



CENTRE DE RECERCA MATEMÀTICA

Preprint núm. 1239

May 2018

Weighted gradient inequalities and
unique continuation problems

L. De Carli, D.V. Gorbachev, S. Yu. Tikhonov

WEIGHTED GRADIENT INEQUALITIES AND UNIQUE CONTINUATION PROBLEMS

LAURA DE CARLI, DMITRY GORBACHEV, AND SERGEY TIKHONOV

ABSTRACT. We use Pitt inequalities for the Fourier transform to prove the following weighted gradient inequality

$$\|e^{-\tau\ell(\cdot)}u^{\frac{1}{q}}f\|_q \leq c_\tau \|e^{-\tau\ell(\cdot)}v^{\frac{1}{p}}\nabla f\|_p, \quad f \in C_0^\infty(\mathbb{R}^n).$$

This inequality is a Carleman-type estimate that yields unique continuation results for solutions of first order differential equations and systems.

1. INTRODUCTION

The main purpose of this paper is to prove that the following weighted Sobolev gradient inequality holds for every linear function $\ell: \mathbb{R}^n \rightarrow \mathbb{R}$, every $f \in C_0^\infty(\mathbb{R}^n)$ and every $\tau \geq 0$, with suitable weights u, v and exponents $1 < p, q < \infty$.

$$(1.1) \quad \|e^{-\tau\ell(\cdot)}u^{\frac{1}{q}}f\|_q \leq c_\tau \|e^{-\tau\ell(\cdot)}v^{\frac{1}{p}}\nabla f\|_p$$

Here, c_τ is a finite constant that may depend on τ but does not depend on ℓ and f . We have denoted with $\|f\|_r = \left(\int_{\mathbb{R}^n} |f(x)|^r dx\right)^{\frac{1}{r}}$ the norm in $L^r(\mathbb{R}^n)$ and with $\langle x, y \rangle = x_1y_1 + \dots + x_ny_n$ and $|x| = \langle x, x \rangle^{\frac{1}{2}}$ the standard inner product and norm in \mathbb{R}^n .

When $\tau > 0$, we prove in Theorem 1.1 that $c_\tau = \max(\tau^{-1}, 1)C$; here and throughout the paper, C denotes a generic constant that depends only on non-essential parameters, i.e. $C = C_{u,v,p,q,n}$. In particular, $c_\tau = C$ when $\tau \geq 1$. Inequalities like (1.1) are often called *Carleman inequalities* in literature. In Sections 3 and 4 we will discuss Carleman inequalities and their connection with unique continuation problems and we will prove new unique continuation results for systems of partial differential equations and inequalities.

When $\tau = 0$ in (1.1), we obtain a standard weighted Sobolev gradient inequality (also called *weighted Poincaré-Sobolev inequality*)

$$(1.2) \quad \|u^{\frac{1}{q}}f\|_q \leq c_0 \|v^{\frac{1}{p}}\nabla f\|_p, \quad f \in C_0^\infty(\mathbb{R}^n).$$

2010 *Mathematics Subject Classification.* Primary: 42B10; Secondary: 35B60.

Key words and phrases. Pitt inequalities, weighted Carleman estimates, first order systems of PDE.

D. G. was supported by the Russian Science Foundation under grant 18-11-00199. S. T. was partially supported by MTM 2017-87409-P, 2017 SGR 358, and by the CERCA Programme of the Generalitat de Catalunya.

These inequalities have deep applications in partial differential equations. For example, the case $p = 2 < q$ of (1.2) arises in Harnack's inequality and regularity estimates for degenerate second order differential operators in divergence form. They also have applications in the study of the stable solutions of the Laplace and the p -Laplace operators in the Euclidean space, the Laplace-Kohn operator in the Heisenberg group, the sub-Laplace operator in the Engel group, etc.; see e.g., [22, 49, 57] and the references cited in these papers; see also [10].

Conditions on the weights u and v and the exponents p, q for which (1.2) holds have been investigated by several authors. The most natural approach to study (1.2) is based on the following pointwise inequality (see e.g., [19, 46])

$$|f(x)| \leq CI_1(|\nabla f|)(x), \quad x \in \mathbb{R}^n,$$

where $I_\alpha \phi(x) = \int_{\mathbb{R}^n} \frac{\phi(y)}{|x-y|^{n-\alpha}} dy$, $\alpha < n$, is the *Riesz potential*. This inequality follows from the classical Sobolev integral representation and is proved e.g. in [42].

If the weighted inequality

$$(1.3) \quad \|u^{\frac{1}{q}} I_1 f\|_q \leq C \|v^{\frac{1}{p}} f\|_p$$

holds for the weights u and v , we also have

$$\|u^{\frac{1}{q}} f\|_q \leq C \|u^{\frac{1}{q}} I_1(|\nabla f|)\|_q \leq C \|v^{\frac{1}{p}} |\nabla f|\|_p.$$

E. Sawyer obtained in [48] a complete characterization of the weights u and v for which the gradient inequality (1.3) holds with $p \leq q$. However, in some cases, the conditions in [48] are difficult to verify. When $p = q = 2$, a full characterization of the weights for which (1.2) holds is also in [41], but also the conditions in this paper are difficult to verify.

H. P. Heinig showed in [25] that weighted norm inequalities for the Fourier transform (or: *Pitt-type inequalities*) in the form of

$$(1.4) \quad \|\widehat{f} u^{\frac{1}{q}}\|_q \leq C \|f w^{\frac{1}{p}}\|_p, \quad f \in C_0^\infty(\mathbb{R}^n),$$

can be used to prove weighted gradient inequalities. The Fourier transform is defined as $\widehat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-i\langle x, y \rangle} dx$.

To prove (1.2) from (1.4), we observe that $\widehat{I_\alpha f}(y) = c_\alpha |y|^{-\alpha} \widehat{f}(y)$, where c_α is an explicit constant; we can see at once that (1.3) is equivalent to

$$\|u^{\frac{1}{q}} (|y|^{-1} \widehat{f})^\vee\|_q \leq C \|v^{\frac{1}{p}} f\|_p$$

where \vee denotes the inverse Fourier transform. We can apply Pitt's inequality twice (with a suitable weight w and an exponent $\gamma \in (1, \infty)$) to obtain

$$\|u^{\frac{1}{q}} (|y|^{-1} \widehat{f})^\vee\|_q \leq C \|w^{\frac{1}{\gamma}} |y|^{-1} \widehat{f}\|_\gamma \leq C \|v^{\frac{1}{p}} f\|_p.$$

Taking $w = |y|^\gamma$ and $\gamma = q$ and assuming conditions on the weights that ensure that both Pitt's inequalities hold we obtain the main theorem in [25], which was proved differently; see Theorem 2.1 in Section 2.

1.1. Main results. Throughout this paper, we will often write $A \lesssim B$ when $A \leq CB$ with a constant $C > 0$. We will also write $A \asymp B$ when there exists a constant $C > 0$, called the *constant of equivalence*, such that $C^{-1}A \leq B \leq CA$. As usual, we let g^* be the non-increasing rearrangement of g . We let $p' = \frac{p}{p-1}$ be the dual exponent of $p \in (1, \infty)$.

Our main result can be stated as follows.

Theorem 1.1. *Let $u \not\equiv 0$ and $v \not\equiv +\infty$ be weights on \mathbb{R}^n , $n \geq 1$.*

(a) *Let $1 < p \leq q < \infty$. If there exists $\gamma > 0$ that satisfies $\max(p, p') \leq \gamma \leq q$, for which*

$$(1.5) \quad \begin{cases} A_u^q(0) := \sup_{s>0} s^{1-q(\frac{1}{\gamma'} - \frac{1}{n})} u^*(s) < \infty, & \frac{1}{n} < \frac{1}{\gamma'} \leq \frac{1}{n} + \frac{1}{q}, \\ A_u^q(\tau) := \sup_{s>0} \int_0^s u^*(t) dt \left(\int_0^{\frac{1}{s}} (t + \tau^n)^{-\frac{\gamma'}{n}} dt \right)^{\frac{q}{\gamma'}} < \infty, & \tau > 0, \end{cases}$$

and

$$(1.6) \quad A_v^p := \sup_{s>0} s^{\frac{p}{\gamma'} - 1} (1/v)^*(s) < \infty,$$

the inequality

$$(1.7) \quad \|e^{-\tau \ell(\cdot)} u^{\frac{1}{q}} f\|_q \leq c_\tau \|e^{-\tau \ell(\cdot)} v^{\frac{1}{p}} \nabla f\|_p, \quad f \in C_0^\infty(\mathbb{R}^n),$$

holds for every $\tau \geq 0$ and every linear function $\ell(x) = \langle \mathbf{a}, x \rangle + b$, $\mathbf{a} \in \mathbb{R}^n$, $|\mathbf{a}| = 1$, $b \in \mathbb{R}$, with the constant

$$(1.8) \quad c_\tau = CA_u(\tau)A_v,$$

where $C = C_{p,q,\gamma,n}$ is some positive constant. Moreover,

$$(1.9) \quad A_u(\tau) \leq \max(\tau^{-1}, 1)A_u(1), \quad \tau > 0.$$

(b) *Let $1 < q < p < \infty$. If there exists $\gamma > 0$ that satisfies*

$$\begin{cases} \frac{n}{n-1} < \gamma \leq q, & \tau = 0, \\ 1 < \gamma \leq q, & \tau > 0, \end{cases}$$

for which (1.5) holds and

$$\tilde{A}_v^r := \int_0^\infty s^{-\frac{r}{\gamma} - 1} \left(\int_0^s (1/v)^*(t)^{\frac{1}{p-1}} dt \right)^{\frac{r}{p'}} ds < \infty,$$

with $\frac{1}{r} = \frac{1}{\gamma} - \frac{1}{p}$, the inequality (1.7) holds with the constant

$$(1.10) \quad c_\tau = CA_u(\tau)\tilde{A}_v.$$

Remark 1.1. When $\tau = 0$ and $\gamma = q$ we obtain Theorem 2.4 in [25] with simplified conditions on u and v . The proof of Theorem 1.1 shows that the assumptions $\frac{1}{\gamma'} \leq \frac{1}{n} + \frac{1}{q}$ and $p' \leq \gamma$ are to rule out the trivial weights $u \equiv 0$ and $v \equiv +\infty$.

Remark 1.2. For the applications of Theorem 1.1 it is important to have the uniform boundedness of c_τ as $\tau \rightarrow \infty$. From (1.8), (1.10) and (1.9), we have $c_\tau \leq c_1 \asymp A_u(1)$ whenever $\tau \geq 1$; thus, to prove the boundedness of c_τ , it is sufficient to verify that $A_u(1) < \infty$.

Remark 1.3. It is interesting to compare our weighted gradient inequalities with those proved by G. Sinnamon in [53]. In that paper, a weighted norm inequality in the form of

$$(1.11) \quad \|fu^{\frac{1}{q}}\|_q \leq C \|\langle \nabla f, x \rangle w^{\frac{1}{p}}\|_p, \quad f \in C_0^\infty(\mathbb{R}^n)$$

is considered. If we denote with $\partial_r f = \langle \frac{x}{|x|}, \nabla f \rangle$ the radial derivative of f , the inequality (1.11) is equivalent to

$$\|fu^{\frac{1}{q}}\|_q \leq C \| |x| w^{\frac{1}{p}} \partial_r f \|_p, \quad f \in C_0^\infty(\mathbb{R}^n),$$

and implies (1.2) with $v = |x|^p w$.

In [53, Theorem 4.1], (1.11) is only proved for $p = q$ and $q < p$ under some conditions on u, w ; moreover, in [53, Theorem 3.4] it is proved that when $1 \leq p < q < \infty$ and the weight w is locally integrable on \mathbb{R}^n , the inequality (1.11) holds for every $f \in C_0^\infty(\mathbb{R}^n)$ if and only if $u \equiv 0$ a.e.

When f is radial, $\nabla f(x) = \frac{x}{|x|} \partial_r f(x)$, and so $|\nabla f(x)| = |\partial_r f(x)|$. Thus, our Theorem 1.1 yields (1.11) for radial functions with a nontrivial weight u and with $w = |x|^{-p} e^{-p\tau\ell(x)} v$. We proved in Corollary 1.2 below that we can consider piecewise power weights $v = |x|^{(\beta_1, \beta_2)}$, with $0 \leq \beta_1 \leq n(\frac{p}{\gamma'} - 1)$ (see definition (1.13)). For example, if $\beta_1 = n(\frac{p}{\gamma'} - 1)$, then w is locally integrable for $\frac{1}{n} < \frac{1}{\gamma'}$ because $-p + \beta_1 > -n$. We remark that the counterexample in [53, Theorem 3.4] is not radial.

