^*$ of X^*, and a set at most countable (which may be empty) of pairwise disjoint trajectories $\{\sigma_i\}$ that start and end to infinity.

We call such a region a **Hyperbolic Sector at Infinity** of X_f if the following conditions are satisfied:

(1) for each $z \in [p_1,q_1)^*$, there exists an arc of trajectory $[z,\pi(z)) \subset S_h$ of $(X_f)_t$ starting at $z \in [p_1,q_1)^*$ and ending at $\pi(z) \in [p_2,q_2]^*$; and
(2) the closure $\overline{\bigcup_{z\in[p_1,q_1)^*} [z,\pi(z)]}$ is all S_h.

In a sector at infinity $S_h = S(p_1,p_2;q_1,q_2,\{\sigma_i\})$ we do not have to assume that $[q_1,\infty)$ and $(\infty,q_2]$ are contained on different trajectories. The case $\{(X_f)_t(q_1) : t \in I_{q_1}\} = \{(X_f)_t(q_2) : t \in I_{q_2}\}$ is allowed (homoclinic contour at infinity).

This concept of hyperbolic sector at infinity, communicated to us by Carlos Gutiérrez, has previously been used in our papers [6, 18, 19], in the first one it is called pseudo–hyperbolic sector at infinity.

Definition 4.1 (Saddle at Infinity). Given $X \in \mathfrak{X}^r(\mathcal{M})$, we say that the flow X_t support a *Saddle at Infinity* if there exist (X_f,S_h,H) where S_h is a hyperbolic sector at infinity of $X_f \in \mathfrak{X}^r(P)$ and $H : S_h \to H(S_h)$ is a homeomorphism such that:

(1) $H(S_h) \subset \mathcal{S}^2(\mathcal{M})$ and $H : S_h \to H(S_h)$ is a topological equivalence of the restriction flows $(X_f)_t|S_h$ and $X_t|H(S_h)$.
(2) The segments $\Sigma_1 = H([p_1, q_1]^*)$ and $\Sigma_2 = H([p_2, q_2]^*)$ are contained in $\mathcal{S}^2(\mathcal{M})$ and they are transversal to X.
(3) Both trajectories $H(q_1, \infty)$ and $H(\infty, q_2)$ are unbounded subsets of $\mathcal{S}^2(\mathcal{M})$.

The region $H(S_h)$ has no singularities of X and then we called it the **hyperbolic sector** associated to the saddle at infinity. Moreover, $H(q_1, \infty)$ (resp. $H(\infty, q_2)$) is called the stable (resp. unstable) *separatrix* of the saddle at infinity.

In this definition the existence of (X_f, S_h, H) with $X \in \mathfrak{X}^r(\mathcal{M})$ naturally induces a saddle at infinity supported by the reverse $-X \in \mathfrak{X}^r(\mathcal{M})$. As a consequence, if $\mathcal{M} = P$ Definition 4.1 coincides with the Saddle at Infinity given in [14].

In the particular case of Hamiltonian flows on the plane $(X_f)_t$ with $\mathcal{C}r(f, \mathcal{M}) = \emptyset$, we can study the regular foliation defined by the level sets $\mathcal{L}_c(f, P)$, as in (3.2). Here, the separatrices of a saddle at infinity are contained on two *inseparable leaves* [5]. Moreover, in the proof of Proposition 1.4 of [3] we establish that the inseparable leaves induces a half–Reeb component which are examples of pseudo–hyperbolic sectors at infinity. Therefore, the existence of inseparable leaves is equivalent to the existence of a saddle at infinity.

Proposition 4.2. *A vector field $X \in \mathcal{X}^r(\mathcal{M})$ support at most countably many saddles at infinity.*

Proof. If $\mathcal{M} = P$ we refer the reader to Proposition 1 of [14]. If $\mathcal{M} = D$ it is enough to observe that for every saddle at infinity, its hyperbolic sector at infinity is contained in some connected component of $\mathcal{S}^2(D)$. Therefore, in this second case we can adapt the ideas of [14], and obtain this proposition if $\mathcal{M} = D$.

The hyperbolic sector associated to a saddle at infinity does not contain zeroes of X, hence we can suppose that the induced flow X_t has no fixed points, and to proceed by using the topological circles C_s which are nonzero homotopic. More precisely, if $\mathcal{M} \in \{C, A\}$ and $n \in \mathbb{N}$ we select some $C_n \subset \mathcal{M}$ with the *minimal* number of tangencies to X_t such that its projection $|\Pi_3(C_n)| \geq n$, where $\Pi_3(x, y, z) = z$. For each saddle at infinity there is some C_n where such a saddle induces tangent points in the set $\{\text{tangent points of } C_n : n \in \mathbb{Z}\}$ which is at most countable. Since the correspondence Saddles at Infinty $\to$ $\{\text{tangency with a fixed } C_n\}$ is injective we conclude the proof for any $\mathcal{M}$. □

Remark 4.3. By using a smooth embedding $(f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$, the authors of [5] prove in their Proposition 1 the existence of Hamiltonian flows $(X_f)_t$ and $(X_g)_t$ which have infinitely (countably) many saddles at infinity.

Associated to any saddle – "finite saddle" as the hyperbolic saddle, or saddle at infinity – there are their separatrices, so a **saddle connection** between two saddles (finite of infinity, and not necessarily distinct) is a *trajectory* which is simultaneously a stable separatrix of one saddle and an unstable separatrix of the other. If the saddles are distinct the saddle connection will be called **proper**. The non–proper saddle connections are the **homoclinic contour**: a saddle connection between a finite saddle and itself, and the **homoclinic contour at infinity**: an unbounded trajectory, contained in the boundary of a hyperbolic sector associated to a saddle at infinity such that it includes both separatrices of the saddle at infinity.

Remark 4.4. In the special case of saddle connections of the Hamiltonian flows $(X_f)_t$, the part (a) in Proposition 3.6 shows that the function f is constant along the trajectories. Therefore, for any separatrix ℓ_p on the first saddle p and any separatrix ℓ_q on the other saddle connected to p, we have $f(\ell_p) = f(\ell_q)$. Furthermore, the homoclinic contour for a Hamiltonian flow is a limit of closed orbits.

