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RIESZ POTENTIAL AND MAXIMAL FUNCTION FOR DUNKL TRANSFORM

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Abstract. We study weighted (L^p, L^q)-boundedness properties of Riesz potentials and fractional maximal functions for the Dunkl transform. In particular, we obtain the weighted Hardy–Littlewood–Sobolev type inequality and weighted week (L^1, L^q) estimate. We find a sharp constant in the weighted L^p-inequality, generalizing the results of W. Beckner and S. Samko.

1. Introduction

Let $\mathbb{R}^d$ be the real Euclidean space of d dimensions equipped with a scalar product $\langle x, y\rangle$ and a norm $|x| = \sqrt{\langle x, x\rangle}$. Let $d\mu(x) = (2\pi)^{-d/2}\,dx$ be the normalized Lebesgue measure, $L^p(\mathbb{R}^d)$, $1 \le p < \infty$, be the Lebesgue space with the norm $\|f\|_p = \bigl(\int_{\mathbb{R}^d} |f|^p\,d\mu\bigr)^{1/p}$, and $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space. The Fourier transform is given by

$$\mathcal{F}(f)(y) = \int_{\mathbb{R}^d} f(x)e^{-i\langle x,y\rangle}\,d\mu(x).$$

Throughout the paper, we will assume that $A \lesssim B$ means that $A \le CB$ with a constant C depending only on nonessential parameters. For $p \ge 1$, $p' = \frac{p}{p-1}$ is the Hölder conjugate and χ_E is the characteristic function of a set E.

The Riesz potential operator or fractional integral I_α is defined by

$$I_\alpha f(x) = (\gamma_\alpha)^{-1}\int_{\mathbb{R}^d} f(y)|x-y|^{\alpha-d}\,d\mu(y) = (\gamma_\alpha)^{-1}\int_{\mathbb{R}^d} \tau^{-y}f(x)|y|^{\alpha-d}\,d\mu(y),$$

where $0 < \alpha < d$, $\gamma_\alpha = 2^{\alpha-d/2}\Gamma(\alpha/2)/\Gamma((d-\alpha)/2)$, and $\tau^y f(x) = f(x+y)$ is the translation operator. Such operator was first investigated by O. Frostman [7]. Several important properties of the potential were obtained by M. Riesz [18].

The weighted (L^p, L^q)-boundedness of Riesz potentials is given by the following Stein–Weiss inequality

$$\bigl\||x|^{-\gamma}I_\alpha f(x)\bigr\|_q \le \mathbf{c}(\alpha, \beta, \gamma, p, q, d)\bigl\||x|^{\beta}f(x)\bigr\|_p \tag{1.1}$$

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with the sharp constant $\mathbf{c}(\alpha,\beta,\gamma,p,q,d)$ and $1<p\le q<\infty$. Sufficient conditions for the finiteness of $\mathbf{c}(\alpha,\beta,\gamma,p,q,d)$ are well known.

Theorem 1.1. *Let* $d\in\mathbb{N}$, $1\le p\le q<\infty$, $\gamma<\frac{d}{q}$, $\gamma+\beta\ge 0$, $0<\alpha<d$, *and* $\alpha-\gamma-\beta=d(\frac{1}{p}-\frac{1}{q})$.

(a) *If* $1<p\le q<\infty$ *and* $\beta<\frac{d}{p'}$, *then* $\mathbf{c}(\alpha,\beta,\gamma,p,q,d)<\infty$.

(b) *If* $p=1$, $1<q<\infty$, $\beta\le 0$, *then, for* $f\in\mathcal{S}(\mathbb{R}^d)$ *and* $\lambda>0$,

$$\int_{\{x\in\mathbb{R}^d\colon |x|^{-\gamma}|I_\alpha f(x)|>\lambda\}} d\mu(x)\lesssim\Big(\frac{\||x|^{\beta}f(x)\|_1}{\lambda}\Big)^q.$$

The part (a) in Theorem 1.1 was proved by G.H. Hardy and J.E. Littlewood [12] for $d=1$, S. Sobolev [26] for $d>1$ and $\gamma=\beta=0$, E.M. Stein and G. Weiss [27] in the general case. The conditions for weak boundedness can be found in [9, 25].

The sharp constant $\mathbf{c}(\alpha,0,0,p,q,d)$ in the non-weighted Sobolev inequality was calculated by E.H. Lieb [15] in any of the following cases: (1) $q=p'$, $1<p<2$, (2) $q=2$, $1<p<2$, (3) $p=2$, $2<q<\infty$. Moreover, in these cases there exist maximazing functions. In the weighted Hardy–Littlewood–Sobolev inequality the constant $\mathbf{c}(\alpha,\beta,\gamma,p,q,d)$ is known only for $q=p$.

Theorem 1.2. *If* $d\in\mathbb{N}$, $1<p<\infty$, $\gamma<\frac{d}{p}$, $\beta<\frac{d}{p'}$, $\alpha>0$, *and* $\gamma=\alpha-\beta$, *then*

$$\mathbf{c}(\alpha,\beta,\gamma,p,p,d)=2^{-\alpha}\frac{\Gamma(\frac{1}{2}(\frac{d}{p}-\alpha+\beta))\Gamma(\frac{1}{2}(\frac{d}{p'}-\beta))}{\Gamma(\frac{1}{2}(\frac{d}{p'}+\alpha-\beta))\Gamma(\frac{1}{2}(\frac{d}{p}+\beta))}.$$

Theorem 1.2 was proved by I.W. Herbst [14] for $\beta=0$ and W. Beckner [4] and S. Samko [24] in the general case.

For $\alpha\in\mathbb{R}$, we define the Riesz potential in the distributional sense. Let Φ be the Lizorkin space [16], [23, p. 39], that is, a subspace of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ which consists of functions orthogonal to all polynomials:

$$\int_{\mathbb{R}^d} x^n f(x)\,d\mu(x)=0,\quad n=(n_1,\dots,n_d)\in\mathbb{Z}_+^d.$$

The subspace Φ is invariant with respect to the operator I_α and its inverse $I_\alpha^{-1}=(-\Delta)^{\alpha/2}$:

$$I_\alpha(\Phi)=(-\Delta)^{\alpha/2}(\Phi)=\Phi,$$

where Δ is the Laplacian. Note that Φ is dense in $L_p(\mathbb{R}^d,|x|^{\beta p}\,d\mu)$ for $1<p<\infty$ and $\beta\in(-d/p,\,d/p')$ [23, p. 41].

It is worth mentioning that the Stein–Weisz inequality (1.1) on Φ is equivalent to the Hardy–Rellich inequality

$$\big\||x|^{-\gamma}f(x)\big\|_q\le\mathbf{c}(\alpha,\beta,\gamma,p,q,d)\big\||x|^{\beta}(-\Delta)^{\alpha/2}f(x)\big\|_p.$$

Let $D_jf(x)$ be the usual partial derivative with respect to a variable x_j, $j = 1,\dots,d$, $D = (D_1,\dots,D_d)$, $D^nf(x) = \prod_{j=1}^d D_j^{n_j}f(x)$, $n \in \mathbb{Z}_+^d$. The subspace

$$\Psi = \{\mathcal{F}(f)\colon f \in \Phi\} = \{f \in \mathcal{S}(\mathbb{R}^d)\colon D^nf(0) = 0,\ n \in \mathbb{Z}_+^d\}$$

is invariant with respect to the operator $\mathcal{F}(I_\alpha)$ and $\mathcal{F}((-\Delta)^{\alpha/2})$:

$$\mathcal{F}(I_\alpha)(\Psi) = \mathcal{F}((-\Delta)^{\alpha/2})(\Psi) = \Psi.$$

For a distribution $f \in \Phi'$ and $\alpha \in \mathbb{R}$ we set

$$I_\alpha f = \mathcal{F}^{-1}|\cdot|^{-\alpha}\mathcal{F}(f), \quad (-\Delta)^{\alpha/2}f = \mathcal{F}^{-1}|\cdot|^{\alpha}\mathcal{F}(f).$$

If $\varphi \in \Phi$, then

$$\langle I_\alpha f, \varphi\rangle = \langle f, \mathcal{F}|\cdot|^{-\alpha}\mathcal{F}^{-1}(\varphi)\rangle, \quad \langle(-\Delta)^{\alpha/2}f, \varphi\rangle = \langle f, \mathcal{F}|\cdot|^{\alpha}\mathcal{F}^{-1}(\varphi)\rangle.$$

One of the generalizations of the Fourier transform is the Dunkl transform $\mathcal{F}_k$ (see [6, 21]). Our main goal in this paper is to prove analogues of Theorems 1.1 and 1.2 for the Riesz potential associated with the Dunkl transform. We shall call it *the D-Riesz potential*.

Let a finite subset $R \subset \mathbb{R}^d\setminus\{0\}$ be a root system, let R_+ be positive subsystem of R, let $G(R) \subset O(d)$ be finite reflection group, generated by reflections $\{\sigma_a\colon a \in R\}$, where σ_a is a reflection with respect to hyperplane $\langle a, x\rangle = 0$, let $k\colon R \to \mathbb{R}_+$ be G-invariant multiplicity function. Recall that a finite subset $R \subset \mathbb{R}^d \setminus \{0\}$ is called a root system, if

$$R \cap \mathbb{R}a = \{a, -a\} \quad \text{and} \quad \sigma_aR = R \text{ for all } a \in R.$$

Let

$$v_k(x) = \prod_{a\in R_+} |\langle a, x\rangle|^{2k(a)}$$

be the Dunkl weight. The normalized Macdonald–Metha–Selberg constant is given by

$$c_k^{-1} = \int_{\mathbb{R}^d} e^{-|x|^2/2}v_k(x)\,dx.$$

Let $L^p(\mathbb{R}^d, d\mu_k)$ be the space of complex-valued Lebesgue measurable functions f such that

$$\|f\|_{p,d\mu_k} = \Big(\int_{\mathbb{R}^d} |f|^p\,d\mu_k\Big)^{1/p} < \infty,$$

where $d\mu_k(x) = c_kv_k(x)dx$ is the Dunkl measure. Assume that

$$T_jf(x) = D_jf(x) + \sum_{a\in R_+} k(a)\langle a, e_j\rangle\,\frac{f(x) - f(\sigma_ax)}{\langle a, x\rangle} \tag{1.2}$$

are differential-differences Dunkl operators, $j = 1,\dots,d$, and $\Delta_k = \sum_{j=1}^d T_j^2$ is the Dunkl Laplacian [10].