Remark 1.4. The inequality (1.7) is equivalent to

$$(1.12) \quad \|u^{\frac{1}{q}} f\|_q \leq c_\tau \|v^{\frac{1}{p}} (\tau \mathbf{a} f + \nabla f)\|_p.$$

To see this, it is enough to use the substitution $f_1 = e^{-\tau\ell(\cdot)} f$ and $\nabla(e^{\tau\ell(\cdot)} f_1) = e^{\tau\ell(\cdot)} (\tau \mathbf{a} f_1 + \nabla f_1)$.

Let $\beta_1, \beta_2 \in \mathbb{R}$; we define the piecewise power function $t \mapsto t^{(\beta_1, \beta_2)}$ as follows:

$$(1.13) \quad t^{(\beta_1, \beta_2)} := \begin{cases} t^{\beta_1}, & 0 < t \leq 1, \\ t^{\beta_2}, & t \geq 1. \end{cases}$$

In the following corollary of Theorem 1.1 we consider the important case of piecewise power weights.

Corollary 1.2. *Let $1 < p \leq q < \infty$; let $\gamma > 0$ that satisfies $\max(p, p') \leq \gamma \leq q$ and $\frac{1}{n} < \frac{1}{\gamma'} \leq \frac{1}{n} + \frac{1}{q}$.*

With the notation and the assumptions of Theorem 1.1 (a), the inequality (1.7) holds with $u(x) = |x|^{(-\alpha_1, -\alpha_2)}$, $v(x) = |x|^{(\beta_1, \beta_2)}$, with $\alpha_j, \beta_j \geq 0$, provided that

$$(1.14) \quad \alpha_1 \leq n \left(1 - \frac{q}{\gamma'} + \frac{q}{n} \right), \quad \begin{cases} \alpha_2 \geq n \left(1 - \frac{q}{\gamma'} + \frac{q}{n} \right) & \text{when } \tau = 0, \\ \alpha_2 \geq 0 & \text{when } \tau > 0, \end{cases}$$

$$(1.15) \quad \beta_1 \leq n \left(\frac{p}{\gamma'} - 1 \right), \quad \beta_2 \geq n \left(\frac{p}{\gamma'} - 1 \right).$$

In particular, for power weights $u(x) = |x|^{-\alpha}$, $v(x) = |x|^\beta$ the inequality (1.7) holds if

$$\begin{cases} \alpha = n \left(1 - \frac{q}{\gamma'} + \frac{q}{n} \right) \geq 0 & \text{when } \tau = 0, \\ 0 \leq \alpha \leq n \left(1 - \frac{q}{\gamma'} + \frac{q}{n} \right) & \text{when } \tau > 0, \end{cases} \quad \beta = n \left(\frac{p}{\gamma'} - 1 \right) \geq 0.$$

Moreover, the conditions

$$(1.16) \quad \begin{cases} \frac{\alpha}{q} + \frac{\beta}{p} = n \left(\frac{1}{q} - \frac{1}{p} \right) + 1 & \text{when } \tau = 0, \\ \frac{\alpha}{q} + \frac{\beta}{p} \leq n \left(\frac{1}{q} - \frac{1}{p} \right) + 1 & \text{when } \tau > 0, \end{cases}$$

are necessary for the validity of (1.7).

Letting $\tau = \alpha = \beta = 0$, $1 < p < n$, and $\gamma = q$ in (1.16), we obtain $q = \frac{np}{n-p}$ and Corollary 1.2 yields the classical Sobolev inequality $\|f\|_q \leq C \|\nabla f\|_p$; see also [25, Corollary 2.5].

When $\tau = 0$, we obtain the inequality

$$\left(\int_{\mathbb{R}^n} |f|^q |x|^{(-\alpha_1, -\alpha_2)} dx \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} |\nabla f|^p |x|^{(\beta_1, \beta_2)} dx \right)^{\frac{1}{p}},$$

which was proved by Maz'ya [42] and Caffarelli, Kohn, and Nirenberg [3] for power weights. In [42, Sect. 2.1.6] it was proved that if $1 < p < n$, $p \leq q \leq \frac{pn}{n-p}$, and $-\frac{\alpha}{q} = \frac{\beta}{p} - 1 + n \left(\frac{1}{p} - \frac{1}{q} \right) > -\frac{n}{q}$, then

$$(1.17) \quad \left(\int_{\mathbb{R}^n} |f|^q |x|^{-\alpha} dx \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} |\nabla f|^p |x|^\beta dx \right)^{\frac{1}{p}}.$$

In [28, Lemma 2.1], this inequality was proved for $n \geq 2$, $1 < p < +\infty$, $0 \leq \frac{1}{p} - \frac{1}{q} = n \left(1 - \frac{\beta}{p} - \frac{\alpha}{q} \right)$ and $-\frac{n}{q} < -\frac{\alpha}{q} \leq \frac{\beta}{p}$. Note that the conditions in [42] and [28] are the same, except for the extra condition $p < n$ in [42].

From Corollary 1.2 with $\tau = 0$ we have that $\alpha = n \left(1 - \frac{q}{\gamma'} + \frac{q}{n} \right) \geq 0$, $\beta = n \left(\frac{p}{\gamma'} - 1 \right) \geq 0$, where $\max(p, p') \leq \gamma \leq q$ and $\frac{1}{n} < \frac{1}{\gamma'} \leq \frac{1}{n} + \frac{1}{q}$. These inequalities

imply $\frac{1}{p} - \frac{1}{q} = n\left(1 - \frac{\beta}{p} - \frac{\alpha}{q}\right)$, $-\frac{n}{q} < -\frac{\alpha}{q} \leq \frac{\beta}{p}$, but we also have to assume $\alpha \geq 0$, $\beta \geq 0$ because of our method of the proof.

It is interesting to observe that the best constant in the inequality (1.17) has been evaluated in [57] and also in [22] for special values of α and β .

1.2. Unique continuation. Our Theorem 1.1 can be used to prove unique continuation results for weak solutions (also called *solutions in distribution sense*) of systems of differential equations and inequalities; see Section 3 for definitions and preliminary results.

We consider solutions in weighted Sobolev spaces of distributions: given a domain $D \subset \mathbb{R}^n$, we let $W_0^{m,p,v}(D)$ be the closure of $C_0^\infty(D)$ with respect to the norm

$$\|f\|_{W^{m,p,v}(D)} = \sum_{|\alpha|=0}^m \|v^{\frac{1}{p}} \partial_x^\alpha f\|_{L^p(D)}$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and the $\partial_x^\alpha f = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f$ are the partial derivatives of f . In Section 3 we prove the following

Theorem 1.3. *Let p, q, γ, u and v be as in Theorem 1.1 (a). Let $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Let $f \in W_0^{1,p,v}(\mathbb{R}^n)$ be a solution of the differential inequality*

$$(1.18) \quad |\nabla f| \leq V|f|$$

with $V \in L^r(\text{supp } f, v^{\frac{r}{p}} u^{-\frac{r}{q}} dx)$. If, for some linear function $\ell: \mathbb{R}^n \rightarrow \mathbb{R}$, we have that $\text{supp } f \subset \{x: \ell(x) \geq 0\}$, necessarily $f \equiv 0$.

Note that the condition $V \in L^r(\text{supp } f, v^{\frac{r}{p}} u^{-\frac{r}{q}} dx)$ follows from either $V \in L^r(\mathbb{R}^n, v^{\frac{r}{p}} u^{-\frac{r}{q}} dx)$ if $\text{supp } f$ is unbounded, or from $V \in L_{\text{loc}}^r(\mathbb{R}^n, v^{\frac{r}{p}} u^{-\frac{r}{q}} dx)$ if f has compact support. In particular, for power weights u, v as in Corollary 1.2, the differential inequality (1.18) does not have solutions with compact support if $V \asymp |x|^{-1+\epsilon}$ for some $\epsilon > 0$; see Remark 3.1.

To prove Theorem 1.3 we use a method developed by T. Carleman in [4]. A brief discussion on unique continuation problems and Carleman's method is in Sections 3 and 4.

When D is measurable and v is a suitable weight we consider the Dirichlet problem

$$(1.19) \quad \begin{cases} -\text{div}(v \nabla f |\nabla f|^{p-2}) = v V f |f|^{p-2}, \\ f \in W_0^{1,p,v}(D), \end{cases}$$

where $\text{div}((g_1, \dots, g_n)) = \partial_{x_1} g_1 + \cdots + \partial_{x_n} g_n$ and the potential V is in a suitable L^r space. The operator $\text{div}(v \nabla f |\nabla f|^{p-2})$ is known as *weighted p -Laplacian* in the literature (see e.g. [23, 31]) and is denoted by Δ_p when $v \equiv 1$. The weighted p -Laplacian is nonlinear when $p \neq 2$ and is linear when $p = 2$.

When $v \equiv 1$, (1.19) can be compared to the Sturm-Liouville problem in the form of $-\Delta_p f = (\lambda m - V)f|f|^{p-2}$ (see e.g. [8]). When $n = 1$ and $p = 2$ we have $-(vf')' = vVf$. This problem is related to the classical Sturm-Liouville problem $-(vf')' = (\lambda w - q)f$. See [40].

We prove the following

Theorem 1.4. *Let $f \in W_0^{1,p,v}(D)$ be a solution of the Dirichlet problem (1.19). Let $V_+ = \max\{V, 0\}$. Assume that $|V|^{1/p} \in L^r(D, v^{r/p} u^{-r/q} dx)$, where u, v are as in Theorem 1.1 and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Then, either*

$$c_0 \|u^{-\frac{1}{q}} v^{\frac{1}{p}} V_+^{\frac{1}{p}}\|_{L^r(D)} \geq 1,$$

where c_0 is as in (1.2), or $f \equiv 0$ in D .

Thus, the Dirichlet problem (1.19) has the unique solution $f \equiv 0$ if the weighted L^r norm of $V_+^{1/p}$ on D is small enough.

To the best of our knowledge, the method of proof of Theorem 1.4 has been used for the first time in [13]; it is extensively used in [11] and [17].

2. PROOF OF THEOREM 1.1

In this section we prove our main theorem and a few corollaries.

2.1. Preliminary results. We will use the following theorem due to Heinig [26], Jurkat-Sampson [34], and Muckenhoupt [43].

Theorem 2.1. *Let $n \geq 1$. If $1 < p \leq q < \infty$ and the weights u and w satisfy*

$$\sup_{s>0} \left(\int_0^{\frac{1}{s}} u^*(t) dt \right)^{\frac{1}{q}} \left(\int_0^{\frac{1}{s}} ((1/w)^*(t))^{\frac{1}{p-1}} dt \right)^{\frac{1}{p'}} =: A_1 < \infty,$$

or if $1 < q < p < \infty$, and

$$\sup_{s>0} \left(\int_0^\infty \left(\int_0^{\frac{1}{s}} u^*(t) dt \right)^{\frac{r}{q}} \left(\int_0^s ((1/w)^*(t))^{\frac{1}{p-1}} dt \right)^{\frac{r}{q'}} ((1/w)^*(s))^{\frac{1}{p-1}} ds \right)^{\frac{1}{r}}$$

$$(2.1) \quad =: A_2 < \infty$$

where $r = \frac{qp}{q-p}$, then Pitt's inequality

$$\|\widehat{f} u^{\frac{1}{q}}\|_q \leq C_j \|f w^{\frac{1}{p}}\|_p, \quad f \in C_0^\infty(\mathbb{R}^n), \quad j = 1, 2,$$

holds with $C_j \leq C_{p,q,j} A_j$.