Note that for every saddle of $(X_f)_t$ (finite or infinity, and keep fixed), f is constant on the respective separatrices. In particular, f is constant on any homoclinic contour at infinity.

Proposition 4.5. *If $r \geq 2$ is a positive integer, $\mathfrak{M} \subset \mathcal{H}^r(\mathcal{M})$ as in Proposition 3.11 and*

$$\widetilde{\mathfrak{M}} = \big\{X_f \in \mathfrak{M} : (X_f)_t \text{ has no proper saddle connections}\big\},$$

where the connections might be finite or infinity, then $\widetilde{\mathfrak{M}}$ is a residual set of $\mathcal{H}^r(\mathcal{M})$.

Proof. By Remark 4.4 a proper saddle connection of X_f gives a non–degenerate saddle $p \in \mathcal{S}^2(\mathcal{M})$ such that $f(p) = f(\ell_q)$, for any separatrix ℓ_q of the other saddle $q \notin \mathcal{S}^1(\mathcal{M}) \cup \mathcal{S}^0(\mathcal{M})$. Thus, we can add a small bump function near p to separate the value on these saddles. Therefore, given any pair of non–degenerated critical points we can make a local perturbation at one of them that ensures these two critical points belong to different level sets, and certainly every function near this new one also assigns them different values.

The Baire category argument described in the proof of the Lemma 2 in [14] can be used on each connected component of $\mathcal{S}^2(\mathcal{M})$ and then we conclude by a direct application of Proposition 3.1. □

Since the critical values might be give a dense set, and it is possible to change, by a local perturbation, the value of a function of any specified critical point to any nearby value, it follows that $\widetilde{\mathfrak{M}}$ is not an open set.

4.1. **Stability in the plane.** We start with the main result about Hamiltonian stability in the plane, this is due to Jarque and Nitecky [14].

Theorem 4.6 (Jarque–Nitecky). *For $r \geq 1$, suppose $f \in \mathcal{F}^{r+1}(P)$. Then the Hamiltonian flow $(X_f)_t$ with $X_f \in \mathcal{H}^r(P)$ is Hamiltonian C^r–stable if and only if:*

1. *every equilibrium of $(X_f)_t$ is either a hyperbolic saddle or a non–degenerate center,*
2. *a separatrix of a finite saddle is isolated from the separatrices of all other finite of infinite saddles for $(X_f)_t$.*

The second item means that for each separatrix ℓ_q of a saddle q there exists V_q, an **isolated neighborhood**, this is an open neighborhood of ℓ_q where q is the unique equilibrium point such that

$$V_q \setminus \{\ell : \ell \text{ is a separatrix of } q\}$$

is disjoint from all the separatrices. In particular, it prevents the existence of proper saddle connections at finite saddles. Moreover, the component of $V_q \setminus V_q'$, where $V_q' \subset V_q$ is a special compact neighborhood of q can be obtained as the saturation of a small transversal segment.

[...]

4.1.1. *Hamiltonian vector fields with an invariant line.* We consider the vector fields whose flows have an invariant line. By using the vertical axis we define the topological subspace of $\mathcal{H}^r(P)$ given by the following set

$$H_z^r(P) = \{X \in \mathcal{H}^r(P) : X(z-\text{axis}) \subset z-\text{axis}\}, \tag{4.1}$$

which is particularly contained in the set of all continuous vector fields of $\mathbb{R}^3$ and then we are able to adapting Definition 3.5. Therefore, a vector field $X_f \in H_z^r(P)$ is said to be $H_z^r(P)-$**stable** if given $\mathcal{N}_{co}(K, U; P)$, there is $\mathcal{N}_\varepsilon^r(X_f, P)$ such that every flow Y_t with $Y \in \mathcal{N}_\varepsilon^r(X_f, P) \cap H_z^r(P)$ is topologically equivalent to $(X_f)_t$ and its equivalence $h \in \mathcal{N}_{co}(K, U; \mathcal{M})$.

The topology in $H_z^r(P)$ is induced from $\mathcal{H}^r(P)$, hence its elements which also satisfy both conditions of Theorem 4.6 provide many examples of $H_z^r(P)-$stable vector fields. The next proposition is a direct applications of the methods of [14], and we include the proof for the sake of completeness.

Proposition 4.7. *Suppose that X_f is $H_z^r(P)-$stable then,*

(a) *The flow $(X_f)_t$ only has topologically simple equilibrium points on $P \setminus (z-axis)$.*
(b) *For each separatrix of a saddle $q \in P \setminus (z-axis)$ there exists an open isolated neighborhood $V_q \subset P \setminus (z-axis)$.*
(c) *The boundary of a hyperbolic sector at infinity $S_h \subset P \setminus (z-axis)$ is either connected (homoclinic contour at infinity) or it has infinitely many components.*

Proof. Consider some neighborhood $\mathcal{N}^r_\varepsilon(X_f,P)$ obtained from the definition of $H^r_z(P)-$stable. By the proof of Proposition 3.11 there exist some vector field $Y \in \mathcal{N}^r_\varepsilon(X_f,P)\cap H^r_z(P)$ with simple singularities and then the flows $(X_f)_t$ and Y_t are topologically equivalent. Therefore, Definition 3.12 shows that $(X_f)_t$ only has topologically simple fixed–points, on $P\setminus(z-\text{axis})$.

On the similar way, by using Proposition 4.5 it is possible to show that X_f has no proper connections on the complement of the vertical axis. However.

(b.1) We claim, the flow $(X_f)_t$ has no a saddle connection induced by a saddle $q\in P\setminus(z-\text{axis})$.