The Dunkl kernel $E_k(x,y)$ is a unique solution of the system

$$T_jf(x)=y_jf(x),\quad j=1,\dots,d,\qquad f(0)=1.$$

Let $e_k(x,y)=E_k(x,iy)$. It plays the role of a generalized exponential function. Its properties are similar to those of the classical exponential function $e^{i\langle x,y\rangle}$. Several basic properties follow from the integral representation given by M. Rösler [20]

$$e_k(x,y)=\int_{\mathbb{R}^d}e^{i\langle\xi,y\rangle}\,d\mu_x^k(\xi), \tag{1.3}$$

where μ_x^k is a probability Borel measure, whose support is contained in $\mathrm{co}(\{gx\colon g\in G(R)\})$ the convex hull of the G-orbit of x in $\mathbb{R}^d$. In particular, $|e_k(x,y)|\le 1$ and $\operatorname{supp}\mu_x^k\subset B_{|x|}$, where B_r is the Euclidean ball of radius r centered at 0.

For $f\in L^1(\mathbb{R}^d,d\mu_k)$, the Dunkl transform is defined by the equality

$$\mathcal{F}_k(f)(y)=\int_{\mathbb{R}^d}f(x)\overline{e_k(x,y)}\,d\mu_k(x).$$

If $k\equiv 0$, then $\mathcal{F}_0$ is the Fourier transform $\mathcal{F}$. We note that $\mathcal{F}_k(e^{-|\cdot|^2/2})(y)=e^{-|x|^2/2}$ and $\mathcal{F}_k^{-1}(f)(x)=\mathcal{F}_k(f)(-x)$. The Dunkl transform is isometry in $\mathcal{S}(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d,d\mu_k)$ and $\|f\|_{2,d\mu_k}=\|\mathcal{F}_k(f)\|_{2,d\mu_k}$.

M. Rösler [19] defined the generalized translation operator τ^y, $y\in\mathbb{R}^d$, on $L^2(\mathbb{R}^d,d\mu_k)$ by equality

$$\mathcal{F}_k(\tau^yf)(z)=e_k(y,z)\mathcal{F}_k(f)(z),$$

or

$$\tau^yf(x)=\int_{\mathbb{R}^d}e_k(y,z)e_k(x,z)\mathcal{F}_k(f)(z)\,d\mu_k(z). \tag{1.4}$$

It acts from $L^2(\mathbb{R}^d,d\mu_k)$ to $L^2(\mathbb{R}^d,d\mu_k)$ and $\|\tau^y\|_{2\to 2}=1$.

If $k\equiv 0$, then $\tau^yf(x)=f(x+y)$. If $f\in\mathcal{S}(\mathbb{R}^d)$, then $\tau^yf(x)\in\mathcal{S}(\mathbb{R}^d)\times\mathcal{S}(\mathbb{R}^d)$ and equality (1.4) holds pointwise. K. Trimèche extended τ^y on $C^\infty(\mathbb{R}^d)$ [30]. For example, $\tau^y1=1$. In general, τ^y is not positive operator and the question of its L_p-boundedness remains open.

First, we define the D-Riesz potential for distributions. Let

$$\Phi_k=\Big\{f\in\mathcal{S}(\mathbb{R}^d)\colon\ \int_{\mathbb{R}^d}x^nf(x)\,d\mu_k(x)=0,\quad n\in\mathbb{Z}_+^d\Big\}$$

be the weighted Lizorkin space,

$$\Psi_k=\{\mathcal{F}_k(f)\colon f\in\Phi_k\}.$$

For $\alpha\in\mathbb{R}$, we define the D-Riesz potential on Φ_k by equality

$$I_\alpha^kf=\mathcal{F}_k^{-1}|\cdot|^{-\alpha}\mathcal{F}_k(f).$$

In Section 6, we will prove that $\Psi_k = \Psi$ (see Theorem 6.1) and then $I_\alpha^k(\Phi_k) = \Phi_k$ and $\mathcal{F}_k(I_\alpha^k)(\Psi_k) = \Psi_k$. Therefore, we can define the D-Riesz potential I_α^k for $f \in \Phi_k'$ and $\alpha \in \mathbb{R}$ by the same equality $I_\alpha^k f = \mathcal{F}_k^{-1}|\cdot|^{-\alpha}\mathcal{F}_k(f)$ as follows

$$\langle I_\alpha f, \varphi\rangle = \langle f, \mathcal{F}_k|\cdot|^{-\alpha}\mathcal{F}_k^{-1}(\varphi)\rangle, \quad \varphi \in \Phi_k.$$

We will also prove (see Theorem 6.3) that Φ_k is dense in $L_p(\mathbb{R}^d, |x|^{\beta p}\,d\mu_k)$ for $1 < p < \infty$ and $\beta \in (-d_k/p, d_k/p')$, where

$$d_k = 2\lambda_k + 2, \quad \lambda_k = \frac{d}{2} - 1 + \sum_{a\in R_+} k(a). \tag{1.5}$$

S. Thangavelu and Y. Xu defined [29] the D-Riesz potential on Schwartz space as follows

$$I_\alpha^k f(x) = (\gamma_\alpha^k)^{-1}\int_{\mathbb{R}^d} \tau^{-y} f(x)|y|^{\alpha-d_k}\,d\mu_k(y), \tag{1.6}$$

where $0 < \alpha < d_k$ and $\gamma_\alpha^k = 2^{\alpha-d_k/2}\Gamma(\alpha/2)/\Gamma((d_k-\alpha)/2)$.

We are interested in the Stein–Weiss inequality for the D-Riesz potential

$$\big\||x|^{-\gamma} I_\alpha^k f(x)\big\|_{q,d\mu_k} \le \mathbf{c}_k(\alpha,\beta,\gamma,p,q,d)\big\||x|^\beta f(x)\big\|_{p,d\mu_k}, \quad f \in \mathcal{S}(\mathbb{R}^d), \tag{1.7}$$

with the sharp constant $\mathbf{c}_k(\alpha,\beta,\gamma,p,q,d)$ and $1 < p \le q < \infty$. On Φ_k, it is equivalent to the Hardy–Rellich type inequality

$$\big\||x|^{-\gamma} f(x)\big\|_{q,d\mu_k} \le \mathbf{c}_k(\alpha,\beta,\gamma,p,q,d)\big\||x|^\beta(-\Delta_k)^{\alpha/2} f(x)\big\|_{p,d\mu_k}.$$

Our main results read as follows.

Theorem 1.3. *If $d \in \mathbb{N}$, $1 < p < \infty$, $\gamma < \frac{d_k}{p}$, $\beta < \frac{d_k}{p'}$, $\alpha > 0$, and $\alpha = \gamma + \beta$, then*

$$\begin{aligned}\mathbf{c}_k(\alpha,\beta,\gamma,p,p,d) &= 2^{-\alpha}\,\frac{\Gamma(\frac12(\frac{d_k}{p}-\gamma))\Gamma(\frac12(\frac{d_k}{p'}-\beta))}{\Gamma(\frac12(\frac{d_k}{p'}+\gamma))\Gamma(\frac12(\frac{d_k}{p}+\beta))}\\ &= \mathbf{c}(\alpha,\beta,\gamma,p,p,d_k).\end{aligned}$$

Theorem 1.4. *Let $d \in \mathbb{N}$, $1 \le p \le q < \infty$, $\gamma < \frac{d_k}{q}$, $\gamma + \beta \ge 0$, $0 < \alpha < d_k$, and $\alpha - \gamma - \beta = d_k(\frac1p - \frac1q)$.*

(a) *If $1 < p \le q < \infty$ and $\beta < \frac{d_k}{p'}$, then $\mathbf{c}_k(\alpha,\beta,\gamma,p,q,d) < \infty$.*

(b) *If $p = 1$, $1 < q < \infty$, $\beta \le 0$, and $\lambda > 0$, then*

$$\int_{\{x\in\mathbb{R}^d:\, |x|^{-\gamma}|I_\alpha^k f(x)|>\lambda\}} d\mu_k(x) \lesssim \Big(\big\||x|^\beta f(x)\big\|_{1,d\mu_k}/\lambda\Big)^q, \qquad f \in \mathcal{S}(\mathbb{R}^d).$$

For $k \equiv 0$, Theorems 1.3 and 1.4 become Theorems 1.1 and 1.2, therefore it is enough to consider the case $k \not\equiv 0$, i.e., $\lambda_k = \frac{d}{2} - 1 + \sum_{a\in R_+} k(a) > -1/2$ and $d_k = 2\lambda_k + 2 > 1$. It is clear that d_k plays the role of the generalized dimension of the space $(\mathbb{R}^d, d\mu_k)$.

For the reflection group $\mathbb{Z}_2^d$ and $\gamma = \beta = 0$, Theorem 1.4 was proved in [29]. For arbitrary reflection group G and $\gamma = \beta = 0$, it was proved by S. Hassani, S. Mustapha and M. Sifi [13]. Following an idea from [29], we have recently given another proof in [11]. Regarding the weighted setting, part (a) was proved in [1] in the case $q = p$ under more restrictive conditions $1 < p < \infty$, $0 < \gamma < \frac{d_k}{p}$, $0 < \beta < \frac{d_k}{p'}$, and $\alpha > 0$.

To estimate the L^p-norm of operator I_α^k, S. Thangavelu and Y. Xu [29] used the maximal function, defined for $f \in \mathcal{S}(\mathbb{R}^d)$ as follows

$$M^k f(x) = \sup_{r>0} \frac{|\int_{\mathbb{R}^d} \tau^{-y} f(x)\chi_{B_r}(y)\, d\mu_k(y)|}{\int_{B_r} d\mu_k},$$

where $B_r = \{x\colon |x| \le r\}$. They proved the strong L^p-boundedness of M^k for $1 < p < \infty$ and the weak boundedness for $p = 1$ [28].

We will use Theorem 1.4 to obtain weighted boundedness of the fractional maximal function $M_\alpha^k f$, $0 \le \alpha < d_k$, given by

$$\begin{aligned} M_\alpha^k f(x) &= \sup_{r>0} r^{\alpha - d_k} \Big| \int_{\mathbb{R}^d} \tau^{-y} f(x)\chi_{B_r}(y)\, d\mu_k(y) \Big| \\ &= \sup_{r>0} r^{\alpha - d_k} \Big| \int_{\mathbb{R}^d} f(x)\tau^{-y}\chi_{B_r}(x)\, d\mu_k(y) \Big|. \end{aligned}$$

If $\alpha = 0$, then M_0^k coincides with M^k up to a constant. Since τ^y is a positive operator on radial functions [22, 28], and using

$$M_\alpha^k f(x) \le M_\alpha^k |f|(x) \lesssim I_\alpha^k |f|(x),$$

Theorem 1.4 implies the boundedness conditions of the fractional maximal function.

Theorem 1.5. *Let* $d \in \mathbb{N}$, $1 \le p \le q < \infty$, $\gamma < \frac{d_k}{q}$, $\gamma + \beta \ge 0$, $0 < \alpha < d_k$, $\alpha - \gamma - \beta = d_k(\frac{1}{p} - \frac{1}{q})$, *and* $f \in \mathcal{S}(\mathbb{R}^d)$.