Recall that the non-increasing rearrangement of a measurable radially decreasing function $f(x) = f_0(|x|)$ is defined as follows: let for $\lambda > 0$

$$\mu_f(\lambda) = \mu\{x : |f(x)| > \lambda\} = \mu\{x : |x| < f_0^{-1}(\lambda)\} = (f_0^{-1}(\lambda))^n V_n,$$

where V_n is the volume of the unit ball $B^n = \{x \in \mathbb{R}^n: |x| \leq 1\}$. Then for $t > 0$

$$f^*(t) = \inf\{\lambda > 0: \mu_f(\lambda) < t\} = f_0((t/V_n)^{\frac{1}{n}}).$$

Note that the conditions on u and w are also necessary when u and w are radial, i.e., $u = u_0(|x|)$ and $w(x) = w_0(|x|)$, with $u_0(r)$ non-increasing and $w_0(r)$ non-decreasing. See [26] and also [12, Theorem 1.2] for simpler and more general necessary conditions on the weight u and w . We should also mention [38, Theorem 2.1] where a necessary condition similar to that in [26], with u replaced by a measure $d\mu$, was proved.

We also need the following

Lemma 2.2. *Let $\psi \not\equiv 0$ be a non-increasing non-negative function; let $\beta_1, \beta_2 > 0$ and let $\beta'_2 = \min(\beta_2, 1)$. If either*

$$A = \sup_{s>0} s^{(-\beta_1, -\beta_2)} \int_0^s \psi(t) dt < \infty,$$

or

$$B = \sup_{s>0} s^{(1-\beta_1, 1-\beta'_2)} \psi(s) < \infty,$$

then $\beta_1 \leq 1$ and $A \asymp B$.

Proof. Assume $A < \infty$; then, for every $s > 0$, we have that $\int_0^s \psi(t) dt \leq As^{(\beta_1, \beta_2)}$. Since ψ is non-increasing, $s\psi(s) \leq \int_0^s \psi(t) dt$, so $\psi(s) \leq As^{(\beta_1-1, \beta_2-1)}$. If $\beta_1 > 1$, then $\lim_{s \rightarrow 0^+} \psi(s) = 0$ and consequently $\psi \equiv 0$; since we assumed $\psi \not\equiv 0$, necessarily $\beta_1 \leq 1$.

Furthermore, from $\psi(s) \leq \psi(1)$ for $s \geq 1$ we can see at once that $\psi(s) \lesssim As^{\beta'_2-1}$ and so $B \lesssim A$.

If we assume $B < \infty$, for every $s > 0$ we have that $\psi(s) \leq Bs^{(\beta_1-1, \beta'_2-1)}$. As above we conclude that $\beta_1 \leq 1$. For $0 < s \leq 1$ we have $\int_0^s \psi(t) dt \lesssim Bs^{\beta_1}$. If $s \geq 1$, then

$$\int_0^s \psi(t) dt = \int_0^1 \psi(t) dt + \int_1^s \psi(t) dt \lesssim B + B \int_1^s t^{\beta'_2-1} dt \lesssim Bs^{\beta'_2} \leq Bs^{\beta_2}.$$

Thus, $\sup_{s \geq 1} s^{-\beta_2} \int_0^s \psi(t) dt \lesssim B$ and $A \lesssim B$. \square

2.2. Proof of Theorem 1.1. We can assume $\ell(x) = \langle \mathbf{a}, x \rangle$, $|\mathbf{a}| = 1$, without loss of generality.

(a) Let $p \leq \gamma \leq q$.

Step 1. For $\tau \geq 0$ and $\xi \in \mathbb{R}^n$, define

$$w_\tau(\xi) = |\xi - i\tau\mathbf{a}|^\gamma = (|\xi|^2 + \tau^2)^{\frac{\gamma}{2}}.$$

By Theorem 2.1 (a), the inequality

$$(2.2) \quad \left(\int_{\mathbb{R}^n} |\widehat{g}(x)|^q u(x) dx \right)^{\frac{1}{q}} \lesssim A_{u, w_\tau} \left(\int_{\mathbb{R}^n} w_\tau(\xi) |g(\xi)|^\gamma d\xi \right)^{\frac{1}{\gamma}}$$

holds with

$$A_{u,w_\tau} = \sup_{s>0} \left(\int_0^s u^*(t) dt \right)^{\frac{1}{q}} \left(\int_0^{\frac{1}{s}} ((1/w_\tau)^*(t))^{\frac{1}{\gamma-1}} dt \right)^{\frac{1}{\gamma'}} < \infty.$$

The weight w_τ is radially increasing, so

$$(1/w_\tau)^*(t) = ((t/V_n)^{\frac{2}{n}} + \tau^2)^{-\frac{\gamma}{2}} \asymp (t + \tau^n)^{-\frac{\gamma}{n}}$$

with the constant of equivalence independent of τ . This implies

$$\int_0^{\frac{1}{s}} ((1/w_\tau)^*(t))^{\frac{1}{\gamma-1}} dt \asymp \int_0^{\frac{1}{s}} (t + \tau^n)^{-\frac{\gamma'}{n}} dt, \quad s > 0.$$

Therefore, for $\tau \geq 0$,

$$(2.3) \quad A_{u,w_\tau}^q \asymp \sup_{s>0} \int_0^s u^*(t) dt \left(\int_0^{\frac{1}{s}} (t + \tau^n)^{-\frac{\gamma'}{n}} dt \right)^{\frac{q}{\gamma'}} = A_u^q(\tau).$$

Since $(t + \tau^n)^{-1} \leq \max(\tau^{-n}, 1)(t + 1)^{-1}$ for $t, \tau > 0$, from (2.3) we conclude that

$$A_u(\tau) \leq \max(\tau^{-1}, 1)A_u(1), \quad \tau > 0.$$

We can give a simple expression for $A_u^q(0)$. Observing that $I := \int_0^{1/s} t^{-\frac{\gamma'}{n}} dt$ is finite when $-\frac{\gamma'}{n} > -1$ or, equivalently, $\frac{n}{n-1} < \gamma$, we have that $I \asymp s^{\frac{\gamma'}{n}-1}$. Therefore, (2.3) can be rewritten as

$$(2.4) \quad A_u^q(0) \asymp \sup_{s>0} s^{-q(\frac{1}{\gamma'} - \frac{1}{n})} \int_0^s u^*(t) dt.$$

By (2.4) and Lemma 2.2 with $\beta_1 = \beta_2 = q(\frac{1}{\gamma'} - \frac{1}{n})$, there holds that $q(\frac{1}{\gamma'} - \frac{1}{n}) \leq 1$ or $\frac{1}{\gamma'} \leq \frac{1}{n} + \frac{1}{q}$ and we can redefine $A_u^q(0)$ as follows.

$$A_u^q(0) = \sup_{s>0} s^{-q(\frac{1}{\gamma'} - \frac{1}{n})} \int_0^s u^*(t) dt \asymp \sup_{s>0} s^{1-q(\frac{1}{\gamma'} - \frac{1}{n})} u^*(s).$$

Step 2. Let $g(x) = e^{-\langle \tau \mathbf{a}, x \rangle} f(x)$. Then $g \in C_0^\infty(\mathbb{R}^n)$ and

$$(2.5) \quad \widehat{g}(\xi) = \int_{\mathbb{R}^n} g(x) e^{-i\langle \xi, x \rangle} dx = \int_{\mathbb{R}^n} f(x) e^{-i\langle \xi, x \rangle - \langle \tau \mathbf{a}, x \rangle} dx = \widehat{f}(\xi - i\tau \mathbf{a}).$$

Since for $g \in C_0^\infty(\mathbb{R}^n)$ the Fourier inversion formula holds, (2.2) and (2.3) imply

$$(2.6) \quad \begin{aligned} \left(\int_{\mathbb{R}^n} |g(x)|^q u(x) dx \right)^{\frac{1}{q}} &\lesssim A_u(\tau) \left(\int_{\mathbb{R}^n} |\xi - i\tau \mathbf{a}|^\gamma |\widehat{g}(\xi)|^\gamma d\xi \right)^{\frac{1}{\gamma}} \\ &= A_u(\tau) \left(\int_{\mathbb{R}^n} |(\xi - i\tau \mathbf{a}) \widehat{f}(\xi - i\tau \mathbf{a})|^\gamma d\xi \right)^{\frac{1}{\gamma}}. \end{aligned}$$

Note that \widehat{f} is entire analytic (and so it is defined at $\xi - i\tau\mathbf{a}$) because f has compact support. Since $\widehat{\nabla f}(\xi) = i\xi\widehat{f}(\xi)$, from (2.5) with $h(x) = (h_1(x), \dots, h_n(x)) = e^{-\langle \tau\mathbf{a}, x \rangle} \nabla f(x)$ we get

$$\widehat{h}(\xi) = \widehat{\nabla f}(\xi - i\tau\mathbf{a}) = i(\xi - i\tau\mathbf{a})\widehat{f}(\xi - i\tau\mathbf{a}).$$

Hence

$$\begin{aligned} & \left(\int_{\mathbb{R}^n} |(\xi - i\tau\mathbf{a})\widehat{f}(\xi - i\tau\mathbf{a})|^\gamma d\xi \right)^{\frac{1}{\gamma}} \\ &= \left(\int_{\mathbb{R}^n} |\widehat{h}(\xi)|^\gamma d\xi \right)^{\frac{1}{\gamma}} = \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^n |\widehat{h}_j(\xi)|^2 \right)^{\frac{\gamma}{2}} d\xi \right)^{\frac{1}{\gamma}} \\ &\leq \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^n |\widehat{h}_j(\xi)| \right)^\gamma d\xi \right)^{\frac{1}{\gamma}} \leq \sum_{j=1}^n \left(\int_{\mathbb{R}^n} |\widehat{h}_j(\xi)|^\gamma d\xi \right)^{\frac{1}{\gamma}}, \end{aligned}$$

where the first inequality holds trivially and the second is Minkowski's inequality.

Let us use Pitt's inequality with $p \leq \gamma$:

$$(2.7) \quad \|\widehat{f}\|_\gamma \leq C_{p,\gamma} A_{1,v} \|f v^{\frac{1}{p}}\|_p,$$

where

$$(2.8) \quad A_{1,v} := \sup_{s>0} \left(\int_0^{\frac{1}{s}} dt \right)^{\frac{1}{\gamma}} \left(\int_0^s (1/v)^*(t)^{\frac{1}{p-1}} dt \right)^{\frac{1}{p'}} < \infty.$$

As in Step 1, we apply Lemma 2.2 with $\beta_1 = \beta_2 = \frac{p'}{\gamma}$. We obtain $\frac{p'}{\gamma} \leq 1$ and

$$A_{1,v}^{p'} \asymp \sup_{s>0} s^{-\frac{p'}{\gamma}} \int_0^s (1/v)^*(t)^{\frac{1}{p-1}} dt \asymp \sup_{s>0} s^{1-\frac{p'}{\gamma}} (1/v)^*(s)^{\frac{1}{p-1}}.$$

It follows that

$$A_{1,v}^p = A_{1,v}^{p'(p-1)} \asymp \sup_{s>0} s^{\frac{p}{\gamma}-1} (1/v)^*(s) = A_v^p < \infty.$$

Applying (2.7) with f replaced by h_j , $j = 1, \dots, n$, we gather

$$\begin{aligned} & \left(\int_{\mathbb{R}^n} |\widehat{h}_j(\xi)|^\gamma d\xi \right)^{\frac{1}{\gamma}} \lesssim A_v \left(\int_{\mathbb{R}^n} |h_j(x)|^p v(x) dx \right)^{\frac{1}{p}} \\ & \lesssim A_v \left(\int_{\mathbb{R}^n} \left(\sum_{k=1}^n |h_k(x)|^2 \right)^{\frac{p}{2}} v(x) dx \right)^{\frac{1}{p}} \\ (2.9) \quad & = A_v \left(\int_{\mathbb{R}^n} |e^{-\langle \tau\mathbf{a}, x \rangle} \nabla f(x)|^p v(x) dx \right)^{\frac{1}{p}}. \end{aligned}$$

This, together with (2.6) proves part (a) of the theorem.

(b) Let $1 < \gamma \leq q < p$. We proceed as in the proof of part (a) to obtain (2.2), provided that (2.3) holds. We note that we assume $\frac{n}{n-1} < \gamma$ when $\tau = 0$.