In fact, given $q\in P\setminus(z-\text{axis})$, a saddle of X_f we keep fixed p in some separatrix ℓ_q of q and proceed as in [14]. We consider $U\subset P\setminus(z-\text{axis})$ a neighborhood of q containing no other zeroes of X_f, bounded by transversals to four separatrices of q and trajectory segments joint their endpoints. We can assume that one of these transversals goes through $p\in\ell_q$, and take V and W two compact neighborhoods of q such that

$$V\subset \operatorname{Int}(W) \quad \text{and} \quad W\subset \operatorname{Int}(U).$$

By narrowing $\mathcal{N}^r_\varepsilon(X_f,P)\cap H^r_z(P)$ we can found a constant $\alpha_0>0$ such that for every value $0<\delta_0<\alpha_0$ the smooth function $\delta\colon P\to\mathbb{R}$ with

$$\delta=0 \text{ outside } W,\quad 0\le\delta\le\delta_0 \text{ on } W,\quad \delta=\delta_0 \text{ on } V,$$

defines a perturbation

$$g(x,0,z)=f(x,0,z)+\delta(x,0,z)$$

which has no other critical points in U and $X_g\in\mathcal{N}^r_\varepsilon(X_f,P)\cap H^r_z(P)$. Moreover, $g(q)=f(\ell_q)+\delta_0$, q is a saddle for X_g whose separatrices leave U by points into four separatrices where $f=f(\ell_q)+\delta_0$ and $X_g=X_f$ outside W.

By Proposition 4.2, $\mathcal{V}=\big\{f(\ell)\in\mathbb{R} : \ell \text{ is a } (X_f)_t\text{–separatrix, finite or infinity}\big\}$ is at most countable. Thus, for some $0<\delta_0<\alpha_0$ the constant $f(\ell_q)+\delta_0=c+\delta_0\notin\mathcal{V}$ and then X_g has no a saddle connection at q (finite or infinity), because a $(X_g)_t-$separatrix of q intersects $P\setminus W$ in the level set $P\{f=c+\delta_0\}$. In this context, if $\mathcal{N}_{co}(K,U;P)$, a compact–open neighborhood of the identity on P, is small enough the equivalence $h\in\mathcal{N}_{co}(K,U;P)$ between $(X_f)_t$ and $(X_g)_t$ satisfies $h(q)=q$. Therefore, $(X_f)_t$ has no a saddle connections at q and (b.1) holds.

(b.2) We claim, a separatrix of the finite saddle $q\in P\setminus(z-\text{axis})$ is isolated from the separatrices in $P\setminus(z-\text{axis})$ of all other finite of infinite saddles.

On the contrary, some saddle has a separatrix ℓ – either stable or unstable, with different type of ℓ_q, some separatrix of q – such that $\delta_0=f(\ell)-f(\ell_q)\in(-\alpha_0,\alpha_0)\setminus\{0\}$ and both functions $f+\delta$ and $f-\delta$ induce hamiltonian vector fields which belong to the neighborhood $\mathcal{N}^r_\varepsilon(X_f,P)\cap H^r_z(P)$ described above. If $\delta_0>0$ (resp. $\delta_0<0$) we can select $g=f+\delta$ (resp. $g=f-\delta$) and then obtain that $(X_g)_t$ has a saddle connection between q and the saddle in ℓ. Thus $(X_f)_t$ has a saddle connection. This contradiction with (b.1) concludes the proof of (b.2), and (b) is true.

In order to prove (c) consider $S_h \subset P \setminus (z-\text{axis})$ an hyperbolic sector at infinity whose boundary has at least two components. By using the proof of (b) this system admits a local perturbation with a perturbed sector at infinity which lost a component in its boundary, and so this might not be topologically equivalent to the original one. Therefore, a $H^r_z(P)-$stable system satisfies (c). This concludes the proof. □

Remark 4.8. Notice that, the perturbation argument described in the proof of (b) in Proposition 4.7 does not work if $q \in (z-\text{axis}) \cap V \neq \emptyset$, because $X_g \in H^r_z(P)$ implies that g is constant on $z-$ axis. However, in the particular case that $\varepsilon(x,0,z) \geq \varepsilon_0 > 0$ for some neighborhood $\mathcal{N}^r_\varepsilon(X_f,P)$ obtained from the definition of $H^r_z(P)-$stable, all the equilibrium points of the flow $(X_f)_t$ are isolated. It is true, because the continuous existence of equilibrium points on the vertical axis of P prevents the equivalence between $(X_f)_t$ and $(X_g)_t$ with $g(x,0,z) = f(x,0,z) + \frac{\varepsilon_0}{4}x$ even when $X_g \in \mathcal{N}^r_\varepsilon(X_f,P) \cap H^r_z(P)$.

Theorem 4.9. *A vector field $X_f \in H^r_z(P)$, with $r \geq 1$ is $H^r_z(P)-$stable if the following conditions hold.*

(a) *The flow $(X_f)_t$ only has topologically simple singular point on $P\setminus(z-\text{axis})$.*
(b) *For each separatrix ℓ_q, with $q \in P$ it satisfies one of the following conditions:*
   - *ℓ_q is isolated from the separatrices of all other finite of infinite saddles.*
   - *ℓ_q is unbounded and admits an open isolated neighborhood $\ell_q \subset V_q \subset P$ with $q \in z-\text{axis}$.*

(c) *If either the $z-$ axis has no isolated singularities or the $z-$ axis is free of singuarities, then this vertical axis admits a open neighborhood obtained as the saturation of some small transversal open segment.*

Proof. This is obtained as a direct application of [14] and so, we only give the main ideas of the proof.

(a.1) We assume that statement (c) never hold, so the $z-$ axis has at most a saddle point.