(a) *If* $1 < p \le q < \infty$ *and* $\beta < \frac{d_k}{p'}$, *then*

$$\big\| |x|^{-\gamma} M_\alpha^k f(x) \big\|_{q,d\mu_k} \lesssim \big\| |x|^{\beta} f(x) \big\|_{p,d\mu_k}.$$

(b) *If* $p = 1$, $1 < q < \infty$, $\beta \le 0$, *and* $\lambda > 0$, *then*

$$\int_{\{x\in\mathbb{R}^d\colon |x|^{-\gamma}|M_\alpha^k f(x)|>\lambda\}} d\mu_k(x) \lesssim \Big(\big\| |x|^{\beta} f(x) \big\|_{1,d\mu_k} / \lambda \Big)^q.$$

In the case $\gamma = \beta = 0$ Theorem 1.5 was proved in [13].

The paper is organized as follows. In the next section, we obtain the sharp inequalities for Mellin convolution and investigate the following representation of the Riesz potential

$$I_\alpha^k f(x) = \int_{\mathbb{R}^d} f(y)\Phi(x, y)\, d\mu_k(y)$$

and basic properties of the kernel

$$\Phi(x,y)=\frac{2^{d_k/2-\alpha}}{\Gamma(\alpha/2)}\int_0^\infty s^{(d_k-\alpha)/2-1}\tau^{-y}(e^{-s|\cdot|^2})(x)\,ds,\quad (x,y)\neq(0,0).$$

In Section 3, we prove sharp (L_p,L_p) Hardy's inequalities with weights for the averaging operator $Hf(x)=\int_{|y|\le|x|}f(y)\,d\mu_k(y)$. In the classical setting ($k=0$), this result was proved by M. Christ and L. Grafakos [5] and Z.W. Fu, L. Grafakos, S.Z. Lu and F.Y. Zhao [8]. Sections 4 and 5 are devoted to the proofs of Theorems 1.3 and 1.4 corespondingly. We finish with Section 6, which contains some important properties of the spaces Φ_k and Ψ_k.

2. Notations and auxiliary statements

Set as usual $\mathbb{R}_+=[0,\infty)$, $\mathbb{S}^{d-1}=\{x\in\mathbb{R}^d\colon |x|=1\}$ and $x=rx'\in\mathbb{R}^d$, $r=|x|\in\mathbb{R}_+$, $x'\in\mathbb{S}^{d-1}$. Let

$$d\nu(r)=r^{-1}\,dr,\quad d\nu_\lambda(r)=b_\lambda r^{2\lambda+1}\,dr,\quad b_\lambda^{-1}=2^\lambda\Gamma(\lambda+1),\quad \lambda\ge-1/2,$$

be the measures on $\mathbb{R}_+$,

$$d\sigma_k(x')=a_kv_k(x')\,dx'$$

be the probability measure on $\mathbb{S}^{d-1}$, and

$$d\mu_k(x)=c_kv_k(x)\,dx,\quad dm_k(x)=d\nu(r)\,d\sigma_k(x')$$

be the measures on $\mathbb{R}^d$.

Note that

$$d\mu_k(x)=d\nu_{\lambda_k}(r)\,d\sigma_k(x')=b_{\lambda_k}|x|^{d_k}\,dm_k(x), \tag{2.1}$$

where λ_k and d_k are defined in (1.5).

Let $L^p(X,d\mu)$, $1\le p\le\infty$, be the Banach space with the norm

$$\|f\|_{p,d\mu}=\begin{cases}\Big(\int_X|f|^p\,d\mu\Big)^{1/p}, & p<\infty,\\ \operatorname{supvrai}_X|f|, & p=\infty.\end{cases}$$

Depending on the context, we assume that $L^p=L^p(X,d\mu)$ and $\|f\|_p=\|f\|_{p,d\mu}$.

2.1. **Convolution inequalities.**

The Mellin convolution is given by

$$A_gf(r)=(f*g)(r)=\int_0^\infty f(r/t)g(t)\,d\nu(t).$$

We will frequently use the fact that

$$(f*g)(r)=(g*f)(r). \tag{2.2}$$

Lemma 2.1. *Let $L^p = L^p(\mathbb{R}_+, d\nu)$, $1 \le p \le \infty$. If $f \in L^p$, $h \in L^{p'}$, $g \in L^1$, then*

$$\|f * g\|_p \le \|g\|_1 \|f\|_p,$$

or

$$\Big|\int_0^\infty \int_0^\infty h(r)f(t)g(r/t)\,d\nu(t)\,d\nu(r)\Big| \le \|g\|_1\|h\|_{p'}\|f\|_p.$$

If $g \ge 0$, then

$$\|A_g\|_{p\to p} = \|g\|_1, \tag{2.3}$$

or

$$\sup_{\|f\|_p\le 1}\ \sup_{\|h\|_{p'}\le 1}\Big|\int_0^\infty h(r)A_g f(r)\,d\nu(r)\Big| = \|g\|_1.$$

Proof. For the classical convolution on the $\mathbb{R}$, see, e.g., [14]. We sketch the proof here for completeness of the exposition. For $1 < p < \infty$, using Hölder's inequality, we obtain

$$\Big|\int_0^\infty f(r/t)g(t)\,d\nu(t)\Big| \le \Big(\int_0^\infty |f(r/t)|^p\,|g(t)|\,d\nu(t)\Big)^{1/p}\Big(\int_0^\infty |g(t)|\,d\nu(t)\Big)^{1/p'}$$

and

$$\begin{aligned}\|f * g\|_p &\le \Big(\int_0^\infty \int_0^\infty |f(r/t)|^p\,|g(t)|\,d\nu(t)\,d\nu(r)\Big)^{1/p}\|g\|_1^{1/p'}\\ &= \Big(\int_0^\infty |g(t)|\int_0^\infty |f(r/t)|^p\,d\nu(r)\,d\nu(t)\Big)^{1/p}\|g\|_1^{1/p'} = \|f\|_p\,\|g\|_1.\end{aligned}$$

Let $g \ge 0$. If $p = 1$, $f \in L^1$, $f \ge 0$, then

$$\begin{aligned}\|A_g f\|_1 &= \int_0^\infty \int_0^\infty f(r/t)g(t)\,d\nu(t)\,d\nu(r)\\ &= \int_0^\infty f(r)\,d\nu(r)\int_0^\infty g(t)\,d\nu(t) = \|g\|_1\|f\|_1,\end{aligned}$$

which gives (2.3). If $p = \infty$, we define $f = \chi_{[\lambda,1/\lambda]}$, $0 < \lambda < 1$. Then $\|f\|_\infty = 1$ and, for $r \in [1, 2]$,

$$\begin{aligned}A_g f(r) &= \int_0^\infty f(t)g(r/t)\,d\nu(t) = \int_\lambda^{1/\lambda} g(r/t)\,d\nu(t)\\ &= \int_{r\lambda}^{r/\lambda} g(t)\,d\nu(t) \ge \int_{2\lambda}^{1/\lambda} g(t)\,d\nu(t) \to \|g\|_1, \qquad \lambda \to 0.\end{aligned}$$

If $1 < p < \infty$, $f = (2\lambda)^{-1/p}\chi_{[e^{-\lambda},e^{\lambda}]}$, and $h = (2\lambda)^{-1/p'}\chi_{[e^{-\lambda},e^{\lambda}]}$, then $\|f\|_p = \|h\|_{p'} = 1$ and by the Lebesgue dominated convergence theorem

$$\begin{aligned}\|A_g\|_{p\to p} &\geq \lim_{\lambda\to\infty}\Big\{(2\lambda)^{-1}\int_{e^{-\lambda}}^{e^{\lambda}}\int_{e^{-\lambda}}^{e^{\lambda}} g(r/t)\,d\nu(r)\,d\nu(t)\Big\}\\ &= \lim_{\lambda\to\infty}\Big\{(2\lambda)^{-1}\int_{e^{-\lambda}}^{e^{\lambda}}\int_{e^{-\lambda}/t}^{e^{\lambda}/t} g(r)\,d\nu(r)\,d\nu(t)\Big\}\\ &= \lim_{\lambda\to\infty}(2\lambda)^{-1}\Big\{\int_{e^{-2\lambda}}^{1}\int_{e^{-\lambda}/r}^{e^{\lambda}}\frac{dt}{t}\,g(r)\,\frac{dr}{r}+\int_{1}^{e^{2\lambda}}\int_{e^{-\lambda}}^{e^{\lambda}/r}\frac{dt}{t}\,g(r)\,\frac{dr}{r}\Big\}\\ &= \lim_{\lambda\to\infty}(2\lambda)^{-1}\Big\{\int_{e^{-2\lambda}}^{1} g(r)(2\lambda+\ln r)\,\frac{dr}{r}+\int_{1}^{e^{2\lambda}} g(r)(2\lambda-\ln r)\,\frac{dr}{r}\Big\}\\ &= \int_0^\infty g(r)\,\frac{dr}{r}=\|g\|_1. \qquad \square\end{aligned}$$

2.2. **A representation of the Riesz potential.** We will use the following representation (see [1]), which is different from the definition (1.6):

$$I_\alpha^k f(x)=\int_{\mathbb{R}^d} f(y)\Phi(x,y)\,d\mu_k(y), \tag{2.4}$$

where

$$\Phi(x,y)=\frac{2^{d_k/2-\alpha}}{\Gamma(\alpha/2)}\int_0^\infty s^{(d_k-\alpha)/2-1}\tau^{-y}(e^{-s|\cdot|^2})(x)\,ds,\quad (x,y)\neq(0,0). \tag{2.5}$$

To verify (2.4), we first remark that the convolution

$$(f*_k g)(x)=\int_{\mathbb{R}^d}\tau^{-y}f(x)\,g(y)\,d\mu_k(y)$$

is commutative, i.e., $(f*_k g)(x)=(g*_k f)(x)$. Indeed, we have the following

Lemma 2.2. *If $f\in\mathcal{S}(\mathbb{R}^d)$, $g\in L^1(\mathbb{R}^d,d\mu_k)$, and $f_t(x)=f(tx)$, then*

$$\int_{\mathbb{R}^d}\tau^{-y}f(x)\,g(y)\,d\mu_k(y)=\int_{\mathbb{R}^d}f(y)\tau^{-y}g(x)\,d\mu_k(y), \tag{2.6}$$

$$\mathcal{F}_k(f_t)(z)=\frac{1}{t^{d_k}}\mathcal{F}_k(f)\Big(\frac{z}{t}\Big),\quad \tau^y(f_t)(x)=\tau^{ty}f(tx). \tag{2.7}$$

Relation (2.6) has been recently proved in [11]. Equalities (2.7) can be verified by simple calculations.