Analogously, we get (2.9), but instead of (2.8) we use (2.1) with $u = 1, w = v$ and $\gamma < p$. Then we have

$$\begin{aligned} A_{1,v}^r &= \int_0^\infty s^{-\frac{r}{\gamma}} \left(\int_0^s (1/v)^*(t)^{\frac{1}{p-1}} dt \right)^{\frac{r}{\gamma'}} (1/v)^*(s)^{\frac{1}{p-1}} ds \\ &\asymp \int_0^\infty s^{-\frac{r}{\gamma}} \frac{d}{ds} \left(\int_0^s (1/v)^*(t)^{\frac{1}{p-1}} dt \right)^{\frac{r}{\gamma'}+1} ds, \end{aligned}$$

where $\frac{1}{r} = \frac{1}{\gamma} - \frac{1}{p}$. After integrating by parts, we get

$$A_{1,v}^r \asymp \int_0^\infty s^{-\frac{r}{\gamma}-1} \left(\int_0^s (1/v)^*(t)^{\frac{1}{p-1}} dt \right)^{\frac{r}{\gamma'}} ds = \tilde{A}_v^r < \infty.$$

This proves part (b) of the theorem.

2.3. Corollaries and remarks. Let us first discuss the conditions on γ in Theorem 1.1. We recall that in part (a) of Theorem 1.1 we assume $1 < p \leq q < \infty$ and $\max(p, p') \leq \gamma \leq q$; when $\tau = 0$ we assume also $\frac{1}{n} < \frac{1}{\gamma'}$.

Note that this extra assumption on γ is not necessary when $n \geq 3$. Indeed, from $\max(p, p') \leq \gamma \leq q$ follows that $2 \leq \gamma \leq q$ and $q' \leq \gamma' \leq 2$; thus, $\frac{1}{n} < \frac{1}{\gamma'}$ whenever $n \geq 3$.

When $n = 1$, the inequality $\frac{1}{n} < \frac{1}{\gamma'}$ (or: $\gamma > \frac{n}{n-1}$) can never be satisfied by γ' and only the case $\frac{1}{n} \geq \frac{1}{\gamma'}$ is possible. In fact, the condition $\max(p, p') \leq \gamma \leq q$ always implies $\frac{1}{2} \leq \frac{1}{\gamma'} < 1$.

If $n = 2$ we can either have $\frac{1}{n} < \frac{1}{\gamma'}$ or $\frac{1}{n} \geq \frac{1}{\gamma'}$. Note that $\frac{1}{2} \geq \frac{1}{\gamma'}$ implies that $p = \gamma = 2$.

For applications, it is important to simplify the expression for $A_u^q(1)$ in (1.5). Recall that, when $\tau > 0, A_u(\tau) \leq \max(\tau^{-1}, 1)A_u(1)$ (see Remark 1.2). We prove the following

Corollary 2.3. *Let $1 < p \leq q < \infty$ and let $\max(p, p') \leq \gamma \leq q$.*

(i) *If $n \geq 2$ and $\frac{1}{n} < \frac{1}{\gamma'}$, then $\frac{1}{\gamma'} \leq \frac{1}{n} + \frac{1}{q}$ and*

$$A_u^q(1) \asymp \sup_{s>0} s^{(1-q(\frac{1}{\gamma'}-\frac{1}{n}),0)} u^*(s).$$

(ii) *If $n = 2$ and $p = \gamma = 2$, then*

$$A_u^q(1) = \sup_{s>0} (\ln(s^{-1} + 1))^{q/2} \int_0^s u^*(t) dt.$$

(iii) *If $n = 1$, then*

$$A_u^q(1) \asymp \sup_{s>0} s^{(0,-\frac{q}{\gamma'})} \int_0^s u^*(t) dt.$$

Proof. (i) Recall that

$$A_u^q(\tau) = \sup_{s>0} \int_0^s u^*(t) dt \left(\int_0^{\frac{1}{s}} (t + \tau^n)^{-\frac{\gamma'}{n}} dt \right)^{\frac{q}{\gamma'}}, \quad \tau > 0.$$

For $\tau = 1$ and $\frac{1}{n} < \frac{1}{\gamma'}$, we have

$$\begin{aligned} \left(\int_0^{\frac{1}{s}} (t + 1)^{-\frac{\gamma'}{n}} dt \right)^{\frac{q}{\gamma'}} &\asymp \left((s^{-1} + 1)^{1-\frac{\gamma'}{n}} - 1 \right)^{\frac{q}{\gamma'}} \\ &\asymp s^{(-q(\frac{1}{\gamma'} - \frac{1}{n}), -\frac{q}{\gamma'})}, \quad s > 0. \end{aligned}$$

Hence

$$A_u^q(1) \asymp \sup_{s>0} s^{(-q(\frac{1}{\gamma'} - \frac{1}{n}), -\frac{q}{\gamma'})} \int_0^s u^*(t) dt,$$

where $q(\frac{1}{\gamma'} - \frac{1}{n}) > 0$ and $\frac{q}{\gamma'} \geq 1$, since $\gamma' \leq 2 \leq q$.

Now we can apply Lemma 2.2 with $\beta_1 = q(\frac{1}{\gamma'} - \frac{1}{n})$, $\beta_2 = \frac{q}{\gamma'} \geq 1$. We obtain $q(\frac{1}{\gamma'} - \frac{1}{n}) \geq 1$ or $\frac{1}{\gamma'} \leq \frac{1}{n} + \frac{1}{q}$ and

$$A_u^q(1) \asymp \sup_{s>0} s^{(1-q(\frac{1}{\gamma'} - \frac{1}{n}), 0)} u^*(s).$$

Part (ii) is obvious. To prove part (iii), we note that $\gamma' > 1$, which gives $\int_0^{\frac{1}{s}} (t + 1)^{-\gamma'} dt \asymp s^{(0, -1)}$. \square

Proof of Corollary 1.2. Recall that in this corollary $u(x) = |x|^{(-\alpha_1, -\alpha_2)}$, $v(x) = |x|^{(\beta_1, \beta_2)}$ with $\alpha_j, \beta_j \geq 0$. We consider the case when $1 < p \leq q < \infty$ and $\gamma \in [\max(p, p'), q]$, with $\frac{1}{n} < \frac{1}{\gamma} \leq \frac{1}{n} + \frac{1}{q}$, and we let $\tau = 0$ or $\tau = 1$.

Since $w^*(s) \asymp w_0(s^{\frac{1}{n}})$, $s > 0$, for any non-increasing radial weight function $w(x) = w_0(|x|)$ we have

$$u^*(s) \asymp s^{(-\frac{\alpha_1}{n}, -\frac{\alpha_2}{n})}, \quad (1/v)^*(s) \asymp s^{(-\frac{\beta_1}{n}, -\frac{\beta_2}{n})}.$$

whenever $\alpha_j, \beta_j \geq 0$.

The expression (1.5), Corollary 2.3 (i), and (1.6) imply that for $s > 0$

$$u^*(s) \lesssim \begin{cases} s^{q(\frac{1}{\gamma'} - \frac{1}{n}) - 1}, & \tau = 0, \\ s^{(q(\frac{1}{\gamma'} - \frac{1}{n}) - 1, 0)}, & \tau = 1, \end{cases} \quad (1/v)^*(s) \lesssim s^{1 - \frac{p}{\gamma'}}.$$

It is easy to see that when $a_j, b_j \geq 0$, the inequality $s^{(-a_1, -a_2)} \lesssim s^{(-b_1, -b_2)}$, holds if and only if $a_1 \leq b_1$, $a_2 \geq b_2$. It follows that

$$\alpha_1 \leq n \left(1 - \frac{q}{\gamma'} + \frac{q}{n} \right), \quad \begin{cases} \alpha_2 \geq n \left(1 - \frac{q}{\gamma'} + \frac{q}{n} \right), & \tau = 0, \\ \alpha_2 \geq 0, & \tau = 1, \end{cases}$$

and

$$0 \leq \beta_1 \leq n \left(\frac{p}{\gamma'} - 1 \right), \quad \beta_2 \geq n \left(\frac{p}{\gamma'} - 1 \right)$$

which proves (1.14) and (1.15).

To prove (1.16) we use a standard homogeneity argument. Let us consider (1.12) (which by Remark 1.4 is equivalent to (1.7)) with $f = f_\lambda(x) = f(\lambda x)$ for some $f \in C_0^\infty(\mathbb{R}^n)$ and $\lambda > 0$. We obtain

$$\| |x|^{-\frac{\alpha}{q}} f_\lambda \|_q \leq c_\tau \| |x|^{\frac{\beta}{p}} (\tau \mathbf{a} f_\lambda + \lambda (\nabla f)_\lambda) \|_p.$$

After the change of variables $x \mapsto \lambda^{-1}x$, we get

$$(2.10) \quad \lambda^{\frac{\alpha}{q} - \frac{n}{q} + \frac{\beta}{p} + \frac{n}{p} - 1} \| |x|^{-\frac{\alpha}{q}} f \|_q \leq c_\tau \| |x|^{\frac{\beta}{p}} (\lambda^{-1} \tau \mathbf{a} f + \nabla f) \|_p.$$

The limits of the two sides of the inequality (2.10), as $\lambda \rightarrow 0$ or as $\lambda \rightarrow \infty$, must be the same. If $\tau = 0$ the right-hand side of (2.10) does not depend on λ , so we must have $\frac{\alpha}{q} - \frac{n}{q} + \frac{\beta}{p} + \frac{n}{p} - 1 = 0$.

If $\tau > 0$, we must have

$$\lambda^{\frac{\alpha}{q} - \frac{n}{q} + \frac{\beta}{p} + \frac{n}{p} - 1} \lesssim \begin{cases} \lambda^{-1}, & \lambda \rightarrow 0, \\ 1, & \lambda \rightarrow \infty. \end{cases}$$

so necessarily $\frac{\alpha}{q} - \frac{n}{q} + \frac{\beta}{p} + \frac{n}{p} - 1 \leq 0$. \square

3. UNIQUENESS PROBLEMS

In this section and in Section 4 we use the inequality (1.1) to prove uniqueness questions for solutions of partial differential equations and systems. First, we state some definitions and preliminary results.

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a vector with non-negative integer components; we use the notation $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\partial_x^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f$.

Let $D \subset \mathbb{R}^n$ open and connected and let $1 \leq p < \infty$. Recall that $W_0^{m,p}(D)$ is the closure of $C_0^\infty(D)$ with respect to the Sobolev norm $\|f\|_{W_0^{m,p}(D)} = \sum_{|\alpha|=0}^m \|\partial_x^\alpha f\|_p$. When $m = 1$, and D is bounded in at least one direction, the classical Poincaré inequality states that $\|f\|_{L^p(D)} \leq C \|\nabla f\|_{L^p(D)}$ (see e.g. [2]); thus, the Sobolev norm in $W_0^{1,p}(D)$ is equivalent to $\|\nabla f\|_{L^p(D)}$.

Given the weight $v: D \rightarrow [0, \infty]$ and $1 \leq p < \infty$, we let $W_0^{m,p,v}(D)$ be the closure of $C_0^\infty(D)$ with respect to the norm $\|f\|_{W_0^{m,p,v}(D)} = \sum_{|\alpha|=0}^m \|v^{\frac{1}{p}} \partial_x^\alpha f\|_p$. We use the standard notation $L^{p,v}(D)$ or $L^p(D, v dx)$ for the closure of $C_0^\infty(D)$ with respect to the norm $\|v^{\frac{1}{p}} f\|_p$.

Let $P(\partial) = \sum_{|\alpha|=0}^m a_\alpha \partial_x^\alpha$ be a linear partial differential operator of order $m > 0$ with complex constant coefficients. We let $P(-\partial)u = \sum_{|\alpha|=0}^m \bar{a}_\alpha (-1)^{|\alpha|} \partial_x^\alpha u$.

A *weak solution* (or: a *solution in distribution sense*) of the equation $P(\partial)f = 0$ on a domain $D \subset \mathbb{R}^n$ is a distribution $f \in W^{m,p}(D)$ that satisfies $\int_D f(x) P(-\partial)\phi(x) dx = 0$ for every $\phi \in C_0^\infty(D)$. Weak solutions for non

linear partial differential operators can be defined on a case-by-case basis. See e.g. [18] or other standard textbooks on partial differential equations for details. We will often consider differential inequalities in the form of $|P(\partial)f| \leq |Vf|$ on a given domain D ; by that we mean that the inequality $|P(\partial)f(x)| \leq |Vf(x)|$ is satisfied a.e. in D , i.e., it is satisfied pointwise with the possible exception of a set of measure zero.