Under these conditions, it is possible to have open disjoint neighborhoods $V(p) \subset P$ by selecting the connected component that contain a critical point $p \in P$ of $f^{-1}(c_-,c_+)$, where (c_-,c_+) is some interval with $c_-(p) = c_- < f(p) < c_+ = c_+(p)$. Thus, when (c_-,c_+) is small enough this $V(p)$ contains no other critical points of f and $V(p) \setminus \{\ell : \ell \text{ is a separatrix of } p\}$ is disjoint from all the separatrices. In particular, if $p \notin (z-\text{axis})$ then $V(p) \subset P \setminus (z-\text{axis})$. The boundary of $V(p)$ is either compact or unbounded (see for instance a center). In the last case, the existence of isolated neighborhoods in P, given by (b) implies that the components of $V(p)\setminus V'(p)$, where $V'(p) \subset V(p)$ is a small neighborhood of p, can be obtained as the saturation of a respective small transversal segment. From these the ideas of [14, Lemma 7] let us to foliate the set of regular points by curves T transverse to X_f so that, whenever $x \in V(p) \setminus \{p\}$ for some critical point p, the curve T_x thorough x intersects the boundary of $V(p)$ in at least one point $\sigma_\pm(x) \in \mathcal{L}_{c\pm}(f, P)$, the level set as in (3.2), and if not at two, then the other end of T_x is p. Consequently, the proof of [14, Proposition 4] shows that

(a.2) Given $K \subset P$ compact and $\delta > 0$, there exist $\varepsilon : P \to (0, +\infty)$ such that for every every flow Y_t with $Y \in \mathcal{N}^r_\varepsilon(X_f, P) \cap H^r_z(P)$ there is some homeomorphism $h : P \to P$ mapping directed $(X_f)_t$−trajectories to directed $(Y)_t$−trajectories with $|h(x) - (x)| < \delta$ whenever $x \in K$.

Therefore, this theorem holds in the case (a.1).

(b.1) When (c) holds then we consider V_z the open neighborhood of the $z-$axis obtained as the saturation of some small transversal open segment.

In this new case, the complement set $P \setminus V_z$ has the same decomposition described in the first part and then it is possible to obtain a similar result as in (a.2) Therefore, this theorem is true. □

4.2. **Consequences of Hamiltonian stability.** In this subsection we present some necessary conditions in order to have the general Hamiltonian stability on $\mathcal{M}$. To this end the following definition will be needed.

Definition 4.10. We say that the separatrix ℓ_q of a saddle q is *isolated* if it admits an open isolated neighborhood. This is an open neighborhood of ℓ_q where q is the unique equilibrium point such that

$$V_q \setminus \big\{\ell : \ell \text{ is a separatrix of } q\big\}$$

is disjoint from all the separatrices.

Theorem 4.11. *Suppose that the Hamiltonian flow $(X_f)_t$ with $X_f \in \mathcal{H}^r(\mathcal{M})$ is* Hamiltonian C^r–stable *then,*

(a) *The flow $(X_f)_t$ only has topologically simple equilibrium points on the respective smooth part, $\mathcal{S}^1(\mathcal{M}) \sqcup \mathcal{S}^2(\mathcal{M})$.*
(b) *For each separatrix of a saddle $q \in \mathcal{S}^2(\mathcal{M})$ there exists an open isolated neighborhood $V_q \subset \mathcal{S}^2(\mathcal{M})$. Moreover, the vertex cone does not admit a saddle connection.*
(c) *The boundary of a hyperbolic sector associated to a saddle at infinity is either connected (homoclinic contour at infinity) or it has infinitely many components.*

Proof. The statement (a) has been proved by [14], in the special case of the plane, $\mathcal{M} = P$. That proof can be adapted directly to the cases of $\mathcal{M} \in \{C, A\}$, where $\mathcal{S}^1(\mathcal{M})$ is the empty set. Similarly, Theorem 4.9 gives (a) in the case $\mathcal{M} = D$, because every Hamiltonian flow on D induces some planar Hamiltonian flows with an invariant line. Therefore, (a) holds.

Theorem 4.6 gives (b) if $\mathcal{M} = P$. Analogously, a direct application of Theorem 4.9 gives (b) if $\mathcal{M} = D$. Since the Hamiltonian flow on A has neither limit cycles or saddle connections, it is not difficult to prove this statement (b) in the case $\mathcal{M} = A$. Similarly, for each separatrix of a saddle $q \in \mathcal{S}^2(C)$ there exists an open isolated neighborhood $V_q \subset \mathcal{S}^2(C)$. To obtain the second part of (b) we claim that

(b.1) The Hamiltonian flow on the cone, $(X_f)_t$ has no a saddle connection on the vertex.

In fact, to prove (b.1) we proceed by contradiction. We assume the existence of a saddle connection on the vertex, and also that this vertex has a separatrix $\ell_0 \subset C^+ = \{(x,y,z) \in C : z > 0\} \setminus \{0\}$. Thus, the Hamiltonian flow $(X_f)_t$ on the Cone joint to the diffeomorphism $\pi\colon C^+ \cup \{0\} \to \tilde{P} = \{z = 0\}$, $\pi(x,y,z) = (x,y,0)$ naturally defines a unique Hamiltonian flow on the plane $\tilde{P}$ induced by the map $\tilde{f}\colon \tilde{P} \to \mathbb{R}$ defined by the equality $\tilde{f}(x,y,0) = f\big(x,y,\sqrt{x^2+y^2}\big)$. In this context, it is not difficult to see that ℓ_0 gives a saddle connection on the Hamiltonian flow $(X_{\tilde{f}})_t$ and also that the map

$$\mathcal{H}^r(C) \ni X_f \longmapsto X_{\tilde{f}} \in \mathcal{H}^r(\tilde{P})$$

preservers the Hamiltonian stability on the spaces. But, Theorem 4.6 implies that $(X_{\tilde{f}})_t$ does not admit a saddle connection. This contradiction show that (b.1) holds. Therefore, statement (b) is true.

In order to prove (c) consider $H(S_h) \subset \mathcal{S}^2(\mathcal{M})$ an hyperbolic sector associated to a saddle at infinity whose boundary has at least two components. By using the proof of Proposition 4.7 this system admits a local perturbation with a perturbed sector at infinity which lost a component in its boundary, and so this might not be topologically equivalent to the original one. Therefore, a Hamiltonian C^r–stable system satisfies (c). This concludes the proof. ☐

5. The Cone and the Open Infinity Cylinder

In this subsection we present sufficient conditions under which the Hamiltonian stability holds, according with the Definition 3.5, but in the particular case of the cone C and the cylinder A. This is complemented in Section 6, where we consider the Double Crossing Plane.