Remark 2.1. It is worth mentioning that if the convolution is defined by

$$(f*_k g)(x)=\int_{\mathbb{R}^d}\tau^{x}f(y)\,g(y)\,d\mu_k(y)$$

(see [22]), it is not commutative:

$$\int_{\mathbb{R}^d}\tau^{x}f(y)\,g(y)\,d\mu_k(y)=\int_{\mathbb{R}^d}f(y)\tau^{-x}g(y)\,d\mu_k(y).$$

Completing the proof of (2.4), we use (1.6), (2.6) and the fact that (see [29])

$$\frac{1}{|y|^{d_k-\alpha}}=\frac{1}{\Gamma((d_k-\alpha)/2)}\int_0^\infty s^{(d_k-\alpha)/2-1}e^{-s|y|^2}\,ds, \tag{2.8}$$

to obtain

$$\begin{aligned} I_\alpha^k f(x) &= (\gamma_k^\alpha)^{-1}\int_{\mathbb{R}^d}\tau^{-y}f(x)|y|^{\alpha-d_k}\,d\mu_k(y)\\ &= \frac{2^{d_k/2-\alpha}}{\Gamma(\alpha/2)}\int_{\mathbb{R}^d}\tau^{-y}f(x)\int_0^\infty s^{(d_k-\alpha)/2-1}e^{-s|y|^2}\,ds\,d\mu_k(y)\\ &= \frac{2^{d_k/2-\alpha}}{\Gamma(\alpha/2)}\int_0^\infty s^{(d_k-\alpha)/2-1}\int_{\mathbb{R}^d}\tau^{-y}f(x)e^{-s|y|^2}\,d\mu_k(y)\,ds\\ &= \frac{2^{d_k/2-\alpha}}{\Gamma(\alpha/2)}\int_{\mathbb{R}^d}f(y)\int_0^\infty s^{(d_k-\alpha)/2-1}\tau^{-y}(e^{-s|\cdot|^2})(x)\,ds\,d\mu_k(y). \end{aligned}$$

The interchange of the order of integration is legitimate, since, for any $x\in\mathbb{R}^d$, the iterated integral

$$\int_{\mathbb{R}^d}\left|\tau^{-y}f(x)\right|\int_0^\infty s^{(d_k-\alpha)/2-1}e^{-s|y|^2}\,ds\,d\mu_k(y)$$

converges, where we have used the fact that $\tau^y f(x)\in\mathcal{S}(\mathbb{R}^d)\times\mathcal{S}(\mathbb{R}^d)$ whenever $f\in\mathcal{S}(\mathbb{R}^d)$.

2.3. **Properties of the kernel $\Phi(x,y)$.** We will need the following notation. Let $\lambda\ge -1/2$, $J_\lambda(t)$ be the classical Bessel function of degree λ and

$$j_\lambda(t)=2^\lambda\Gamma(\lambda+1)t^{-\lambda}J_\lambda(t)$$

be the normalized Bessel function. The Hankel transform is defined as follows

$$\mathcal{H}_\lambda(f_0)(r)=\int_0^\infty f_0(t)j_\lambda(rt)\,d\nu_\lambda(t),\quad r\in\mathbb{R}_+.$$

It is a unitary operator in $L^2(\mathbb{R}_+,d\nu_\lambda)$ and $\mathcal{H}_\lambda^{-1}=\mathcal{H}_\lambda$ [3, Chap. 7]. If $\lambda=\lambda_k$, the Hankel transform is a restriction of the Dunkl transform on radial functions. Recall that we assume that $\lambda_k>-1/2$.

For $\lambda>-1/2$, let us consider the Gegenbauer-type translation operator (see, e.g., [17])

$$G^s f_0(r)=c_\lambda\int_0^\pi f_0(\sqrt{r^2+s^2-2rs\cos\varphi})\sin^{2\lambda}\varphi\,d\varphi, \tag{2.9}$$

where $c_\lambda=\frac{\Gamma(\lambda+1)}{\Gamma(1/2)\Gamma(\lambda+1/2)}$. If $f_0\in\mathcal{S}(\mathbb{R}_+)$, then

$$G^s f_0(r)=\int_0^\infty j_\lambda(rt)j_\lambda(st)\mathcal{H}_\lambda(f_0)(t)\,d\nu_\lambda(t). \tag{2.10}$$

We will also need the following partial case of the Funk–Hecke formula [31]

$$\int_{\mathbb{S}^{d-1}} e_k(x, ty')\, d\sigma_k(y') = j_{\lambda_k}(t|x|). \tag{2.11}$$

Let $x = rx'$, $y = ty'$, $r, t > \mathbb{R}_+$, and $x', y' \in \mathbb{S}^{d-1}$.

Lemma 2.3. *The kernel* $\Phi(x, y)$ *satisfies the following properties*

(1) $\Phi(x, y) = \Phi(y, x)$;

(2) $\Phi(rx', ty') = r^{\alpha-d_k}\Phi(x', (t/r)y')$;

(3) $\int_{\mathbb{S}^{d-1}} \Phi(rx', ty')\, d\sigma_k(x') = \Phi_0(r, t)$, *where*

$$\Phi_0(r, t) := (\gamma_\alpha^k)^{-1} c_{\lambda_k} \int_0^\pi \big(r^2 + t^2 - 2rt\cos\varphi\big)^{(\alpha-d_k)/2} \sin^{d_k-2}\varphi\, d\varphi;$$

(4) $\Phi(x, y) = (\gamma_\alpha^k)^{-1}\tau^{-y}(|\cdot|^{\alpha-d_k})(x)$ *or, equivalently,*

$$\Phi(x, y) = (\gamma_\alpha^k)^{-1} \int_{\mathbb{R}^d} (|x|^2 + |y|^2 - 2\langle y, \eta\rangle)^{(\alpha-d_k)/2}\, d\mu_x^k(\eta),$$

where μ_x^k *is a probability measure from* (1.3).

Proof. Recall that $E_k(x, y)$ is the Dunkl kernel. Using $E_k(\lambda x, y) = E_k(x, \lambda y)$, $\lambda \in \mathbb{C}$, we have from [19, Sec. 4.9] that

$$\int_{\mathbb{R}^d} e_k(x, z) e_k(-y, z) e^{-|z|^2/2}\, d\mu_k(z) = e^{-\frac{|x|^2+|y|^2}{2}} E_k(x, y).$$

This, (2.7), and the fact that $\mathcal{F}_k(e^{-|\cdot|^2/2})(y) = e^{-|x|^2/2}$ imply that

$$\tau^{-y}(e^{-s|\cdot|^2})(x) = e^{-s(|x|^2+|y|^2)} E_k(\sqrt{2s}\, x, \sqrt{2s}\, y).$$

Since $E_k(x, y) = E_k(y, x)$, the property (1) follows and, moreover,

$$\Phi(x, y) = \frac{2^{d_k/2-\alpha}}{\Gamma(\alpha/2)} \int_0^\infty s^{(d_k-\alpha)/2-1} e^{-s(|x|^2+|y|^2)} E_k(\sqrt{2s}\, x, \sqrt{2s}\, y)\, ds.$$

Changing variables $s \to u/r^2$, we obtain the property (2):

$$\begin{aligned}\Phi(rx', ty') &= r^{\alpha-d_k} \int_0^\infty u^{(d_k-\alpha)/2-1} e^{-u(1+(t/r)^2)} E_k(\sqrt{2u}\, x', \sqrt{2u}\,(t/r)y')\, du \\ &= r^{\alpha-d_k}\Phi(x', (t/r)y').\end{aligned}$$

Since, by (2.7) and (1.4), we have

$$\tau^{-ty'}(e^{-s|\cdot|^2})(rx') = \int_{\mathbb{R}^d} e_k(\sqrt{2s}rx', z) e_k(-\sqrt{2s}ty', z) e^{-|z|^2/2}\, d\mu_k(z),$$

then taking into account (2.1), (2.9), (2.10), and (2.11), we obtain

$$\int_{\mathbb{S}^{d-1}} \tau^{-ty'}(e^{-s|\cdot|^2})(rx')\, d\sigma_k(x')$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} e_k(-\sqrt{2s}ty', z)e^{-|z|^2/2} \int_{\mathbb{S}^{d-1}} e_k(\sqrt{2s}rx', z)\, d\sigma_k(x')\, d\mu_k(z) \\
&= \int_0^\infty j_{\lambda_k}(\sqrt{2s}ru)e^{-u^2/2} \int_{\mathbb{S}^{d-1}} e_k(-\sqrt{2s}ty', uz')\, d\sigma_k(z')\, d\nu_{\lambda_k}(u) \\
&= \int_0^\infty j_{\lambda_k}(\sqrt{2s}ru) j_{\lambda_k}(\sqrt{2s}tu)e^{-u^2/2}\, d\nu_{\lambda_k}(u) \\
&= c_{\lambda_k} \int_0^\pi e^{-s(r^2+t^2-2rt\cos\varphi)} \sin^{d_k-2}\varphi\, d\varphi.
\end{aligned}
$$

This and (2.5) imply that

$$
\begin{aligned}
&\int_{\mathbb{S}^{d-1}} \Phi(rx', ty')\, d\sigma_k(x') \\
&= \frac{2^{d_k/2-\alpha}}{\Gamma(\alpha/2)} c_{\lambda_k} \int_0^\infty s^{(d_k-\alpha)/2-1} \int_0^\pi e^{-s(r^2+t^2-2rt\cos\varphi)} \sin^{d_k-2}\varphi\, d\varphi\, ds.
\end{aligned}
$$

Finally, applying (2.8) gives

$$
\begin{aligned}
&\int_{\mathbb{S}^{d-1}} \Phi(rx', ty')\, d\sigma_k(x') \\
&= (\gamma_\alpha^k)^{-1} c_{\lambda_k} \int_0^\pi \big(r^2+t^2-2rt\cos\varphi\big)^{(\alpha-d_k)/2} \sin^{d_k-2}\varphi\, d\varphi = \Phi_0(r,t),
\end{aligned}
$$

i.e., the property (3) follows.