3.1. Unique continuation and Carleman method. Let $P(\partial)$ be a homogeneous partial differential operator of order $m \geq 1$. Clearly, $f \equiv 0$ is a solution of the equation $P(\partial)f = 0$ on any domain $D \subset \mathbb{R}^n$. It is natural to ask whether this equation has also nontrivial solutions, i.e., distributions in some suitable Sobolev space that satisfy the equation in distribution sense and are not identically $= 0$. In particular it is natural to ask whether (1), (2) or (3) below are satisfied or not on a given domain D .

(1) *Uniqueness for the Dirichlet problem.* The only solution of the Dirichlet

$$\text{problem } \begin{cases} P(\partial)f = 0, \\ f \in W_0^{m,p}(D) \end{cases} \text{ is } f \equiv 0.$$

(2) *Weak unique continuation property* (or: unique continuation from an open set). Every solution of the equation $P(\partial)f = 0$ which is $\equiv 0$ on an open subset of D is $\equiv 0$.

(3) *Strong continuation property* (or: unique continuation from a point). Let $x_0 \in D$. Every solution of the equation $P(\partial)f = 0$ that satisfies

$$\lim_{r \rightarrow 0} r^{-N} \int_{|x-x_0| < r} |f(x)|^2 dx = 0$$

for every $N > 0$ is $\equiv 0$.

For other relevant unique continuation problems see the survey paper [55].

Historically, the study of unique continuation originated from the uniqueness for the Cauchy problem; an equally strong motivation arose from some fundamental questions in mathematical physics, with the study of the eigenvalues of the time-independent Schrödinger operator $H = -\Delta + V$ as a notable example. See [50] and [51] and also [36] and the references cited there.

In 1939 T. Carleman used in [4] a new weighted Sobolev inequality to show that the Schrödinger operator $H = -\Delta + V$ has the strong unique continuation property when $n = 2$ and V is bounded. Carleman's original idea has permeated the large majority of results on unique continuation. The weighted Sobolev inequality that he used in his proof has been widely generalized and applied to a vast array of problems in unique continuation and control theory.

A *Carleman-type inequality* for a differential operator $P(\partial)$ is a weighted inequality of the form of

$$(3.1) \quad \|\eta^{\tau_k} f\|_q \leq C \|\eta^{\tau_k} P(\partial)f\|_p, \quad f \in C_0^\infty(D),$$

where $\eta: D \rightarrow [0, 1)$, the sequence $\{\tau_k\}_{k \in \mathbb{N}} \subset (0, \infty)$ increases to $+\infty$, the constant C is independent of the sequence of the τ_k and of f , and $1 \leq p \leq q < \infty$. If (3.1) holds with a suitable function η , a version of the argument used in the proof of Theorem 1.3 can be applied to show that the operator $Q(\partial) = P(\partial) - V(x)$ has the unique continuation property (2) or (3) (or some variation of these properties) whenever $V \in L^{\frac{pq}{q-p}}(D)$.

The literature on Carleman inequalities and unique continuation is very extensive. A sample of references on unique continuation problems for second order elliptic operators include the important [32, 33, 35, 52] and the survey papers [58, 37, 55].

The inequality (1.7) in Theorem 1.1 can be viewed as a weighted Carleman-type inequality for the operator $P(\partial)f = |\nabla f|$. To the best of our knowledge, the inequality (1.7) is new in the literature, even when $u(x) \asymp v(x) \asymp 1$.

3.2. Proof of Theorem 1.3. In this section we prove Theorem 1.3 and some corollary.

Proof of Theorem 1.3. Assume for simplicity that $f \equiv 0$ when $x_n < 0$ (the proof is similar in the general case). It is enough to show that $f \equiv 0$ also on the strip $S_\epsilon = \{x: 0 < x_n < \epsilon\}$, where $\epsilon > 0$ will be determined during the proof. Using Theorem 1.1 (a) with $\mathbf{a} = (0, \dots, 0, 1)$, $\tau \geq 1$ and $c_\tau \leq c_1$ (see Remark 1.2), the differential inequality (1.18) and Hölder's inequality with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, we can write the following chain of inequalities:

$$\begin{aligned} \|e^{-\tau x_n} f u^{\frac{1}{q}}\|_{L^q(S_\epsilon)} &\leq c_1 \|e^{-\tau x_n} \nabla f v^{\frac{1}{p}}\|_{L^p(\mathbb{R}^n)} \\ &\leq c_1 \|e^{-\tau x_n} \nabla f v^{\frac{1}{p}}\|_{L^p(S_\epsilon)} + c_1 \|e^{-\tau x_n} \nabla f v^{\frac{1}{p}}\|_{L^p(\{x_n > \epsilon\})} \\ &\leq c_1 \|e^{-\tau x_n} f V v^{\frac{1}{p}}\|_{L^p(S_\epsilon)} + c_1 e^{-\tau \epsilon} \|\nabla f v^{\frac{1}{p}}\|_{L^p(\{x_n > \epsilon\})} \\ &\leq c_1 \|V v^{\frac{1}{p}} u^{-\frac{1}{q}}\|_{L^r(S_\epsilon \cap \text{supp } f)} \|e^{-\tau x_n} f u^{\frac{1}{q}}\|_{L^q(S_\epsilon)} + C' e^{-\tau \epsilon}. \end{aligned}$$

Here, $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ and we have let $C' = c_1 \|\nabla f v^{\frac{1}{p}}\|_{L^p(\{x_n > \epsilon\})}$. Note that C' does not depend on τ .

Since $V \in L^r(\text{supp } f, v^{\frac{r}{p}} u^{-\frac{r}{q}} dx)$ we can chose $\epsilon > 0$ so that $c_1 \|V v^{\frac{1}{p}} u^{-\frac{1}{q}}\|_{L^r(S_\epsilon \cap \text{supp } f)} < \frac{1}{2}$. From the chain of inequalities above, follows that

$$\|e^{-\tau x_n} f u^{\frac{1}{q}}\|_{L^q(S_\epsilon)} \leq \frac{1}{2} \|e^{-\tau x_n} f u^{\frac{1}{q}}\|_{L^q(S_\epsilon)} + C' e^{-\tau \epsilon}.$$

We gather

$$\frac{1}{2} \|e^{\tau(\epsilon - x_n)} f u^{\frac{1}{q}}\|_{L^q(S_\epsilon)} \leq C'.$$

Since $\epsilon - x_n > 0$ on S_ϵ , if $f \not\equiv 0$ the left-hand side of this inequality goes to infinity when τ goes to infinity; this is a contradiction because C' does not depend on τ and so necessarily $f \equiv 0$ in S_ϵ . \square

Corollary 3.1. *Let p , q and γ be as in Theorem 1.1 (a). Let $u = |x|^{(-\alpha_1, -\alpha_2)}$ and $v = |x|^{(\beta_1, \beta_2)}$, with $0 \leq \alpha_1 \leq n(1 - \frac{q}{\gamma} + \frac{q}{n})$, $\alpha_2 \geq 0$ and $0 \leq \beta_1 \leq n(\frac{p}{\gamma} - 1)$, $\beta_2 \geq n(\frac{p}{\gamma} - 1)$. Let $V = |x|^{(s_1, s_2)}$, with*

$$(3.2) \quad s_1 > -\frac{n}{r} - \frac{\alpha_1}{q} - \frac{\beta_1}{p},$$

and, if $\text{supp } f$ is unbounded,

$$(3.3) \quad s_2 < -\frac{\alpha_2}{q} - \frac{\beta_2}{p} - \frac{n}{r}.$$

Then, every solution of the differential inequality $|\nabla f| \leq V|f|$ is $\equiv 0$.

Proof. The weights u and v are as in Corollary 1.2, so the inequality (1.7) holds with $\tau > 0$. By Theorem 1.3, every solution of the differential inequality $|\nabla f| \leq V|f|$ is $\equiv 0$ whenever $Vv^{\frac{1}{p}}u^{-\frac{1}{q}} \in L^r(\text{supp } f)$. We can see at once that $Vv^{\frac{1}{p}}u^{-\frac{1}{q}} = |x|^{(t_1, t_2)} \in L^r(\text{supp } f)$ if and only if $t_1 = s_1 + \frac{\alpha_1}{q} + \frac{\beta_1}{p} > -\frac{n}{r}$ and, if $\text{supp } f$ is unbounded, $t_2 = s_2 + \frac{\alpha_2}{q} + \frac{\beta_2}{p} < -\frac{n}{r}$, which is equivalent to (3.2) and (3.3). This concludes the proof. \square

Remark 3.1. From the inequalities above and the assumptions on α_j , β_j , and γ' (see Corollary 1.2) follows that

$$t_1 \leq s_1 + \frac{n}{q} \left(1 - \frac{q}{\gamma'} + \frac{q}{n}\right) + \frac{n}{p} \left(\frac{p}{\gamma'} - 1\right) = s_1 - \frac{n}{r} + 1.$$

$$t_2 \geq s_2 + \frac{n}{p} \left(\frac{p}{\gamma'} - 1\right) = s_2 + \frac{n}{\gamma'} - \frac{n}{p} > s_2 - \frac{n}{p} + 1.$$

The condition $t_1 > -\frac{n}{r}$ yields $s_1 > -1$. We can see at once that $t_2 < -\frac{n}{r}$ yields $s_2 < \frac{n}{q} - 1$. In particular, $V = |x|^{-1+\epsilon}$ with $0 < \epsilon < \frac{n}{q}$, satisfies the assumptions of Corollary 3.1. If f has compact support, then we can omit the condition on t_2 and assume only $\epsilon > 0$.

Potentials $V(x) = C|x|^{-s}$, with $s, C > 0$ are known as *Hardy potentials* in the literature. They appear in the relativistic Schrödinger equations and in problem of stability of relativistic matter in magnetic fields. See e.g. [27] and the introduction to [20] and [21], just to cite a few.

It is proved in [16] that when \mathcal{L} is the Dirac operator in dimension $n \geq 2$ (see Section 4.2) the differential inequality $|\mathcal{L}f| \leq C|x|^{-1}|f|$ has the strong unique continuation property from the point $x_0 = 0$ whenever $C \leq 1$. We conjecture that also the differential inequalities $|\nabla f| \leq C|x|^{-1}|f|$ has the strong unique continuation property from the origin when C is sufficiently small.

3.3. Proof of Theorem 1.4. Recall that the solution f of the Dirichlet problem (1.19) is intended in distribution sense, i.e., f satisfies

$$(3.4) \quad \int_D \langle \nabla \psi, \nabla f \rangle |\nabla f|^{p-2} v \, dx = \int_D \psi V f |f|^{p-2} v \, dx$$

for every $\psi \in C_0^\infty(D)$. To prove Theorem 1.4 we need two important lemmas:

Lemma 3.2. *Suppose that the weighted gradient inequality*

$$(3.5) \quad \|u^{\frac{1}{q}} f\|_q \leq c_0 \|v^{\frac{1}{p}} \nabla f\|_p, \quad f \in C_0^\infty(D)$$

holds with exponents $1 \leq p, q < \infty$. Then the space $W_0^{1,p,v}(D)$ embeds into $L^q(D, u dx)$ and $\|f\|_{L^q(D, u dx)} \leq c_0 \|\nabla f\|_{L^p(D, v dx)}$.