Theorem 5.1. *Consider a Hamiltonian vector field $X_f \in \mathcal{H}^r(\mathcal{M})$, with $r \geq 1$ and $\mathcal{M} \in \{C, A\}$. The induced Hamiltonian flow $(X_f)_t$ is* Hamiltonian C^r–stable *if the following conditions hold.*

(a) *The flow $(X_f)_t$ only has topologically simple singular point on $\mathcal{S}^2(\mathcal{M})$.*
(b) *For each separatrix ℓ_q, with $q \in \mathcal{M}$ it satisfies one of the following conditions:*
  - *ℓ_q is isolated from the separatrices of all other finite of infinite saddles.*

- ℓ_q *is unbounded and admits an open isolated neighborhood* $\ell_q \subset V_q \subset C$.

Proof. In the case that $X_f \in \mathcal{H}^r(A)$, we observe that the Hamiltonian flow $(X_f)_t$ has no limit cycles. Thus, by using the assumption (a) and the first part of (b) it is not difficult to see that for every singularity p it is possible to select some open neighbourhood $V(p) \subset A$ such that $p \neq q$ implies that $V(p) \cap V(q) = \emptyset$ and $V(p) \setminus \{\ell : \ell \text{ is a separatrix of } p\}$ is disjoint from all the separatrices. Consequently, the proof of [14, Proposition 4] shows that

(a) Given $K \subset A$ compact and $\delta > 0$, there exist $\varepsilon : A \to (0, +\infty)$ such that for every every flow Y_t with $Y \in \mathcal{N}^r_\varepsilon(X_f, A) \cap \mathcal{H}^r(A)$ there is some homeomorphism $h\colon A \to A$ mapping directed $(X_f)_t$–trajectories to directed $(Y)_t$–trajectories with $|h(x) - (x)| < \delta$ whenever $x \in K$.

Therefore, the Hamiltonian flow $(X_f)_t$ is *Hamiltonian C^r–stable* in the cylinder A in agreement to Definition 3.5.

In the case $X_f \in \mathcal{H}^r(C)$, we have the extra condition given bye the second part of (b). Consequently, $C \setminus V_q$ has a similar decomposition on isolated neighbourhoods, and it is possible to show that the Hamiltonian flow $(X_f)_t$ is *Hamiltonian C^r–stable* in the cylinder C. □

6. The Double Crossing Plane.

The Double Crossing Plane is the open singular surface defined as $D = \{(x, y, z) \in \mathbb{R}^3 : xy = 0\}$. We consider, on D, the System of Directions $\overline{\mathfrak{D}} = (\overline{U}, \overline{V})$ given by

$$\overline{U}(x, y, z) = \begin{cases} \big(|x|, 0, 0\big) & y = 0; \\ \big(0, |y|, 0\big) & x = 0; \end{cases} \quad \text{and} \quad \overline{V}(x, y, z) = (0, 0, 1). \tag{6.1}$$

On a similar way, on the Plane $P = \{(x, y, z) \in \mathbb{R}^3 : y = 0\}$ the Tangent Vector Fields $(x, 0, z) \mapsto (1, 0, 0)$ and $(x, 0, z) \mapsto (0, 0, 1)$ define a System of Directions.

Lemma 6.1. *Given $X \in \mathfrak{X}^r(D)$, for each connected component $S \subset \mathcal{S}^2(D)$ we then have:*

(a) *The restriction $X|\colon S \to \mathbb{R}^3$ defines a section on S.*
(b) *There exist a diffeomorphism $H_S\colon P \to S$ and a vector field $X_S \in \mathfrak{X}^r(P)$ such that H_S sends trajectories of X_S onto trajectories of $X|\colon S \to \mathbb{R}^3$ preserving the orientations.*
(c) *The pair $(\overline{U}_S, \overline{V}_S)$, induced by (6.1), is the System of Directions on P; more precisely, $\overline{U}_S(x, 0, z) = (1, 0, 0)$ and $\overline{V}_S(x, 0, z) = (0, 0, 1)$.*

Proof. Statement (a) is directly obtained from the definition of Tangent Vector Field on D, see § 2. Therefore, (a) is true.

The following notations of the connected components of $\mathcal{S}^2(D)$ will be needed:

$$
\begin{aligned}
&S_x^+ = \big\{(x,y,z) \in D : x > 0\big\}, && S_x^- = \big\{(x,y,z) \in D : x < 0\big\},\\
&S_y^+ = \big\{(x,y,z) \in D : y > 0\big\} \quad \text{and} && S_y^- = \big\{(x,y,z) \in D : y < 0\big\}.
\end{aligned} \tag{6.2}
$$

To prove (b) we consider two smooth diffeomorphisms $\alpha : \mathbb{R} \to (0,+\infty)$ and $\widetilde{\alpha} : \mathbb{R} \to (-\infty, 0)$ whose derivative satisfy $\alpha'(t) > 0$ and $\widetilde{\alpha}'(t) > 0$, for all $t \in \mathbb{R}$. Under these conditions, we take into account the mappings given by the following rules:

$$
\begin{aligned}
&H_x^+(x,0,z) = \big(\alpha(x),0,z\big) \in S_x^+, && H_x^-(x,0,z) = \big(\widetilde{\alpha}(x),0,z\big) \in S_x^-,\\
&H_y^+(x,0,z) = \big(0,\alpha(x),z\big) \in S_y^+ \quad \text{and} && H_y^-(x,0,z) = \big(0,\widetilde{\alpha}(x),z\big) \in S_y^-.
\end{aligned} \tag{6.3}
$$

Furthermore, for each $H \in \{H_x^+, H_x^-, H_y^+, H_y^-\}$, $S \in \{S_x^+, S_x^-, S_y^+, S_y^-\}$ and $p \in S$, we define $X_S(p)$ as the pullback of $X(H(p))$ by H. Thus, the derivative of H at $q \in P$ sends $X_S(q)$ on $X(H(q))$. Since H is a diffeomorphism, (b) holds.