Let us prove the property (4). Since for radial functions $f(x) = f_0(|x|) \in \mathcal{S}(\mathbb{R}^d)$ [22, 28]

$$
\tau^{-y} f(x) = \int_{\mathbb{R}^d} f_0\big(\sqrt{|x|^2+|y|^2-2\langle y,\eta\rangle}\big)\, d\mu_x^k(\eta),
$$

where μ_x^k is the probability measure in (1.3), we derive

$$
\tau^{-y}(e^{-s|\cdot|^2})(x) = \int_{\mathbb{R}^d} e^{-s(|x|^2+|y|^2-2\langle y,\eta\rangle)}\, d\mu_x^k(\eta)
$$

and

$$
\begin{aligned}
\Phi(x,y) &= \frac{2^{d_k/2-\alpha}}{\Gamma(\alpha/2)} \int_0^\infty s^{(d_k-\alpha)/2-1} \int_{\mathbb{R}^d} e^{-s(|x|^2+|y|^2-2\langle y,\eta\rangle)}\, d\mu_x^k(\eta)\, ds \\
&= \frac{2^{d_k/2-\alpha}}{\Gamma(\alpha/2)} \int_{\mathbb{R}^d} \int_0^\infty s^{(d_k-\alpha)/2-1} e^{-s(|x|^2+|y|^2-2\langle y,\eta\rangle)}\, ds\, d\mu_x^k(\eta) \\
&= (\gamma_\alpha^k)^{-1} \int_{\mathbb{R}^d} (|x|^2+|y|^2-2\langle y,\eta\rangle)^{(\alpha-d_k)/2}\, d\mu_x^k(\eta),
\end{aligned}
$$

where we have used the Tonelli–Fubini Theorem for nonnegative functions. □

3. Sharp Hardy's inequalities

Define the Hardy and Bellman operators as follows

$$Hf(x)=\int_{|y|\le|x|}f(y)\,d\mu_k(y)$$

and

$$Bf(x)=\int_{|y|\ge|x|}f(y)\,d\mu_k(y).$$

Let $1\le p\le\infty$. We are interested in the weighted Hardy inequalities of the form

$$\big\||x|^{-a}Hf(x)\big\|_{p,d\mu_k}\le \mathbf{c}_k^H(a,b,p,d)\big\||x|^{b}f(x)\big\|_{p,d\mu_k} \tag{3.1}$$

and

$$\big\||x|^{-a}Bf(x)\big\|_{p,d\mu_k}\le \mathbf{c}_k^B(a,b,p,d)\big\||x|^{b}f(x)\big\|_{p,d\mu_k} \tag{3.2}$$

with the sharp constants $\mathbf{c}_k^H(a,b,p,d)$ and $\mathbf{c}_k^B(a,b,p,d)$.

In the classical setting ($k\equiv 0$), the sharp constants were calculated by M. Christ and L. Grafakos [5] in the non-weighted case ($b=0$, $a=d$) and later by Z.W. Fu, L. Grafakos, S.Z. Lu and F.Y. Zhao [8] in the general case. We extend these results for the Dunkl setting. Recall that

$$\lambda_k=\frac{d}{2}-1+\sum_{a\in R_+}k(a),\quad d_k=2\lambda_k+2,\quad b_{\lambda_k}=\frac{1}{2^{\lambda_k}\Gamma(\lambda_k+1)}.$$

Theorem 3.1. *Let $d\in\mathbb{N}$ and $1\le p\le\infty$. Inequality* (3.1) *holds with $\mathbf{c}_k^H(a,b,p,d)<\infty$ if and only if $\frac{a}{p'}>\frac{b}{p}$ and $a+b=d_k$. Moreover,*

$$\mathbf{c}_k^H(a,b,p,d)=\frac{b_{\lambda_k}}{\frac{a}{p'}-\frac{b}{p}}.$$

Proof. Assume that $\frac{a}{p'}>\frac{b}{p}$ and $a+b=d_k$. We consider

$$\widetilde{H}f(x)=\int_{|y|\le|x|}|y|^{d_k/p'-b}f(y)\,dm_k(y).$$

According to (2.1), inequality (3.1) is equivalent to the following estimate

$$b_{\lambda_k}\big\||x|^{-a+d_k/p}\widetilde{H}f(x)\big\|_{p,dm_k}\le \mathbf{c}_k^H(a,b,p,d)\|f\|_{p,dm_k}.$$

If $x=rx'$, $y=ty'$, then changing variables $y\to(r/t)y'$ yields

$$\widetilde{H}f(x)=r^{d_k/p'-b}\int_{\mathbb{R}^d}f((r/t)y')g_0(t)\,dm_k(ty'),$$

where

$$g_0(t)=t^{b-d_k/p'}\chi_{[1,\infty)}(t).$$

Hence, by (2.2), we have

$$|x|^{-a+d_k/p}\widetilde{H}f(x)=\int_{\mathbb{R}^d}f((r/t)y')g_0(t)\,dm_k(ty')=\int_{\mathbb{R}^d}f(ty')g_0(r/t)\,dm_k(ty').$$

Let us consider the integral

$$\begin{aligned}J&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}h(rx')f(ty')g_0(r/t)\,dm_k(x)\,dm_k(y)\\&=\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\int_0^\infty\int_0^\infty h(rx')f(ty')g_0(r/t)\,d\nu(t)\,d\nu(r)\,d\sigma_k(x')\,d\sigma_k(y').\end{aligned}$$

Using Hölder's inequality and Lemma 2.1, we obtain

$$\begin{aligned}|J|&\le\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\Big(\int_0^\infty|h(rx')|^{p'}\,d\nu(r)\Big)^{1/p'}\Big(\int_0^\infty|f(ty')|^{p}\,d\nu(t)\Big)^{1/p}\\&\qquad\int_0^\infty g_0(r/t)\,d\nu(t)\,d\sigma_k(x')\,d\sigma_k(y')\\&\le\|g_0\|_1\Big(\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\int_0^\infty|h(rx')|^{p'}\,d\nu(r)\,d\sigma_k(x')\,d\sigma_k(y')\Big)^{1/p'}\\&\qquad\Big(\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\int_0^\infty|f(ty')|^{p}\,d\nu(t)\,d\sigma_k(x')\,d\sigma_k(y')\Big)^{1/p}\\&=\|g_0\|_1\|h\|_{p',dm_k}\|f\|_{p,dm_k}.\end{aligned}$$

Hence,

$$\mathbf{c}_k^H(a,b,p,d)\le b_{\lambda_k}\|g_0\|_1=b_{\lambda_k}\int_1^\infty t^{b-d_k/p'}\,\frac{dt}{t}=\frac{b_{\lambda_k}}{\frac{a}{p'}-\frac{b}{p}}.$$

Considering radial functions $f(x)=f_0(|x|)=f_0(r)$, we note that

$$\widetilde{H}f(x)=r^{d_k/p'-b}\int_{\mathbb{R}^d}f(r/t)g_0(t)\,\frac{dt}{t}$$

and

$$|x|^{a-d_k/p}\widetilde{H}f(x)=\int_{\mathbb{R}^d}f(r/t)g_0(t)\,\frac{dt}{t}.$$

Thus, Lemma 2.1 yields that

$$\mathbf{c}_k^H(a,b,p,d)=b_{\lambda_k}\|g_0\|_1=\frac{b_{\lambda_k}}{\frac{a}{p'}-\frac{b}{p}}.$$

Note that, in particular, this implies that the condition $\frac{a}{p'}>\frac{b}{p}$ is necessary for $\mathbf{c}_k^H(a,b,p,d)<\infty$ to hold. Moreover, if $f_t(x)=f(tx)$, then

$$Hf_t(x)=t^{-d_k}(Hf)_t(x),\quad \big\||x|^bf_t(x)\big\|_{p,d\mu_k}=t^{-b-d_k/p}\big\||x|^bf(x)\big\|_{p,d\mu_k}$$

and inequality (3.1) can be written as

$$t^{-d_k(1+1/p)+a}\big\||x|^{-a}Hf(x)\big\|_{p,d\mu_k}\le t^{-b-d_k/p}\mathbf{c}_k^H(a,b,p,d)\big\||x|^bf(x)\big\|_{p,d\mu_k},$$

which gives the condition $a+b=d_k$. □

Similarly, we prove the sharp Hardy's inequality for Bellman transform.

Theorem 3.2. *Let* $d \in \mathbb{N}$ *and* $1 \le p \le \infty$. *Inequality* (3.2) *holds with* $\mathbf{c}_k^B(a,b,p,d) < \infty$ *if and only if* $\frac{a}{p'} < \frac{b}{p}$ *and* $a+b=d_k$. *Moreover,*

$$\mathbf{c}_k^B(a,b,p,d) = \frac{b_{\lambda_k}}{\frac{b}{p}-\frac{a}{p'}}.$$

Proof. We only sketch the proof. Considering

$$\widetilde{B}f(x) = \int_{|y|\ge|x|} |y|^{d_k/p'-b} f(y)\, dm_k(y)$$

and (2.1), we rewrite inequality (3.2) as follows

$$b_{\lambda_k}\||x|^{-a+d_k/p}\widetilde{B}f(x)\|_{p,dm_k} \le \mathbf{c}_k^B(a,b,p,d)\||f\|_{p,dm_k}.$$

Then we have

$$\widetilde{B}f(x) = r^{d_k/p'-b}\int_{\mathbb{R}^d} f((r/t)y')g_0(t)\, dm_k(ty'),$$

where

$$g_0(t) = t^{b-d_k/p'}\chi_{[0,1]}(t).$$

Finally,

$$\mathbf{c}_k^B(a,b,p,d) = b_{\lambda_k}\|g_0\|_1 = b_{\lambda_k}\int_0^1 t^{b-d_k/p'}\,\frac{dt}{t} = \frac{b_{\lambda_k}}{\frac{b}{p}-\frac{a}{p'}}.$$ □

4. Proof of Theorem 1.3

Recall that we consider the case $k \not\equiv 0$, $\lambda_k > -1/2$ and $d_k > 1$. Let $1 < p < \infty$, $\gamma < \frac{d_k}{p}$, $\beta < \frac{d_k}{p'}$, $\alpha > 0$, and $\alpha = \gamma + \beta$. Consider the modified operator

$$\widetilde{I}_\alpha^k f(x) = \int_{\mathbb{R}^d} f(y)|y|^{d_k/p'-\beta}\Phi(x,y)\, dm_k(y).$$

According to (2.1), inequality (1.7) for $q = p$ is equivalent to

$$b_{\lambda_k}\big\||x|^{-\gamma+d_k/p}\widetilde{I}_\alpha^k f(x)\big\|_{p,dm_k} \le \mathbf{c}_k(\alpha,\beta,\gamma,p,p,d)\|f(x)\|_{p,dm_k}.$$

If $x = rx'$, $y = ty'$, then using the change of variables $y \to (r/t)y'$ and applying the properties (1), (2) in Lemma 2.3, we have

$$\widetilde{I}_\alpha^k f(x) = r^{-\beta+\alpha-d_k/p}\int_{\mathbb{R}^d} f((r/t)y')\Phi_1(t,x',y')\, dm_k(ty'),$$

where

$$\Phi_1(t,x',y') = t^{d_k/p-\alpha+\beta}\Phi(tx',y').$$

Hence, by (2.2),

$$|x|^{-\gamma+d_k/p}\widetilde{I}_\alpha^k f(x) = \int_{\mathbb{R}^d} f(ty')\Phi_1(r/t,x',y')\, dm_k(ty').$$

We set

$$J := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(rx')f(ty')\Phi_1(r/t,x',y')\,dm_k(x)\,dm_k(y)$$
$$= \int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\int_0^\infty\int_0^\infty h(rx')f(ty')\Phi_1(r/t,x',y')\,d\nu(t)\,d\nu(r)\,d\sigma_k(x')\,d\sigma_k(y').$$