Proof. Fix $f \in W_0^{1,p,v}(D)$; let $\{f_n\}_{n \in \mathbb{N}} \subset C_0^\infty(D)$ be a sequence that converges to f in the Sobolev norm $\|\cdot\|_{W_0^{1,p,v}(D)}$. Thus, $\{f_n\}$ is a Cauchy sequence in $W_0^{1,p,v}(D)$; for every $\epsilon > 0$ we can chose $N > 0$ such that

$$\|f_n - f_m\|_{W_0^{1,p,v}(D)} = \|v^{\frac{1}{p}}(f_n - f_m)\|_{L^p(D)} + \|v^{\frac{1}{p}} \nabla(f_n - f_m)\|_{L^p(D)} < \epsilon$$

whenever $n, m > N$; thus, $\|v^{\frac{1}{p}} \nabla(f_n - f_m)\|_{L^p(D)} < \epsilon$. By (3.5),

$$\|u^{\frac{1}{q}}(f_n - f_m)\|_{L^q(D)} \leq c_0 \|v^{\frac{1}{p}} \nabla(f_n - f_m)\|_{L^p(D)} < c_0 \epsilon.$$

We have proved that $\{f_n\}$ is a Cauchy sequence in $L^q(D, u dx)$ (which is complete) and so it converges to f also in $L^q(D, u dx)$. We gather

$$\begin{aligned} \|f\|_{L^q(D, u dx)} &= \lim_{n \rightarrow \infty} \|f_n\|_{L^q(D, u dx)} \leq c_0 \lim_{n \rightarrow \infty} \|\nabla f_n\|_{L^p(D, v dx)} \\ &= c_0 \|\nabla f\|_{L^p(D, v dx)} \end{aligned}$$

as required. \square

Lemma 3.3. *Suppose that the weighted gradient inequality (3.5) holds with $1 < p < q$. Let f be a solution to the Dirichlet problem (1.19), with $|V|^{\frac{1}{p}} \in L^r(D, v^{\frac{r}{p}} u^{-\frac{r}{q}} dx)$. We have*

$$\int_D |\nabla f|^p v dx = \int_D V |f|^p v dx.$$

Proof. Let $\{\psi_n\}$ be a sequence of functions in $C_0^\infty(D)$ that converges to \bar{f} , the complex conjugate of f , in $W_0^{1,p,v}(D)$. We show first that $\lim_{n \rightarrow \infty} \int_D \langle \nabla \psi_n, \nabla f \rangle |\nabla f|^{p-2} v dx = \int_D |\nabla f|^p v dx$. Indeed,

$$\begin{aligned} &\int_D (\langle \nabla \psi_n, \nabla f \rangle |\nabla f|^{p-2} - |\nabla f|^p) v dx \\ &= \int_D (\langle \nabla \psi_n, \nabla f \rangle |\nabla f|^{p-2} - \langle \nabla \bar{f}, \nabla f \rangle |\nabla f|^{p-2}) v dx \\ &= \int_D \langle \nabla \psi_n - \nabla \bar{f}, \nabla f |\nabla f|^{p-2} \rangle v dx \\ &\leq \|(\nabla \psi_n - \nabla \bar{f}) v^{\frac{1}{p}}\|_p \| |\nabla f|^{p-1} v^{\frac{1}{p}} \|_{p'} \\ &= \|\nabla(\psi_n - \bar{f}) v^{\frac{1}{p}}\|_p \| |\nabla f| v^{\frac{1}{p}} \|_{\frac{p}{p'}} \end{aligned}$$

and $\lim_{n \rightarrow \infty} \|\nabla(\psi_n - \bar{f}) v^{\frac{1}{p}}\|_p = 0$, as required.

In view of (3.4), we have that

$$\int_D \langle \nabla \psi_n, \nabla f \rangle |\nabla f|^{p-2} v \, dx = \int_D \psi_n V f |f|^{p-2} v \, dx;$$

to complete the proof it suffices to show that

$$\lim_{n \rightarrow \infty} \int_D \psi_n V f |f|^{p-2} v \, dx = \int_D V |f|^p v \, dx$$

when $|V|^{\frac{1}{p}} \in L^r(D, v^{\frac{r}{p}} u^{-\frac{r}{q}} dx)$. By Lemma 3.2, ψ_n converges to \bar{f} in $L^q(D, u \, dx)$. Using Holder's inequality with $\frac{r}{r} + \frac{q}{q} = 1$, we gather

$$\begin{aligned} \int_D (\psi_n V f |f|^{p-2} - V |f|^p) v \, dx &\leq \int_D |V v u^{-\frac{p}{q}}| |f|^{p-1} |\psi_n - \bar{f}| u^{\frac{p}{q}} \, dx \\ &\leq \left(\int_D |v V u^{-\frac{p}{q}}|^{\frac{r}{p}} \, dx \right)^{\frac{p}{r}} \left(\int_D |f|^{(p-1)\frac{q}{p}} |\psi_n - \bar{f}|^{\frac{q}{p}} u \, dx \right)^{\frac{p}{q}} \\ &= \left(\int_D (|V|^{\frac{1}{p}} v^{\frac{1}{p}} u^{-\frac{1}{q}})^r \, dx \right)^{\frac{p}{r}} \left(\int_D |f|^{\frac{q}{p'}} |\psi_n - \bar{f}|^{\frac{q}{p}} u \, dx \right)^{\frac{p}{q}}. \end{aligned}$$

We let $C = \left(\int_D (|V|^{\frac{1}{p}} v^{\frac{1}{p}} u^{-\frac{1}{q}})^r \, dx \right)^{\frac{p}{r}}$ and we apply Hölder's inequality (with $\frac{1}{p} + \frac{1}{p'} = 1$) to the remaining integral. We obtain

$$\begin{aligned} \int_D (V f |f|^{p-2} \psi_n - V |f|^p) v \, dx &\leq C \left(\int_D |f|^q u \, dx \right)^{\frac{p}{q p'}} \left(\int_D |\psi_n - \bar{f}|^q u \, dx \right)^{\frac{1}{q}} \\ (3.6) \qquad \qquad \qquad &= C \|f u^{\frac{1}{q}}\|_q^{p-1} \|(\psi_n - \bar{f}) u^{\frac{1}{q}}\|_q. \end{aligned}$$

By assumption, $\lim_{n \rightarrow \infty} \|(\psi_n - \bar{f}) u^{\frac{1}{q}}\|_q = 0$; by Lemma 3.2, $\|f u^{\frac{1}{q}}\|_q < \infty$, and so the right-hand side of (3.6) goes to zero when $n \rightarrow \infty$ as required. \square

Proof of Theorem 1.4. Since the weights u and v are as in Theorem 1.1, the weighted gradient inequality (3.5) holds. By Lemma 3.3 and Hölder's inequality (with $\frac{p}{q} + \frac{p}{r} = 1$) we have the following chain of inequalities

$$\begin{aligned} \|f u^{\frac{1}{q}}\|_{L^q(D)}^p &\leq c_0^p \|\nabla f v^{\frac{1}{p}}\|_{L^p(D)}^p = c_0^p \int_D v |\nabla f|^p \, dx \\ &= c_0^p \int_D V v |f|^p \, dx \leq c_0^p \int_D V_+ v u^{-\frac{p}{q}} |f|^p u^{\frac{p}{q}} \, dx \\ &\leq c_0^p \left(\int_D V_+^{\frac{r}{p}} v^{\frac{r}{p}} u^{-\frac{r}{q}} \, dx \right)^{\frac{p}{r}} \left(\int_D |f|^q u \, dx \right)^{\frac{p}{q}} \\ &\leq c_0^p \|V_+^{\frac{1}{p}}\|_{L^r(D, v^{\frac{r}{p}} u^{-\frac{r}{q}} \, dx)}^p \|f u^{\frac{1}{q}}\|_{L^q(D)}^p. \end{aligned}$$

We obtain $\|f u^{\frac{1}{q}}\|_{L^q(D)} (1 - c_0^p \|V_+^{\frac{1}{p}}\|_{L^r(D, v^{\frac{r}{p}} u^{-\frac{r}{q}} dx)}) \leq 0$; this inequality is possible only if either $c_0^p \|V_+^{\frac{1}{p}}\|_{L^r(D, v^{\frac{r}{p}} u^{-\frac{r}{q}} dx)} \geq 1$ or $f \equiv 0$ in D . \square

4. LINEAR SYSTEMS OF PDE AND THE DIRAC OPERATOR

We use the following notation: If $\vec{p} = (p_1, \dots, p_m) \in \mathbb{R}^m$, we let $|\vec{p}| = (p_1^2 + \dots + p_m^2)^{\frac{1}{2}}$. If \mathbf{A} is a matrix with rows A_1, \dots, A_N , we will let $|\mathbf{A}| = (|A_1|^2 + \dots + |A_N|^2)^{\frac{1}{2}}$. Note that, by Cauchy Schwartz inequality,

$$|\mathbf{A}\vec{p}| = (\langle A_1, \vec{p} \rangle^2 + \dots + \langle A_N, \vec{p} \rangle^2)^{\frac{1}{2}} \leq (|A_1|^2 + \dots + |A_N|^2)^{\frac{1}{2}} |\vec{p}| = |\mathbf{A}| |\vec{p}|.$$

Let $\vec{F} = (f_1, \dots, f_N) \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^N)$. We denote with $\nabla \vec{F}$ the $N \times n$ matrix whose rows are $\nabla f_1, \dots, \nabla f_N$.

Unless otherwise specified, we assume that p, q, u and v are as in Theorem 1.1 (a) and that $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$.

In this section we use the Carleman inequality (1.1) to prove unique continuation properties of systems of linear partial differential equations of the first order.

4.1. Linear systems of PDE. Most of the first order systems considered in the literature are in the form of

$$(4.1) \quad \sum_{j=1}^n \mathbf{L}_j(x) \partial_{x_j} \vec{F} = V(x) \vec{F},$$

where $\vec{F} = (f_1, \dots, f_N)$ and the $\mathbf{L}_j(x)$ and V are $M \times N$ matrices defined in a domain $D \subset \mathbb{R}^n$. We let $\mathbf{L}(x)(\vec{F}) = \sum_{j=1}^n \mathbf{L}_j(x) \partial_{x_j} \vec{F}$. Differential inequalities in the form of

$$(4.2) \quad |\mathbf{L}(x)\vec{F}| \leq |\mathbf{V}(x)\vec{F}|$$

are also considered. In some of early papers on the subject, it is proved that solutions of elliptic systems in the form of (4.1) that vanish of sufficiently high order at the origin are $\equiv 0$; see [7, 15, 47] and the references cited in these papers for definitions of elliptic systems. A classical method of proof is to reduce the systems to (quasi-) diagonal form; this approach requires conditions on the regularity and the multiplicity of the eigenvalues of the system that are often difficult to check; see [9, 24, 29, 56]. The strong continuation properties of systems of complex analytic vector fields in the form of $\vec{L}u = 0$ defined on a real-analytic manifold is proved in [1].

We have found only a few papers in the literature where the Carleman method is used to prove unique continuation properties of first-order systems. The Carleman method often allows to prove unique continuation results for the differential

inequality (4.2), often with a singular potential V . In [14, Theorem 4.1] Carleman estimates are used to prove that (4.2) has the weak unique continuation property when \vec{L} is a system of vector fields on a pseudoconcave Cauchy-Riemann (CR) with some specified conditions and V is bounded. In [44] and [45] T. Okaji considers systems in two independent variables, Maxwell's equations, and the Dirac operator; he proved that the differential inequalities (4.2) with $|V(x)| \asymp |x|^{-1}$ has the strong unique continuation property using sophisticated $L^2 \rightarrow L^2$ Carleman estimates. See also [54], which improves results in [44].

We prove the following

Theorem 4.1. *Let $\vec{F} \in W_0^{1,p,v}(\mathbb{R}^n, \mathbb{R}^N)$ be a solution of the differential inequality (4.2). Assume that \vec{F} satisfies also*

$$(4.3) \quad |\nabla \vec{F}| \lesssim |\mathbf{L}(x)\vec{F}|.$$

If $|\mathbf{V}| \in L^r(\text{supp } \vec{F}, u^{-\frac{r}{q}} v^{\frac{r}{p}} dx)$, with $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$ and \vec{F} vanishes on one side of a hyperplane, then $\vec{F} \equiv 0$.

In particular, for power weights u, v as in Remark 3.1, the differential inequality (4.2) does not have solutions with compact support support that satisfy also (4.3) if $V \asymp |x|^{-1+\epsilon}$ for some $\epsilon > 0$.

Our unique continuation result is weaker than other results in the literature, but it applies to first-order systems of linear partial differential equations that satisfy only the assumptions (4.3). Furthermore, we consider solutions in weighted Sobolev spaces and potential in weighted L^r spaces that, to the best of our knowledge, have not been considered in other papers.

Before proving Theorem 4.1 we prove the following Lemma, which is an easy consequence of Theorem 1.1.

Lemma 4.2. *Let \mathbf{A} be a $N \times N$ invertible matrix. Under the assumptions of Theorem 1.1 (a), the following inequality holds for all $\vec{F} \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^N)$ and $\tau \geq 0$*

$$(4.4) \quad \|e^{-\tau\ell(x)} u^{\frac{1}{q}} \vec{F}\|_q \leq c_{\tau,N,\mathbf{A}} \|e^{-\tau\ell(x)} v^{\frac{1}{p}} \mathbf{A} \nabla \vec{F}\|_p,$$

where $c_{\tau,N,\mathbf{A}} = N C_{\mathbf{A}} c_\tau$ and c_τ is the constant in (1.7).