In order to prove (c), we consider the diffeomorphism $H \in \{H_x^+, H_x^-, H_y^+, H_y^-\}$ above, and also (6.1). By a direct computation of the derivative of H at $q \in P$ and (b), we obtain that

$$\overline{U}_S(x,0,z) = (1,0,0) \quad \text{and} \quad \overline{V}_S(x,0,z) = (0,0,1).$$

This shows (c) and concludes the proof. □

Corollary 6.2. *If $X \in \mathfrak{X}^r(D)$ is Hamiltonian and $S \subset \mathcal{S}^2(D)$ is a connected component, then the respective X_S is also Hamiltonian, that is $X_S \in \mathcal{H}^r(P)$.*

Proof. Since $X \in \mathcal{H}^r(D)$, there is $f \in \mathcal{F}^{r+1}(D)$ such that

$$X(p) = \frac{\partial f}{\partial^* \overline{u}}|_p - \frac{\partial^* f}{\partial \overline{v}}|_p, \quad \forall p \in S,$$

where in the right–hand side, the vector fields are respect to (6.1). Therefore, the function $f \circ H$, where $H : P \to D$ is the respective diffeomorphism given in Lemma 6.1 also belongs to $\mathcal{F}^{r+1}(D)$.

To proceed we suppose that $S = S_x^+$ and $H = H_x^+$, (resp. $S = S_x^-$ and $H = H_x^-$) see (6.2) and (6.3).

In this context, it is not difficult to prove the existence of a smooth function $\beta \colon \mathbb{R} \to \mathbb{R}$ (resp. $\widetilde{\beta} \colon \mathbb{R} \to \mathbb{R}$) such that the function $g \colon P \to \mathbb{R}$ given by $g(x,0,z) = f \circ H_x^+\big(\beta(x),0,z\big)$ (resp. $g(x,0,z) = f \circ H_x^-\big(\widetilde{\beta}(x),0,z\big)$) belongs to $\mathcal{F}^{r+1}(P)$ and satisfies

$$\frac{\partial^* g}{\partial u}|_p - \frac{\partial^* g}{\partial v}|_p = X_S(q), \quad \forall q \in P,$$

where the $\mathfrak{D}$–partial derivatives are given by the system $(\overline{U}_S, \overline{V}_S)$ as in the part (c) of Lemma 6.1. This gives the corollary. □

Corollary 6.3. *For each connected component $S \subset \mathcal{S}^2(D)$, the following mapping, introduced in Corollary 6.2,*

$$\mathcal{H}^r(D) \ni X \longmapsto X_S \in \mathcal{H}^r(P)$$

is continuous.

Proof. We only proof the case $S = S_x^+$ of (6.2). Therefore, the map is given by

$$\mathcal{H}^r(D) \ni X_f \longmapsto X_g \in \mathcal{H}^r(P),$$

where $g(x,0,z) = f \circ H_x^+\big(\alpha(x),0,z\big)$ and H_x^+ are as in (6.3).

Given $\varepsilon : \mathcal{M} \to (0,+\infty)$ a positive function, we consider the basic neighborhood $\mathcal{N}_\varepsilon^r(f,\mathcal{M})$ defined in (2.4). By using the definition of H_x^+ it is not difficult to show that

$$\hat{f} \in \mathcal{N}_\varepsilon^r(f,D) \Longrightarrow \hat{g} \in \mathcal{N}_\varepsilon^r(f,P),$$

where $\hat{g}(x,0,z) = \hat{f} \circ H_x^+\big(\hat{\alpha}(x),0,z\big)$. This proves the corollary. □

In the definition of the next proposition we do not assume that the topological equivalence belongs to a compact–open neighborhood of the identity.

Theorem 6.4. *Suppose that $X \in \mathcal{H}^r(D)$ and that $X_S \in \mathcal{H}^r(P)$ is Hamiltonian C^r stable on $\mathcal{H}^r(P)$, for each connected component $S \subset \mathcal{S}^2(D)$. Then, $X \in \mathcal{H}^r(D)$ is **Weak Hamiltonian** C^r **stable** on $\mathcal{H}^r(D)$: there exists a basic neighborhood $\mathcal{N}_\varepsilon^r(X,D)$ of X such that every Hamiltonian flow Y_t with $Y \in \mathcal{N}_\varepsilon^r(X,D) \cap \mathcal{H}^r(D)$ is dynamically equivalent to X_t.*

Proof. Since X_S is is Hamiltonian C^r stable, there is $\widetilde{\varepsilon} = \widetilde{\varepsilon}(S) : \mathcal{M} \to (0,+\infty)$ a positive function such that the condition: $Z \in \mathcal{N}_{\widetilde{\varepsilon}}^r(X_S,P) \cap \mathcal{H}^r(P)$ implies that Z_t is dynamically equivalent to $(X_S)_t$.

By Corollary 6.3 the four sets $\{Y \in \mathcal{H}^r(D) : Y_S \in \mathcal{N}_{\widetilde{\varepsilon}}^r(X_S,P) \cap \mathcal{H}^r(P)\}$ are open. Thus, the intersection

$$\bigcap_S \big\{Y \in \mathcal{H}^r(D) : Y_S \in \mathcal{N}_{\widetilde{\varepsilon}}^r(X_S,P) \cap \mathcal{H}^r(P)\big\}$$

is a neighborhood of X in $\mathcal{H}^r(D)$. Therefore, there exists a basic neighborhood $\mathcal{N}_\varepsilon^r(X,D)$ of X which satisfies this proposition. This concludes the proof. □

The definition of Weak Hamiltonian C^r stable is naturally extended to any $\mathcal{M}$.