In light of Lemma 2.1 and Hölder's inequality, we have

$$|J| \le \int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\Big(\int_0^\infty |h(rx')|^{p'}\,d\nu(r)\Big)^{1/p'}\Big(\int_0^\infty |f(ty')|^p\,d\nu(t)\Big)^{1/p}$$
$$\int_0^\infty \Phi_1(t,x',y')\,d\nu(t)\,d\sigma_k(x')\,d\sigma_k(y')$$
$$= \int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\Big(\int_0^\infty |h(rx')|^{p'}\,d\nu(r)\int_0^\infty \Phi_1(t,x',y')\,d\nu(t)\Big)^{1/p'}$$
$$\Big(\int_0^\infty |f(ty')|^p\,d\nu(t)\int_0^\infty \Phi_1(t,x',y')\,d\nu(t)\Big)^{1/p}\,d\sigma_k(x')\,d\sigma_k(y')$$
$$\le \Big(\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\int_0^\infty |h(rx')|^{p'}\,d\nu(r)\int_0^\infty \Phi_1(t,x',y')\,d\nu(t)\,d\sigma_k(x')\,d\sigma_k(y')\Big)^{1/p'}$$
$$\Big(\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\int_0^\infty |f(ty')|^p\,d\nu(t)\int_0^\infty \Phi_1(t,x',y')\,d\nu(t)\,d\sigma_k(x')\,d\sigma_k(y')\Big)^{1/p}.$$

Taking into account the properties (1) and (3) of Lemma 2.3, we have

$$\int_{\mathbb{S}^{d-1}}\Phi_1(t,x',y')\,d\sigma_k(x') = t^{d_k/p-\alpha+\beta}\int_{\mathbb{S}^{d-1}}\Phi(tx',y')\,d\sigma_k(x') = t^{d_k/p-\alpha+\beta}\Phi_0(t,1)$$

and

$$\int_{\mathbb{S}^{d-1}}\Phi_1(t,x',y')\,d\sigma_k(y') = t^{d_k/p-\alpha+\beta}\Phi_0(1,t) = t^{d_k/p-\alpha+\beta}\Phi_0(t,1).$$

Then, changing the order of integration implies

$$|J| \le \Big(\int_0^\infty t^{d_k/p-\alpha+\beta}\Phi_0(t,1)\,d\nu(t)\Big)^{1/p'}\Big(\int_{\mathbb{S}^{d-1}}\int_0^\infty |h(rx')|^{p'}\,d\nu(r)\,d\sigma_k(x')\Big)^{1/p'}$$
$$\Big(\int_0^\infty t^{d_k/p-\alpha+\beta}\Phi_0(t,1)\,d\nu(t)\Big)^{1/p}\Big(\int_{\mathbb{S}^{d-1}}\int_0^\infty |f(ty')|^p\,d\nu(t)\,d\sigma_k(y')\Big)^{1/p}$$
$$= \int_0^\infty t^{d_k/p-\alpha+\beta}\Phi_0(t,1)\,d\nu(t)\,\|h\|_{p',dm_k}\|f\|_{p,dm_k}.$$

Thus,

$$\mathbf{c}_k(\alpha,\beta,\gamma,p,p,d) \le b_{\lambda_k}\int_0^\infty t^{d_k/p-\alpha+\beta}\Phi_0(t,1)\,d\nu(t).$$

Since for radial functions $f(x) = f_0(|x|) = f_0(r)$ we have that

$$|x|^{-\gamma+d_k/p}\widetilde{I}_\alpha^k f(x) = \int_0^\infty f_0(r/t)t^{d_k/p-\alpha+\beta}\Phi_0(t,1)\,\frac{dt}{t},$$

Lemma 2.1 gives

$$\mathbf{c}_k(\alpha,\beta,\gamma,p,p,d)=b_{\lambda_k}\int_0^\infty t^{d_k/p-\alpha+\beta}\Phi_0(t,1)\,d\nu(t).$$

Let us now prove that the conditions $\gamma<\frac{d_k}{p}$, $\beta<\frac{d_k}{p'}$, and $\gamma+\beta=\alpha>0$ guarantee that $\mathbf{c}_k(\alpha,\beta,\gamma,p,p,d)<\infty$. We have

$$\begin{aligned}&\mathbf{c}_k(\alpha,\beta,\gamma,p,p,d)\\&=(\gamma_\alpha^k)^{-1}c_{\lambda_k}b_{\lambda_k}\int_0^\infty t^{d_k/p-\alpha+\beta}\int_0^\pi\bigl(t^2+1-2t\cos\varphi\bigr)^{(\alpha-d_k)/2}\sin^{d_k-2}\varphi\,d\varphi\,d\nu(t)\\&=(\gamma_\alpha^k)^{-1}c_{\lambda_k}b_{\lambda_k}\int_0^\infty\frac{t^{d_k/p-\alpha+\beta}}{(1+t^2)^{(d_k-\alpha)/2}}\int_0^\pi\Bigl(1-\frac{2t\cos\varphi}{1+t^2}\Bigr)^{(\alpha-d_k)/2}\sin^{d_k-2}\varphi\,d\varphi\,d\nu(t).\end{aligned}$$

The integral with respect to t has singularities at $t=0,1,\infty$. It converges at the origin if and only if $\gamma=\alpha-\beta<\frac{d_k}{p}$. Moreover, the integral converges at ∞ if and only if $\beta<\frac{d_k}{p'}$. Concerning the point $t=1$, we set $r:=2t/(1+t^2)$ and note that, letting $r\to1-0$,

$$\begin{aligned}\psi(r)&:=\int_0^\pi(1-r\cos\varphi)^{(\alpha-d_k)/2}\sin^{d_k-2}\varphi\,d\varphi\\&\asymp\int_0^1(1-r+r\varphi^2/2)^{(\alpha-d_k)/2}\varphi^{d_k-2}\,d\varphi+1\\&\asymp\int_0^{\sqrt{1-r}}(1-r)^{(\alpha-d_k)/2}\varphi^{d_k-2}\,d\varphi+\int_{\sqrt{1-r}}^1\varphi^{\alpha-2}\,d\varphi+1\\&\asymp\begin{cases}(1-r)^{\frac{\alpha-1}{2}}, & 0<\alpha<1,\\-\ln(1-r), & \alpha=1,\\1, & \alpha>1.\end{cases}\end{aligned}$$

Therefore, letting $t\to1$, we have

$$\int_0^\pi\Bigl(1-\frac{2t\cos\varphi}{1+t^2}\Bigr)^{(\alpha-d_k)/2}\sin^{d_k-2}\varphi\,d\varphi\asymp\begin{cases}|1-t|^{\alpha-1}, & 0<\alpha<1,\\-\ln|1-t|, & \alpha=1,\\1, & \alpha>1,\end{cases}$$

which implies that the singularity at the point $t=1$ is integrable.

It remains to calculate the integral $\int_0^\infty t^{d_k/p-\alpha+\beta}\Phi_0(t,1)\,d\nu(t)$. Let $t\ne1$, $r=2t/(1+t^2)$. The series

$$\begin{aligned}(1-r\cos\varphi)^{(\alpha-d_k)/2}&=\Gamma\Bigl(\frac{\alpha-d_k}{2}+1\Bigr)\sum_{n=0}^\infty\frac{(-1)^n}{\Gamma(n+1)\Gamma(\frac{\alpha-d_k}{2}+1-n)}r^n\cos^n\varphi\\&=\frac{1}{\Gamma(\frac{d_k-\alpha}{2})}\sum_{n=0}^\infty\frac{(-1)^n\Gamma(\frac{d_k-\alpha}{2}+n)}{\Gamma(n+1)}r^n\cos^n\varphi\end{aligned}$$

converges uniformly on $[0,\pi]$ and

$$\psi(r)=\int_0^{\pi}(1-r\cos\varphi)^{(\alpha-d_k)/2}\sin^{d_k-2}\varphi\,d\varphi$$
$$=\frac{1}{\Gamma(\frac{d_k-\alpha}{2})}\sum_{m=0}^{\infty}\frac{\Gamma(\frac{d_k-\alpha}{2}+2m)}{\Gamma(2m+1)}r^{2m}\int_0^{\pi}\cos^{2m}\varphi\sin^{d_k-2}\varphi\,d\varphi$$
$$=\frac{1}{\Gamma(\frac{d_k-\alpha}{2})}\sum_{m=0}^{\infty}\frac{\Gamma(m+\frac{1}{2})\Gamma(\frac{d_k-\alpha}{2}+2m)\Gamma(\frac{d_k-1}{2})}{\Gamma(2m+1)\Gamma(\frac{d_k}{2}+m)}r^{2m}.$$

Since a positive series can be integrated term-by-term, it follows that

$$\mathbf{c}_k(\alpha,\beta,\gamma,p,p,d)=b_{\lambda_k}\int_0^{\infty}t^{d_k/p-\alpha+\beta}\Phi_0(t,1)\,d\nu(t)$$
$$=\frac{(\gamma_\alpha^k)^{-1}c_{\lambda_k}b_{\lambda_k}}{\Gamma(\frac{d_k-\alpha}{2})}\sum_{m=0}^{\infty}\frac{2^{2m}\Gamma(m+\frac{1}{2})\Gamma(\frac{d_k-\alpha}{2}+2m)\Gamma(\frac{d_k-1}{2})}{\Gamma(2m+1)\Gamma(\frac{d_k}{2}+m)}\int_0^{\infty}\frac{t^{d_k/p-\alpha+\beta+2m-1}}{(1+t^2)^{(d_k-\alpha)/2+2m}}\,dt.$$

Taking into account that

$$\int_0^{\infty}\frac{t^{d_k/p-\alpha+\beta+2m-1}}{(1+t^2)^{(d_k-\alpha)/2+2m}}\,dt=\frac{\Gamma(\frac{d_k}{2p}+\frac{\beta-\alpha}{2}+m)\Gamma(\frac{d_k}{2p'}-\frac{\beta}{2}+m)}{2\Gamma(\frac{d_k-\alpha}{2}+2m)}$$

and

$$\gamma_\alpha^k=\frac{2^{\alpha-d_k/2}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d_k-\alpha}{2})},\quad c_{\lambda_k}=\frac{\Gamma(\frac{d_k}{2})}{\Gamma(1/2)\Gamma(\frac{d_k-1}{2})},\quad b_{\lambda_k}=\frac{1}{2^{d_k/2-1}\Gamma(\frac{d_k}{2})},$$

we arrive at

$$\mathbf{c}_k(\alpha,\beta,\gamma,p,p,d)=\frac{2^{-\alpha}}{\Gamma(\alpha/2)}\sum_{m=0}^{\infty}\frac{\Gamma(\frac{d_k}{2p}+\frac{\beta-\alpha}{2}+m)\Gamma(\frac{d_k}{2p'}-\frac{\beta}{2}+m)}{\Gamma(m+1)\Gamma(\frac{d_k}{2}+m)}.$$

Letting

$$a=\frac{d_k}{2p}+\frac{\beta-\alpha}{2},\quad b=\frac{d_k}{2p'}-\frac{\beta}{2},\quad c=\frac{d_k}{2},$$

we write

$$\mathbf{c}_k(\alpha,\beta,\gamma,p,p,d)=\frac{2^{-\alpha}}{\Gamma(\alpha/2)}\sum_{m=0}^{\infty}\frac{\Gamma(a+m)\Gamma(b+m)}{\Gamma(1+m)\Gamma(c+m)}.$$