Proof. Using Theorem 1.1 (a), the elementary inequalities

$$|\vec{F}| = (f_1^2 + \dots + f_N^2)^{\frac{1}{2}} \leq |f_1| + \dots + |f_N|, \quad |f_j| \leq |\vec{F}|,$$

and Minkowsky's inequality, we obtain

$$\begin{aligned} \|e^{-\tau\ell(x)} u^{\frac{1}{q}} \vec{F}\|_q &\leq \sum_{j=1}^N \|e^{-\tau\ell(x)} u^{\frac{1}{q}} f_j\|_q \leq c_\tau \sum_{j=1}^N \|e^{-\tau\ell(x)} v^{\frac{1}{p}} \nabla f_j\|_p \\ &\leq c_\tau N \|e^{-\tau\ell(x)} v^{\frac{1}{p}} \nabla \vec{F}\|_p. \end{aligned}$$

If \mathbf{A} is invertible, then, for every $\xi \in \mathbb{R}^n$, we have that $|\mathbf{A}\vec{\xi}| \geq C_{\mathbf{A}}^{-1}|\xi|$ for some $C_{\mathbf{A}} > 0$; thus,

$$\|e^{-\tau\ell(x)}u^{\frac{1}{q}}\vec{F}\|_q \leq c_{\tau}NC_{\mathbf{A}}\|e^{-\tau\ell(x)}v^{\frac{1}{p}}\mathbf{A}\nabla\vec{F}\|_p$$

as required. \square

Proof of Theorem 4.1. We argue as in the proof of Theorem 1.3. Without loss of generality, we can assume that $\vec{F} \equiv 0$ when $x_n < 0$ and $\mathbf{A} = \mathbf{I}$, where \mathbf{I} is the $N \times N$ identity matrix. For simplicity of notation, we denote with c_1 the constant $c_{1,N,\mathbf{I}}$ in Lemma 4.2. We show that $\vec{F} \equiv 0$ also on the strip $S_{\epsilon} = \{x : 0 < x_n < \epsilon\}$, for some $\epsilon > 0$ to be determined during the proof.

Using (4.4) with $\ell(x) = x_n$ and $\tau \geq 1$, the differential inequality (4.3), Hölder's inequality and Remark 1.2, we obtain

$$\begin{aligned} \|e^{-\tau x_n}\vec{F}u^{\frac{1}{q}}\|_{L^q(S_{\epsilon})} &\leq c_1\|e^{-\tau x_n}\nabla\vec{F}v^{\frac{1}{p}}\|_{L^p(\mathbb{R}^n)} \\ &\leq c_1\|e^{-\tau x_n}\nabla\vec{F}v^{\frac{1}{p}}\|_{L^p(S_{\epsilon})} + c_1\|e^{-\tau x_n}\nabla\vec{F}v^{\frac{1}{p}}\|_{L^p(\{x_n>\epsilon\})} \\ &\leq c_1C\|e^{-\tau x_n}\mathbf{L}(x)(\nabla\vec{F})v^{\frac{1}{p}}\|_{L^p(S_{\epsilon})} + c_1e^{-\tau\epsilon}\|\nabla\vec{F}v^{\frac{1}{p}}\|_{L^p(\{x_n>\epsilon\})} \\ &\leq c_1C\|e^{-\tau x_n}\mathbf{V}\vec{F}v^{\frac{1}{p}}\|_{L^p(S_{\epsilon})} + c_1e^{-\tau\epsilon}\|\nabla\vec{F}v^{\frac{1}{p}}\|_{L^p(\{x_n>\epsilon\})} \\ &\leq c_1C\|\mathbf{V}\|v^{\frac{1}{p}}u^{-\frac{1}{q}}\|_{L^r(S_{\epsilon}\cap\text{supp}\vec{F})}\|e^{-\tau x_n}\vec{F}u^{\frac{1}{q}}\|_{L^q(S_{\epsilon})} + C'e^{-\tau\epsilon}, \end{aligned}$$

where we have let $C' = c_1\|\nabla\vec{F}v^{\frac{1}{p}}\|_{L^p(\{x_n>\epsilon\})}$.

Since $\|\mathbf{V}\| \in L^r(\text{supp}\vec{F}, u^{-\frac{r}{q}}v^{\frac{r}{p}}dx)$ we can chose $\epsilon > 0$ so that $c_1C\|\mathbf{V}\|v^{\frac{1}{p}}u^{-\frac{1}{q}}\|_{L^r(S_{\epsilon}\cap\text{supp}\vec{F})} < \frac{1}{2}$. We have obtained

$$\|e^{-\tau x_n}\vec{F}u^{\frac{1}{q}}\|_{L^q(S_{\epsilon})} \leq \frac{1}{2}\|e^{-\tau x_n}\vec{F}u^{\frac{1}{q}}\|_{L^q(S_{\epsilon})} + C'e^{-\tau\epsilon}.$$

In view of $\epsilon - x_n > 0$ on S_{ϵ} , the left-hand side of this inequality goes to infinity with τ unless $\vec{F} \equiv 0$ on S_{ϵ} ; this is a contradiction because C' does not depend on τ , and so $\vec{F} \equiv 0$ in S_{ϵ} . \square

Let $\mathbf{G}_1(x), \dots, \mathbf{G}_n(x)$ be $N \times n$ matrices defined on a domain $D \subset \mathbb{R}^n$. We consider the operator

$$\mathbf{G}(\vec{F}) = \mathbf{G}(f_1, \dots, f_N) = \sum_{j=1}^N \mathbf{G}_j(x)f_j$$

with $f_j \in C_0^{\infty}(D)$.

In [39], systems in the form of $\nabla F = \mathbf{G}\vec{F}$ are considered. These systems can be used to model linear elasticity (in curvilinear coordinates) of linearly elastic shells. See [5] and the references cited there. We prove the following

Theorem 4.3. *Let $\vec{F} \in W_0^{1,p,v}(D, \mathbb{R}^N)$ be a solution of the differential inequality*

$$(4.5) \quad |\nabla \vec{F}| \lesssim |\mathbf{G}\vec{F}|.$$

If $|\mathbf{G}| \in L^r(\text{supp } \vec{F}, u^{-\frac{r}{q}} v^{\frac{r}{p}} dx)$, with $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$, and \vec{F} vanishes on one side of a hyperplane, then $\vec{F} \equiv 0$.

Proof. Assume for simplicity that $\vec{F} \equiv 0$ when $x_n < 0$ (the proof is similar in the general case). We show that $\vec{F} \equiv 0$ also on the strip $0 < x_n < \epsilon$, for some $\epsilon > 0$ to be determined during the proof. As in the proof of Theorem 4.1, we use (4.4) with $\mathbf{A} = \mathbf{I}$, $\ell(x) = x_n$ and $\tau \geq 1$. For each $j = 1, \dots, N$, we use the differential inequality (4.5) and Hölder's inequality in the following chain of inequalities

$$\begin{aligned} \|e^{-\tau x_n} \vec{F} u^{\frac{1}{q}}\|_{L^q(S_\epsilon)} &\leq c_1 \|e^{-\tau x_n} \nabla \vec{F} v^{\frac{1}{p}}\|_{L^p(\mathbb{R}^n)} \\ &\leq c_1 \|e^{-\tau x_n} \nabla \vec{F} v^{\frac{1}{p}}\|_{L^p(S_\epsilon)} + c_1 \|e^{-\tau x_n} \nabla \vec{F} v^{\frac{1}{p}}\|_{L^p(\{x_n > \epsilon\})} \\ &\leq c_1 C \|e^{-\tau x_n} \mathbf{G}\vec{F} v^{\frac{1}{p}}\|_{L^p(S_\epsilon)} + c_1 e^{-\tau \epsilon} \|\nabla \vec{F} v^{\frac{1}{p}}\|_{L^p(\{x_n > \epsilon\})} \\ &\leq c_1 C \sum_{j=1}^N \|e^{-\tau x_n} |\mathbf{G}_j| f_j v^{\frac{1}{p}}\|_{L^p(S_\epsilon)} + c_1 e^{-\tau \epsilon} \|\nabla \vec{F} v^{\frac{1}{p}}\|_{L^p(\{x_n > \epsilon\})} \\ &\leq c_1 C \sum_{j=1}^N \| |\mathbf{G}_j| v^{\frac{1}{p}} u^{-\frac{1}{q}} \|_{L^r(S_\epsilon \cap \text{supp } \vec{F})} \|e^{-\tau x_n} f_j u^{\frac{1}{q}}\|_{L^q(S_\epsilon)} + C' e^{-\tau \epsilon} \\ &\leq c_1 C N \| |\mathbf{G}| v^{\frac{1}{p}} u^{-\frac{1}{q}} \|_{L^r(S_\epsilon \cap \text{supp } \vec{F})} \|e^{-\tau x_n} |\vec{F}| u^{\frac{1}{q}}\|_{L^q(S_\epsilon)} + C' e^{-\tau \epsilon}, \end{aligned}$$

where we have let $C' = c_1 \|\nabla \vec{F} v^{\frac{1}{p}}\|_{L^p(\{x_n > \epsilon\})}$.

We chose $\epsilon > 0$ so that $c_1 C N \| |\mathbf{G}| v^{\frac{1}{p}} u^{-\frac{1}{q}} \|_{L^r(S_\epsilon \cap \text{supp } \vec{F})} < \frac{1}{2}$. We gather

$$\|e^{-\tau x_n} \vec{F} u^{\frac{1}{q}}\|_{L^q(S_\epsilon)} \leq \frac{1}{2} \|e^{-\tau x_n} \vec{F} u^{\frac{1}{q}}\|_{L^q(S_\epsilon)} + C' e^{-\tau \epsilon}$$

which gives

$$\frac{1}{2} \|e^{\tau(\epsilon - x_n)} \vec{F} u^{\frac{1}{q}}\|_{L^q(S_\epsilon)} \leq C',$$

and we can conclude the proof as in Theorem 4.1. \square

Remark 4.1. It is shown in [39] that the $W^{1,1}(D, \mathbb{R}^n)$ solutions of the system $\nabla \vec{F} = \mathbf{G}\vec{F}$, with $\mathbf{G} \in L^1(D, \mathbb{R}^{(n \times n) \times n})$, cannot vanish on an open set. The proof in [39] does not use Carleman inequalities.

4.2. The Dirac operator. Let α_j , $j = 0, \dots, n$, be $N \times N$ matrices which satisfy the following relations.

$$(4.6) \quad \alpha_j^* = \alpha_j, \quad \alpha_j^2 = I, \quad \alpha_j \alpha_k + \alpha_k \alpha_j = 0, \quad j \neq k$$

(we also say that the α_j form a basis of a Clifford algebra). It is known that for (4.6) to hold, N must be in the form $2^{\lfloor \frac{n+1}{2} \rfloor} m$, with $m > 0$ integer

The (n -dimensional) Dirac operator associated to the matrices α_j is a matrix value operator, initially defined on $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{N \times N})$ as follows.

$$\mathcal{L}U = -i \sum_{j=1}^n \alpha_j \partial_{x_j} U.$$

Here, $\partial_{x_i} U$ is a matrix whose entries are the partial derivative of the entries of U . We can use (4.6) to show that $\mathcal{L} \circ \mathcal{L}U = -\Delta U I$, where I is the identity matrix. When $U = fI$, where $f \in C_0^\infty(\mathbb{R}^n)$, we can see at once that $(\mathcal{L}(fI))^2 = -I|\nabla f|^2$. Thus, a Dirac operators can be viewed as a generalization of the gradient operator and a square root of the Laplacian.

There is a lot of literature on the Dirac operator and its role in several domains of mathematics and physics See e.g. [6]. For example, the Dirac equation which describes free relativistic electrons is represented by

$$i\hbar \partial_t \psi(t, x) = H_0 \psi(t, x),$$

where H_0 is given explicitly by the 4×4 matrix-valued differential expression

$$H_0 = -i\hbar c \sum_{j=1}^3 \alpha_j \partial_{x_j} + \alpha_0 m c^2.$$

Here, c is the speed of light, m is a mass of a particle and \hbar is the Planck's constant.