Corollary 6.5. *If $X \in \mathcal{H}^r(D)$ and $X_S \in \mathcal{H}^r(P)$ is Weak Hamiltonian C^r stable on $\mathcal{H}^r(P)$, for each connected component $S \subset \mathcal{S}^2(D)$, then $X \in \mathcal{H}^r(D)$ is also Weak Hamiltonian C^r stable on $\mathcal{H}^r(D)$.*

Proof. We refer the reader to the proof of Theorem 6.4. □

6.0.1. *Maximal invariant planes around the singular part.* To describe the dynamics around the singular part $\mathcal{S}^1(D)$, we consider in Definition 2.3 the pair $\mathfrak{D} = (U, V)$ given by

$$U(x,y,z) = \begin{cases} \big(|x|+1,0,0\big) & y=0, \\ \big(0,|y|+1,0\big) & x=0, \end{cases} \quad \text{and} \quad V = \overline{V}; \tag{6.4}$$

where $\overline{V}$ is as in (6.1).

From (6.2) we define

$$D_x = S_x^+ \cup \mathcal{S}^1(D) \cup S_x^- \quad \text{and} \quad D_y = S_y^+ \cup \mathcal{S}^1(D) \cup S_y^-,$$

which also satisfy $D_x = \{(x,y,z) \in D : y = 0\}$ and $D_y = \{(x,y,z) \in D : x = 0\}$. Moreover, we take $H_x : P \to D_x$ and $H_y \colon P \to D_y$ as follows

$$H_x(x,0,z) = \big(h(x),0,z\big) \quad \text{and} \quad H_y(x,0,z) = \big(0,h(x),z\big),$$

where

$$h(x) = \begin{cases} e^x - 1 & x > 0, \\ 0 & x = 0, \\ -e^{-x}+1 & x < 0. \end{cases}$$

Observe that, $h'(x) = |h(x)| + 1, \forall x$. Furthermore, the C^1 diffeomorphisms H_x and H_y send $z-\text{axis} = \mathcal{S}^1(D) \subset P$ onto $\mathcal{S}^1(D) \subset D$ and outside this vertical axis they are of class C^r, for every positive $r \in \mathbb{N}$.

Lemma 6.6. *Suppose that $X \in \mathfrak{X}^r(D)$ and $a \in \{x,y\}$, then the restriction $X| \colon D_a \to \mathbb{R}^3$ is tangent to D_a. Moreover, there exists $X_a \in \mathfrak{X}^0(P)$, as in* (2.1)*, such that*

(a) *$X_a(q) = X(q)$ for any $q \in \mathcal{S}^1(D)$. Thus, the vertical line $\mathcal{S}^1(D)$ is invariant under X_a; that is, $X_a(q) \in \mathcal{S}^1(D)$, $\forall q \in \mathcal{S}^1(D)$.*

(b) *The homeomorphism $H_a : P \to D_a$ sends trajectories of X_a onto trajectories of $X| \colon D_a \to \mathbb{R}^3$ preserving the orientations.*

(c) *The pair (U_a, V_a) induced by* (6.4) *satisfies $U_a = (1,0,0)$ and $V_a = (0,0,1)$.*

Proof. We only consider the case $a = y$. The first part follows from the definitions introduced in § 2. Moreover, we define the vector field X_a as the pullback of the restriction $X| \colon D_a \to \mathbb{R}^3$ under the diffeomorphism $H_a \colon P \to D_a$. Consequently, $X_a \in \mathfrak{X}^0(P)$.

In order to prove (a) and (b), observe that the condition $q \in \mathcal{S}^1(D)$ implies that $H_a(q) = q$ and then the derivative of H_a at q sends $\mathcal{S}^1(D)$ onto itself. Note that this is an affine 1–dimensional vector space in the Tangent Space of $\mathcal{S}^1(D)$ at q. More precisely, the respective restriction of this derivative is the identity over such affine vector space. This properties and the definition of X_a directly give the first statements, (a) and (b).

Since $h'(x) = |h(x)| + 1$, by using the derivative of H_a and the definition of X_a it is not difficult to obtain (c). Therefore, this lemma holds. □

Corollary 6.7. *Both vector fields X_x and X_y on Lemma 6.6 are Hamiltonian as long as $X \in \mathcal{H}^r(D)$.*

Proof. Take $a = x$. Since

$$X(p) = \frac{\partial^* f}{\partial u}|_p - \frac{\partial^* f}{\partial v}|_p, \quad \forall p \in D_a,$$

there exist $g \in \mathcal{F}^{r+1}(P)$ given by $g(x,0,z) = f \circ H_a\big(h^{-1}(x), 0, z\big)$. Therefore, we obtain the proof by using Lemma 6.6 and the arguments of Corollary 6.2. □

Remark 6.8. In the last corollary both Hamiltonian vector fields have an invariant line. In particular, X_x and X_y belong to (4.1) and they might be $H^r_z(P)-$stable.

In the next theorem we present a sufficient conditions in order to have a characterization of the Hamiltonian satability on the Double Crossing Plane.

Theorem 6.9. *Consider $X_f \in \mathcal{H}^r(D)$, with $r \geq 1$. The induced Hamiltonian flow $(X_f)_t$ is Hamiltonian C^r–stable if both flows $(Y_g)_t$ with $Y_g \in \{X_x, X_y\}$ (as in Lemma 6.6) satisfy the following conditions hold.*

(a) *The flow $(Y_g)_t$ only has topologically simple singular point on $P \backslash (z-axis)$.*
(b) *For each separatrix ℓ_q, with $q \in P$ it satisfies one of the following conditions:*
  - *ℓ_q is isolated from the separatrices of all other finite of infinite saddles.*
  - *ℓ_q is unbounded and admits an open isolated neighborhood $\ell_q \subset V_q \subset P$ with $q \in z-axis$.*

(c) *If either the $z-axis$ has no isolated singularities or the $z-axis$ is free of singuarities, then this vertical axis admits a open neighborhood obtained as the saturation of some small transversal open segment.*

Proof. Consider a compact–open neighborhood $\mathcal{N}_{co}(K, U; D)$ of the identity on D. The restrictions

$$f \mapsto f| \colon D_x \longrightarrow D_x \quad \text{and} \quad f \mapsto f| \colon D_y \longrightarrow D_y$$

Naturally induces some compact–open neighborhoods on the plane, and we can apply the Theorem 4.9. Therefore, there exist two basic neighbourhoods $\mathcal{N}^r_\varepsilon(X_x, D_x) \ni X_x$ and $\mathcal{N}^r_\varepsilon(X_y, D_y) \ni X_y$ where the definition of $H^r_z(P)-$stable hold. More precisely, for every Hamiltonian flow Y_t with $Y \in \mathcal{N}^r_\varepsilon(X_x, D_x) \cap \mathcal{H}^r(P)$ (resp. $Y \in \mathcal{N}^r_\varepsilon(X_y, D_y) \cap \mathcal{H}^r(P)$) is topologically equivalent to $(X_f)_t$ and its equivalence belongs to the respective compact–open neighborhoods on the plane.