Using now the hypergeometric function [2, Ch. II]

$$F(a,b;c;z)=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{m!(c)_m}z^k,\quad (a)_m=\frac{\Gamma(a+m)}{\Gamma(a)},$$

we obtain that

$$\mathbf{c}_k(\alpha,\beta,\gamma,p,p,d)=\frac{2^{-\alpha}}{\Gamma(\alpha/2)}\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}F(a,b;c;1).$$

Finally, since [2, Sect. 2.8, (46)]

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\quad c\neq 0,-1,-2,\dots,\quad c>a+b,$$

we have

$$\mathbf{c}_k(\alpha,\beta,\gamma,p,p,d)=\frac{2^{-\alpha}\Gamma(a)\Gamma(b)\Gamma(c-a-b)}{\Gamma(\alpha/2)\Gamma(c-a)\Gamma(c-b)},$$

where

$$c-a-b=\frac{d_k}{2}-\Big(\frac{d_k}{2p}+\frac{\beta-\alpha}{2}+\frac{d_k}{2p'}-\frac{\beta}{2}\Big)=\frac{\alpha}{2},$$
$$c-a=\frac{d_k}{2}-\Big(\frac{d_k}{2p}+\frac{\beta-\alpha}{2}\Big)=\frac{d_k}{2p'}+\frac{\alpha-\beta}{2},$$
$$c-b=\frac{d_k}{2}-\Big(\frac{d_k}{2p'}-\frac{\beta}{2}\Big)=\frac{d_k}{2p}+\frac{\beta}{2},$$

or, equivalently,

$$\mathbf{c}_k(\alpha,\beta,\gamma,p,p,d)=2^{-\alpha}\,\frac{\Gamma(\frac{1}{2}(\frac{d_k}{p}-\gamma))\Gamma(\frac{1}{2}(\frac{d_k}{p'}-\beta))}{\Gamma(\frac{1}{2}(\frac{d_k}{p'}+\gamma))\Gamma(\frac{1}{2}(\frac{d_k}{p}+\beta))}.\qquad\square$$

Remark 4.1. It is clear that the condition $\alpha=\gamma+\beta$ is necessary for $\mathbf{c}_k(\alpha,\beta,\gamma,p,p,d)<\infty$ to hold. Indeed, setting $f_t(x)=f(tx)$, we have

$$\mathcal{F}_k(f_t)(z)=t^{-d_k}\mathcal{F}_k(f)\Big(\frac{z}{t}\Big),\quad \tau^y f_t(x)=\tau^{ty}f(tx),\quad I_\alpha^k f_t(x)=t^{-\alpha}(I_\alpha^k f)_t(x),$$
$$\||x|^\beta f_t(x)\|_{p,d\mu_k}=t^{-\beta-d_k/p}\||x|^\beta f(x)\|_{p,d\mu_k}.$$

Writing inequality (1.7) with $q=p$ as follows

$$t^{\gamma-\alpha-d_k/p}\||x|^{-\gamma}I_\alpha^k f(x)\|_{p,d\mu_k}\le t^{-\beta-d_k/p}\mathbf{c}_k(\alpha,\beta,\gamma,p,p,d)\||x|^\beta f(x)\|_{p,d\mu_k}$$

implies $\alpha=\gamma+\beta$.

5. Proof of Theorem 1.4

Part (a). Let $1<p<q<\infty$, $\gamma<\frac{d_k}{q}$, $\beta<\frac{d_k}{p'}$, $\gamma+\beta\ge 0$, $0<\alpha<d_k$, and $\alpha-\gamma-\beta=d_k(\frac{1}{p}-\frac{1}{q})$. Note that the case $q=p$ was studied in Theorem 1.3. We will use the representation of the kernel $\Phi(x,y)$ given in Lemma 2.3 and then essentially follow the ideas of [27].

We write

$$\widetilde{I}_\alpha^k f(x)=\int_{\mathbb{R}^d} f(y)|y|^{-\beta}\Phi_\alpha(x,y)\,d\mu_k(y),$$

where

$$f\in\mathcal{S}(\mathbb{R}^d),\quad \Phi_\alpha(x,y)=\int_{\mathbb{R}^d}(|x|^2+|y|^2-2\langle y,\eta\rangle)^{(\alpha-d_k)/2}\,d\mu_x^k(\eta),$$

and

$$\operatorname{supp}\mu_x^k\subset B_{|x|}=\{\eta\colon |\eta|\le|x|\}.$$

We define

$$J:=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)g(x)\,\frac{\Phi_\alpha(x,y)}{|x|^\gamma|y|^\beta}\,d\mu_k(y)\,d\mu_k(x).$$

It is sufficient to prove the inequality

$$J\lesssim \|f\|_{p,d\mu_k}\|g\|_{q',d\mu_k} \tag{5.1}$$

for $f,g\geq 0$.

Recall that in the case $1<p<q<\infty$, $\gamma=\beta=0$, and $\alpha=d_k(\frac{1}{p}-\frac{1}{q})$ inequality (5.1) holds (see [1, 11]). Let

$$\mathbb{R}^d\times\mathbb{R}^d=E_1\sqcup E_2\sqcup E_3,$$

where

$$E_1=\{(x,y)\colon 2^{-1}|y|<|x|<2|y|\},$$
$$E_2=\{(x,y)\colon |x|\leq 2^{-1}|y|\},$$
$$E_3=\{(x,y)\colon |y|\leq 2^{-1}|x|\}.$$

Then

$$J=\iint_{E_1}+\iint_{E_2}+\iint_{E_3}=J_1+J_2+J_3.$$

<u>Estimate of J_1.</u> If $(x,y)\in E_1$, using $|\eta|\leq|x|$, then by conditions $\alpha-\beta-\gamma=d_k(\frac{1}{p}-\frac{1}{q})$, $\gamma+\beta\geq 0$ we have

$$\begin{aligned}(|x|^2+|y|^2-2\langle y,\eta\rangle)^{\frac{\gamma+\beta}{2}}&\leq(|x|^2+4|x|^2+2|x||y|)^{\frac{\gamma+\beta}{2}}\\&\lesssim|x|^{\gamma+\beta}\lesssim|x|^\gamma|y|^\beta\end{aligned}$$

and

$$\begin{aligned}\frac{(|x|^2+|y|^2-2\langle y,\eta\rangle)^{\frac{\alpha-d_k}{2}}}{|x|^\gamma|y|^\beta}&\lesssim(|x|^2+|y|^2-2\langle y,\eta\rangle)^{\frac{\alpha-\beta-\gamma-d_k}{2}}\\&=(|x|^2+|y|^2-2\langle y,\eta\rangle)^{(d_k(\frac{1}{p}-\frac{1}{q})-d_k)/2}.\end{aligned}$$

Set $\widetilde{\alpha}=d_k(\frac{1}{p}-\frac{1}{q})$. By (5.1) with $\gamma=\beta=0$ and $0<\widetilde{\alpha}<d_k$, we have

$$J_1\lesssim\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)g(x)\,\Phi_{\widetilde{\alpha}}(x,y)\,d\mu_k(y)\,d\mu_k(x)\lesssim\|f\|_{p,d\mu_k}\|g\|_{q',d\mu_k}.$$

<u>Estimate of J_2.</u> If $(x,y)\in E_2$, then

$$\sqrt{|x|^2+|y|^2-2\langle y,\eta\rangle}\geq\sqrt{|x|^2+|y|^2-2|x||y|}\geq|y|-|x|\geq 2^{-1}|y|,$$

therefore

$$\begin{aligned}\Phi_\alpha(x,y)&=\int_{\mathbb{R}^d}\frac{1}{(\sqrt{|x|^2+|y|^2-2\langle y,\eta\rangle})^{d_k-\alpha}}\,d\mu_x^k(\eta)\\&\lesssim|y|^{\alpha-d_k}\int_{\mathbb{R}^d}d\mu_x^k(\eta)=|y|^{\alpha-d_k}.\end{aligned}$$

From here and since $E_2 \subset \{(x,y)\colon |x| \le |y|\}$,

$$J_2 \lesssim \iint_{|x|\le|y|} \frac{f(y)g(x)}{|x|^{\gamma}|y|^{\beta-\alpha+d_k}}\, d\mu_k(x)\, d\mu_k(y)$$
$$= \int_{\mathbb{R}^d} f(y)|y|^{\alpha-\beta-d_k} \int_{|x|\le|y|} g(x)|x|^{-\gamma}\, d\mu_k(x)\, d\mu_k(y)$$
$$= \int_{\mathbb{R}^d} f(y)|y|^{\alpha-\beta-\gamma} Vg(y)\, d\mu_k(y),$$

where

$$Vg(y) = |y|^{\gamma-d_k} \int_{|x|\le|y|} g(x)|x|^{-\gamma}\, d\mu_k(x).$$

Note that

$$Vg(y) \le |y|^{\gamma-d_k} \Big(\int_{|x|\le|y|} |x|^{-q\gamma}\, d\mu_k(x)\Big)^{1/q} \|g\|_{q',d\mu_k}$$
$$\lesssim |y|^{\gamma-d_k}|y|^{d_k/q-\gamma}\|g\|_{q',d\mu_k} = |y|^{-d_k/q'}\|g\|_{q',d\mu_k}.$$

Hence

$$|Vg(y)|^{p'-q'}|y|^{(\alpha-\beta-\gamma)p'} \lesssim |x|^{-d_k(p'-q')/q'+(\alpha-\beta-\gamma)p'}\|g\|_{q',d\mu_k}^{p'-q'}.$$

Since

$$-\frac{d_k(p'-q')}{q'} + (\alpha-\beta-\gamma)p' = p'\Big\{\alpha-\beta-\gamma-d_k\Big(\frac{1}{q'}-\frac{1}{p'}\Big)\Big\}$$
$$= p'\Big\{\alpha-\beta-\gamma+d_k\Big(\frac{1}{q}-\frac{1}{p}\Big)\Big\} = 0,$$

it follows that

$$|Vg(y)|^{p'-q'}|y|^{(\alpha-\beta-\gamma)p'} \lesssim \|g\|_{q',d\mu_k}^{p'-q'}. \tag{5.2}$$

On the other hand, by Theorem 3.1 with $a = d_k - \gamma$, $b = \gamma$, $p = q'$, and $\frac{a}{q} > \frac{b}{q'}$ (or, equivalently, $\gamma < \frac{d_k}{q}$), we see that

$$\|Vg\|_{q',d\mu_k} \lesssim \|g\|_{q',d\mu_k}. \tag{5.3}$$

Using (5.2) and (5.3), we have

$$\int_{\mathbb{R}^d} |Vg(y)|^{p'}|y|^{(\alpha-\beta-\gamma)p'}\, d\mu_k(y) = \int_{\mathbb{R}^d} |Vg(y)|^{q'}|Vg(y)|^{p'-q'}|y|^{(\alpha-\beta-\gamma)p'}\, d\mu_k(y)$$
$$\lesssim \int_{\mathbb{R}^d} |Vg(y)|^{q'}\, d\mu_k(y)\, \|g\|_{q',d\mu_k}^{p'-q'} \lesssim \|g\|_{q',d\mu_k}^{p'}.$$