In [16] is proved that the the differential inequality

$$(4.7) \quad |\mathcal{L}U| \leq |VU|$$

where $V(x)$ is a $N \times N$ matrix, has the strong unique continuation property from the origin whenever $V(x) \leq C|x|^{-1}$, with $0 \leq C \leq 1$. It is also proved in [16] that the condition $C \leq 1$ cannot be improved. See also [30] and the references cited there. We prove the following

Theorem 4.4. *Let $f \in W_0^{1,p,v}(D)$ be a solution of the differential inequality (4.7). If $|\mathbf{V}| \in L^r(\text{supp } f, u^{-\frac{r}{q}} v^{\frac{r}{p}} dx)$ with $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$ and f vanishes on one side of a hyperplane, then $f \equiv 0$.*

Proof. Since $\mathcal{L}(fI) \cdot \mathcal{L}(fI) = -I|\nabla f|^2$, we can see at once that

$$|\nabla f| = |\mathcal{L}(fI) \cdot \mathcal{L}(fI)| \leq |\mathcal{L}(fI)|^2$$

With this observation, the proof of Theorem 4.4 is almost a line-by-line repetition of the proof of Theorem 4.1. We leave the details to the reader. □

REFERENCES

- [1] R. F. Barostichi, P. D. Cordaro, and G. Petronilho, *Strong unique continuation for systems of complex vector fields*. Bull. Sci. Math. 138 (2014), no. 4, 457–469.
- [2] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer Science & Business Media (2010).
- [3] L. Caffarelli, R. Kohn, and L. Nirenberg, *First order interpolation inequalities with weight*. Compos. Math. 53 (1984), 259–275.
- [4] T. Carleman, *Sur un probleme d'unicité pur les systemes d'equations aux derivees partielles a deux variables independantes*. Ark. Mat., Astr. Fys. 26 (1939), no. 17, 9 pp.
- [5] P.G. Ciarlet and C. Mardare, *On rigid and infinitesimal rigid displacements in shell theory*. J. Math. Pures Appl. (9) 83 (2004), no. 1, 1–15.
- [6] J. Cnops, *An Introduction to Dirac Operators on Manifolds*. Springer Science & Business Media, 2012.
- [7] C. Cosner, *On the definition of ellipticity for systems of partial differential equations*. J. Math. Anal. Appl. 158 (1991), no. 1, 80–93.
- [8] M. Cuesta, and H. Ramos Quoirin, *A weighted eigenvalue problem for the p -Laplacian plus a potential*. Nonlinear Diff. Eq. Appl. 16 (2009), 469–491.
- [9] A. Douglis, *Uniqueness in Cauchy problems for elliptic systems of equations*. Comm. Pure Appl. Math. 6 (1953), 291–298.
- [10] L. Dupaigne, *Stable Solutions of Elliptic Partial Differential Equations*. Monographs and Surveys in Pure and Applied Mathematics, CRC press, 2011.
- [11] L. De Carli, J. Edward, S. Hudson, and M. Leckband, *Minimal support results for Schrödinger's equation*. Forum Math. 27 (2015), no. 1, 343–371.
- [12] L. De Carli, D. Gorbachev, and S. Tikhonov, *Pitt inequalities and restriction theorems for the Fourier transform*. Rev. Mat. Iberoam. 33 (2017), no. 3, 789–808.
- [13] L. De Carli and S. Hudson, *Geometric remarks on the level curves of harmonic functions*. Bull. London Math. Soc. 42 (2010), no. 1, 83–95.
- [14] L. De Carli and M. Nacinovich, *Unique continuation in abstract pseudoconvex CR manifolds*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 27 (1998), no. 1, 27–46.
- [15] A. Douglis and L. Nirenberg, *Interior estimates for elliptic systems of partial differential equations*. Comm. Pure Appl. Math. 8 (1955), 503–538.
- [16] L. De Carli and T. Okaji, *Strong Unique continuation for the Dirac operator*. Publ. Res. Inst. Math. Sci. 35 (1999), no. 6, 825–846.
- [17] J. Edward, S. Hudson, and M. Leckband, *Existence problems for the p -Laplacian*. Forum Math. 27 (2013), no. 2, 1203–1225.
- [18] L.C. Evans, *Partial Differential Equations*. American Mathematical Society, Providence, 1998.
- [19] E. Fabes, C. Kenig, and R. Serapioni, *The local regularity of solutions of degenerate elliptic equations*. Comm. Partial Differential Equations 7 (1982), no. 1, 77–116.
- [20] M.M. Fall, *Semilinear elliptic equations for the fractional Laplacian with Hardy potential*. arXiv:1109.5530v4. (2011).
- [21] M.M. Fall and V. Felli, *Unique continuation properties for relativistic Schrödinger operators with a singular potential*. Discrete Contin. Dyn. Syst. 35 (2015), no. 12, 5827–5867.
- [22] F. Ferrari and E. Valdinoci, *Some weighted Poincaré inequalities*. Indiana Univ. Math. J. 58 (4) (2009), 1619–1637.
- [23] R. Filippucci, P. Pucci, and M. Rigoli, *Nonlinear weighted p -Laplacian elliptic inequalities with gradient terms*. Commun. Contemp. Math. 12 (2010), no. 3, 501–535.
- [24] K. Hayashida, *Unique continuation theorem of elliptic systems of partial differential equations*. Proc. Japan Acad. 38 (1962), 630–635.

- [25] H.P. Heinig, *Weighted Sobolev inequalities for gradients*. Harmonic analysis and applications, 17–23, Appl. Numer. Harmon. Anal., Birkhäuser, Boston, MA, 2006.
- [26] H.P. Heinig, *Weighted norm inequalities for classes of operators*, Indiana Univ. Math. J. 33 (1984), no. 4, 573–582.
- [27] I.W. Herbst, *Spectral theory of the operator $(p^2 + m^2)^{1/2} - Ze^2/r$* . Comm. Math. Phys. 53 (1977), no. 3, 285–294.
- [28] T. Horiuchi, *Best constant in weighted Sobolev inequality with weights being powers of distance from the origin*. J. Inequal. Appl. 1 (1997), 275–292.
- [29] G.N Hile and M.H. Protter, *Unique continuation and the Cauchy problem for first order systems of partial differential equations*. Comm. Partial Differential Equations 1 (1976), no. 5, 437–465.
- [30] H. Kalf and O. Yamada, *Note on the paper: “Strong unique continuation property for the Dirac equation” by L. De Carli and T. Okaji*. Publ. Res. Inst. Math. Sci. 35 (1999), no. 6, 847–852.
- [31] S.A. Khafagy, *On positive weak solution for a nonlinear system involving weighted p -Laplacian*. J. Adv. Res. Dyn. Control Syst. 4 (2012), no. 4, 50–58.
- [32] D. Jerison, *Carleman inequalities for the Dirac and Laplace operators and unique continuation*. Advances in Math. 62 (1986), no. 2, 118–134.
- [33] D. Jerison and C. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operators*. Ann. of Math. 121 (1985), 463–494.
- [34] W. Jurkat and G. Sampson, *On rearrangement and weight inequalities for the Fourier transform*. Indiana Univ. Math. J. 33 (1984), 257–270.
- [35] C. Kenig, A. Ruiz, and C. D. Sogge, *Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators*. Duke Math. J. 55 (1987), no. 2, 329–347.
- [36] H. Koch and D. Tataru, *Carleman estimates and absence of embedded eigenvalues*. Comm. Math. Phys. 267 (2006), no. 2, 419–449.
- [37] H. Koch and D. Tataru, *Recent results on unique continuation for second order elliptic equations*. Carleman estimates and applications to uniqueness and control theory (Cortona, 1999), 73–84, Progr. Nonlinear Differential Equations Appl., 46, Birkhäuser Boston, Boston, MA, 2001.
- [38] J.D. Lakey, *Weighted Fourier transform inequalities via mixed norm Hausdorff-Young inequalities*. Can. J. Math. 46 (1994), no. 3, 586–601.
- [39] J. Lankeit, P. Neff, and D. Pauly, *Unique continuation for first-order systems with integrable coefficients and applications to elasticity and plasticity*. C. R. Math. Acad. Sci. Paris 351 (2013), no. 5-6, 247–250.
- [40] B.M. Levitan and I.S. Sargsjan, *Introduction to spectral theory: Self adjoint ordinary differential operators*. Transl. Math. Monogr., 39, Amer. Math. Soc., Providence, Rhode Island, 1975.
- [41] R. Long and F. Nie, *Weighted Sobolev inequality and eigenvalue estimates of Schrödinger operators*. Lecture Notes in Mathematics 1494 (1990), 131–141.
- [42] V. G. Maz’ya, *Sobolev spaces*. Springer-Verlag, Berlin, 1985.
- [43] B. Muckenhoupt, *Weighted norm inequalities for the Fourier transform*. Trans. Amer. Math. Soc. 276 (1983), 729–142.
- [44] T. Okaji, *Strong unique continuation property for elliptic systems of normal type in two independent variables*. Tohoku Math. J. (2) 54 (2002), no. 2, 309–318.
- [45] T. Okaji, *Strong unique continuation property for first order elliptic systems*. Carleman estimates and applications to uniqueness and control theory (Cortona, 1999), 149–164, Progr. Nonlinear Differential Equations Appl., 46, Birkhäuser, Boston, MA, 2001.

- [46] C. Pérez, *Sharp L^p -weighted Sobolev inequalities*, Ann. Inst. Fourier (Grenoble) 45 (1995), no. 3, 809–824.
- [47] B. Ruf, *Superlinear elliptic equations and systems*. In: Chipot, M. (ed.) Handbook of Differential Equations. Stationary Partial Differential Equations, vol. 5, pp. 211–276. Elsevier, Amsterdam (2008).
- [48] E.T. Sawyer, *A characterization of two weight norm inequalities for fractional fractional and Poisson integrals*. Trans. Amer. Math. Soc. 308 (1988), 533–545.
- [49] E. Sawyer and R.L. Wheeden, *Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces*, Amer. J. Math. 114 (1992), no. 4, 813–874.
- [50] B. Simon, *On positive eigenvalues of one-body Schrödinger operators*. Commun. Pure Appl. Math. 22 (1969), 531–538.
- [51] B. Simon, *Schrodinger semigroups*. Bull. Amer. Math. Soc. (N.S.) 7 (1982), no. 3, 447–526.
- [52] C.D. Sogge, *Strong uniqueness theorems for second order elliptic differential equations*. Amer. J. Math. 112 (1990), no. 6, 943–984.
- [53] G. Sinnamon, *A weighted gradient inequality*. Proc. Roy. Soc. Edinburgh Sect. A 111 (1989), no. 3–4, 329–335.
- [54] M. Tamura, *A note on strong unique continuation for normal elliptic systems with Gevrey coefficients*. J. Math. Kyoto Univ. 49 (2009), no. 3, 593–601.
- [55] D. Tataru, *Unique continuation problems for partial differential equations*. Geometric methods in inverse problems and PDE control, 239–255, IMA Vol. Math. Appl., 137, Springer, New York, 2004.
- [56] H. Uryu, *The local uniqueness of some characteristic Cauchy problems for the first order systems*. Funkcialaj Ekvacioj 38 (1995), 21–36.
- [57] Q. Yang and B. Lian, *On the best constant of weighted Poincaré inequalities*. J. Math. Anal. Appl. 377 (2011), 207–215.
- [58] T. Wolff, *Recent work on sharp estimates in second order elliptic unique continuation problems*. In *Fourier Analysis and Partial Differential Equations*, Garcia-Cuerva, Hernandez, Soria, Torrea ed., Studies in Advanced Mathematics, CRC press (1995), 99–128.

L. DE CARLI

FLORIDA INTERNATIONAL UNIVERSITY
DEPARTMENT OF MATHEMATICS
MIAMI, FL 33199, USA

E-mail address: decarlil@fiu.edu

D. GORBACHEV

TULA STATE UNIVERSITY
DEPARTMENT OF APPLIED MATHEMATICS AND COMPUTER SCIENCE
300012 TULA, RUSSIA

E-mail address: dvgmail@mail.ru

S. YU. TIKHONOV

ICREA, CENTRE DE RECERCA MATEMÀTICA, AND UAB
CAMPUS DE BELLATERRA, EDIFICI C
08193 BELLATERRA (BARCELONA), SPAIN.

E-mail address: stikhonov@crm.cat