Since the set

$$\big\{Y : Y_x \in \mathcal{N}^r_\varepsilon(X_x, D_x) \ \text{ and } \ Y_y \in \mathcal{N}^r_\varepsilon(X_y, D_y)\big\}$$

is open, there is an open basic neighbourhood $\mathcal{N}^r_\varepsilon(X_f, D)$ such that the condition $Y \in \mathcal{N}^r_\varepsilon(X_f, D) \cap \mathcal{H}^r(D)$ implies that the Hamiltonian flow Y_t is topologically equivalent to $(X_f)_t$ and its equivalence belongs to $\mathcal{N}_{co}(K, U; D)$. Therefore, the Definition 3.5 holds and this theorem is true. □

Notice that this theorem is naturally complemented with the necessary conditions presented in Theorem 4.11. In particular, it generalizes Theorem 4.6.

References

[1] Javier Chavarriga, Hector Giacomin, Jaume Giné, and Jaume Llibre, *Local analytic integrability for nilpotent centers.* Ergodic Theory Dynam. Systems **23** (2003), no. 2, 417–428. MR 1972229 (2004c:34079)

[2] Freddy Dumortier and Douglas S. Shafer, *Restrictions on the equivalence homeomorphism in stability of polynomial vector fields.* J. London Math. Soc. (2) **41** (1990), no. 1, 100–108. MR 1063547 (91g:58145)

[3] Alexandre Fernandes, Carlos Gutierrez, and Roland Rabanal, *Global asymptotic stability for differentiable vector fields of* $\mathbb{R}^2$. J. Differential Equations **206** (2004), no. 2, 470–482. MR 2096702 (2005h:37055)

[4] Hector Giacomini, Jaume Giné, and Jaume Llibre, *The problem of distinguishing between a center and a focus for nilpotent and degenerate analytic systems.* J. Differential Equations **227** (2006), no. 2, 406–426. MR 2237674 (2007c:34043)

[5] C. Gutierrez, X. Jarque, J. Llibre, and M. A. Teixeira, *Global injectivity of* C^1 *maps of the real plane, inseparable leaves and the Palais-Smale condition.* Canad. Math. Bull. **50** (2007), no. 3, 377–389. MR 2344172 (2008i:37034)

[6] Carlos Gutierrez, Benito Pires, and Roland Rabanal, *Asymptotic stability at infinity for differentiable vector fields of the plane.* J. Differential Equations **231** (2006), no. 1, 165–181. MR 2287882 (2008a:34111)

[7] Carlos Gutiérrez and Jorge Sotomayor, *Champs de vecteurs stables sur les variétés à singularités simples.* C. R. Acad. Sci. Paris Sér. A-B, **291** (1980), no. 4, A287–A290. MR 591752 (81j:58052)

[8] ———, *Stable vector fields on manifolds with simple singularities.* Proc. London Math. Soc. (3) **45** (1982), no. 1, 97–112. MR 662665 (84h:58080)

[9] Shuhei Hayashi, *Connecting invariant manifolds and the solution of the* C^1 *stability and* Ω*-stability conjectures for flows.* Ann. of Math. (2), **145** (1997), no. 1, 81–137. MR 1432037 (98b:58096)

[10] Xavier Jarque and Jaume Llibre, *Structural stability of planar Hamiltonian polynomial vector fields.* Proc. London Math. Soc. (3), **68** (1994), no. 3, 617–640. MR 1262311 (94k:58077)

[11] ———, *Polynomial foliations of* $\mathbf{R}^2$. Pacific J. Math. **197** (2001), no. 1, 53–72. MR 1810208 (2003b:37045)

[12] Xavier Jarque, Jaume Llibre, and Douglas S. Shafer, *Structural stability of planar polynomial foliations.* J. Dynam. Differential Equations **17** (2005), no. 3, 573–587. MR 2165559 (2006d:37031)

[13] ———, *Structurally stable quadratic foliations.* Rocky Mountain J. Math. **38** (2008), no. 2, 489–530. MR 2401566 (2009c:37025)

[14] Xavier Jarque and Zbigniew Nitecki, *Hamiltonian stability in the plane.* Ergodic Theory Dynam. Systems **20** (2000), no. 3, 775–799. MR 1764927 (2001d:37019)

[15] Janina Kotus, Michał Krych, and Zbigniew Nitecki, *Global structural stability of flows on open surfaces.* Mem. Amer. Math. Soc. **37** (1982), no. 261, v+108. MR 653093 (83h:58055)

[16] John W. Milnor, *Topology from the differentiable viewpoint.* Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997, Based on notes by David W. Weaver, Revised reprint of the 1965 original. MR 1487640 (98h:57051)

[17] Jacob Palis, Jr. and Welington de Melo, *Geometric theory of dynamical systems.* Springer-Verlag, New York-Berlin, 1982, An introduction, Translated from the Portuguese by A. K. Manning. MR 669541 (84a:58004)

[18] Roland Rabanal, *Center type performance of differentiable vector fields in the plane*. Proc. Amer. Math. Soc. **137** (2009), no. 2, 653–662. MR 2448587 (2010a:34069)

[19] ———, *Asymptotic stability at infinity for bidimensional Hurwitz vector fields*. J. Differential Equations **255** (2013), no. 5, 1050–1066. MR 3062761

[20] Douglas S. Shafer, *Structure and stability of gradient polynomial vector fields*. J. London Math. Soc. (2), **41** (1990), no. 1, 109–121. MR 1063548 (91g:58146)

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