This gives

$$J_2 \lesssim \|f\|_{p,d\mu_k}\big\||y|^{\alpha-\beta-\gamma}Vg(y)\big\|_{p',d\mu_k} \lesssim \|f\|_{p,d\mu_k}\|g\|_{q',d\mu_k}.$$

Note that, for $p=1$, a similar result is valid as well, i.e.,

$$J_2 \lesssim \|f\|_{1,d\mu_k}\|g\|_{q',d\mu_k}, \qquad \gamma < \frac{d_k}{q},\ \beta \le 0, \tag{5.4}$$

since $\alpha-\beta-\gamma = d_k/q'$ and

$$|y|^{\alpha-\beta-\gamma}Vg(y) \lesssim |y|^{\alpha-\beta-\gamma}|y|^{-d_k/q'}\|g\|_{q',d\mu_k} = \|g\|_{q',d\mu_k}.$$

Estimate of J_3. If $(x,y)\in E_3$, we similarly have

$$\sqrt{|x|^2+|y|^2-2\langle y,\eta\rangle} \ge 2^{-1}|x|,$$

$$\Phi_\alpha(x,y) = \int_{\mathbb{R}^d} \frac{1}{(\sqrt{|x|^2+|y|^2-2\langle y,\eta\rangle})^{d_k-\alpha}}\,d\mu_x^k(\eta) \lesssim |x|^{\alpha-d_k}$$

and

$$J_3 \lesssim \iint_{|y|\le|x|} \frac{f(y)g(x)}{|x|^{\gamma-\alpha+d_k}|y|^\beta}\,d\mu_k(x)\,d\mu_k(y) = \int_{\mathbb{R}^d} g(x)|x|^{\alpha-\beta-\gamma}Vf(x)\,d\mu_k(x),$$

where

$$Vf(x) = |x|^{\beta-d_k}\int_{|y|\le|x|} f(y)|y|^{-\beta}\,d\mu_k(y).$$

Since

$$\begin{aligned} Vf(x) &\le |x|^{\beta-d_k}\Big(\int_{|y|\le|x|}|y|^{-p'\beta}\,d\mu_k(y)\Big)^{1/p'}\|f\|_{p,d\mu_k} \\ &\lesssim |x|^{\beta-d_k}|x|^{d_k/p'-\beta}\|f\|_{p,d\mu_k} = |x|^{-d_k/p}\|f\|_{p,d\mu_k}, \end{aligned}$$

we obtain

$$|Vf(x)|^{q-p}|x|^{(\alpha-\beta-\gamma)q} \lesssim |x|^{-d_k(q-p)/p+(\alpha-\beta-\gamma)q}\|f\|_{p,d\mu_k}^{q-p} = \|f\|_{p,d\mu_k}^{q-p}.$$

Taking into account Theorem 3.1 with $a=d_k-\beta$, $b=\beta$, $p=p$, $\frac{a}{p'}>\frac{b}{p}$ (or, $\beta<\frac{d_k}{p'}$), we obtain

$$\|Vf\|_{p,d\mu_k} \lesssim \|f\|_{p,d\mu_k},$$

which implies

$$\begin{aligned} \int_{\mathbb{R}^d}|Vf(x)|^q|x|^{(\alpha-\beta-\gamma)q}\,d\mu_k(x) &= \int_{\mathbb{R}^d}|Vf(x)|^p|Vf(x)|^{q-p}|x|^{(\alpha-\beta-\gamma)q}\,d\mu_k(x) \\ &\lesssim \int_{\mathbb{R}^d}|Vf(x)|^p\,d\mu_k(x)\,\|f\|_{p,d\mu_k}^{q-p} \lesssim \|f\|_{p,d\mu_k}^q. \end{aligned}$$

We finally have

$$J_3 \lesssim \|g\|_{q',d\mu_k}\big\||x|^{\alpha-\beta-\gamma}Vf(x)\big\|_{q,d\mu_k} \lesssim \|f\|_{p,d\mu_k}\|g\|_{q',d\mu_k}.$$

Again, the estimate $J_3 \lesssim \|f\|_{p,d\mu_k}\|g\|_{q',d\mu_k}$ also holds for $p=1$, $\gamma<\frac{d_k}{q}$, and $\beta<0$.

This completes the proof of part (a).

Part (b). Let us prove the weak (L^q, L^1)-boundedness of D-Riesz potential for $1<q<\infty$, $\gamma<\frac{d_k}{q}$, $\beta<0$, $\gamma+\beta\ge 0$, $0<\alpha<d_k$. We will use the notations and assumptions of part (a).

It is sufficient to prove the inequality

$$S=\int_{\{x\in\mathbb{R}^d:\ |x|^{-\gamma}\widetilde{I}_\alpha^k f(x)>\lambda\}} d\mu_k(x)\lesssim\Big(\frac{\|f\|_{1,d\mu_k}}{\lambda}\Big)^q. \tag{5.5}$$

Let us consider the operators

$$A_\alpha^i f(x)=\int_{\mathbb{R}^d} f(y)|y|^{-\beta}\Phi_\alpha(x,y)\chi_{E_i}(x,y)\,d\mu_k(y),\quad i=1,2,3.$$

We have

$$\widetilde{I}_\alpha^k=\sum_{i=1}^3 A_\alpha^i,$$

and

$$S=\sum_{i=1}^3\int_{\{x\in\mathbb{R}^d:\ |x|^{-\gamma}A_\alpha^i f(x)>\lambda/3\}} d\mu_k(x)=S_1+S_2+S_3.$$

<u>Estimate of S_1.</u> Applying the estimate of J_1 and inequality (5.5) with $1<q<\infty$, $\gamma=\beta=0$, and $\alpha=\widetilde{\alpha}=\frac{d_k}{q'}$ (see [1, 11]), we derive

$$A_\alpha^1 f(x)\lesssim\int_{\mathbb{R}^d} f(y)|y|^{-\beta}\Phi_{\widetilde{\alpha}}(x,y)\,d\mu_k(y)=\widetilde{I}_{\widetilde{\alpha}}^k f(x),$$

and

$$\begin{aligned}S_1&=\int_{\{x\in\mathbb{R}^d:\ |x|^{-\gamma}A_\alpha^1 f(x)>\lambda/3\}} d\mu_k(x)\\&\lesssim\int_{\{x\in\mathbb{R}^d:\ |x|^{-\gamma}\widetilde{I}_{\widetilde{\alpha}}^k f(x)\gtrsim\lambda\}} d\mu_k(x)\lesssim\Big(\frac{\|f\|_{1,d\mu_k}}{\lambda}\Big)^q.\end{aligned}$$

<u>Estimate of S_2.</u> Applying the obtained estimate of J_2, we get

$$A_\alpha^2 f(x)\lesssim\int_{|y|\ge|x|}|y|^{\alpha-\beta-d_k}f(y)\,d\mu_k(y)=B_1f(x).$$

Since

$$\begin{aligned}&\int_{\mathbb{R}^d} g(x)|x|^{-\gamma}\int_{|y|\ge|x|}|y|^{\alpha-\beta-d_k}f(y)\,d\mu_k(y)\,d\mu_k(x)\\&=\int_{\mathbb{R}^d} f(y)|y|^{\alpha-\beta-d_k}\int_{|x|\le|y|}g(x)|x|^{-\gamma}\,d\mu_k(x)\,d\mu_k(y),\end{aligned}$$

in light of (5.4) with $p=1$, $\gamma<\frac{d_k}{q}$, and $\beta\le 0$, we have

$$\big\||x|^{-\gamma}B_1f(x)\big\|_{q,d\mu_k}\lesssim\|f\|_{1,d\mu_k}.$$

Hence,

$$S_2 = \int_{\{x\in\mathbb{R}^d\colon\, |x|^{-\gamma}A_\alpha^2 f(x)>\lambda/3\}} d\mu_k(x)$$
$$\lesssim \int_{\{x\in\mathbb{R}^d\colon\, |x|^{-\gamma}B_1 f(x)\gtrsim\lambda\}} d\mu_k(x) \lesssim \Big(\frac{\|f\|_{1,d\mu_k}}{\lambda}\Big)^q.$$

Estimate of S_3. Applying the estimate of J_3, we obtain

$$A_\alpha^3 f(x) \lesssim |x|^{\alpha-d_k}\int_{|y|\le|x|}|y|^{-\beta}f(y)\,d\mu_k(y) = H_1 f(x).$$

Using the estimate $J_3 \lesssim \|f\|_{1,d\mu_k}\|g\|_{q',d\mu_k}$ with $\gamma<\frac{d_k}{q}$ and $\beta<0$ yields

$$\big\||x|^{-\gamma}H_1 f(x)\big\|_{q,d\mu_k} \lesssim \|f\|_{1,d\mu_k}.$$

Thus,

$$S_3 = \int_{\{x\in\mathbb{R}^d\colon\, |x|^{-\gamma}A_\alpha^3 f(x)>\lambda/3\}} d\mu_k(x)$$
$$\lesssim \int_{\{x\in\mathbb{R}^d\colon\, |x|^{-\gamma}H_1 f(x)\gtrsim\lambda\}} d\mu_k(x) \lesssim \Big(\frac{\|f\|_{1,d\mu_k}}{\lambda}\Big)^q. \tag{5.6}$$

If $\beta=0$, $\alpha-\gamma=d_k(1-\frac{1}{q})$, then

$$|x|^{-\gamma}H_1 f(x) = |x|^{\alpha-\gamma-d_k}\int_{|y|\le|x|} f(y)\,d\mu_k(y) = |x|^{-d_k/q}\int_{|y|\le|x|} f(y)\,d\mu_k(y).$$

Since inequality (5.6) is homogeneous, we can assume that $\|f\|_{1,d\mu_k}=1$. Therefore,

$$S_3 \lesssim \int_{\{x\in\mathbb{R}^d\colon\, |x|^{-d_k/q}\int_{|y|\le|x|} f(y)\,d\mu_k(y)\gtrsim\lambda\}} d\mu_k(x)$$
$$\lesssim \int_{\{x\in\mathbb{R}^d\colon\, |x|^{-d_k/q}\gtrsim\lambda\}} d\mu_k(x) \lesssim \lambda^{-q} = \Big(\frac{\|f\|_{1,d\mu_k}}{\lambda}\Big)^q,$$

completing the proof. □

Let us mention that for the so-called B-Riesz potentials the results that are similar to Theorem 1.4 were established in [9].

6. Properties of the spaces Φ_k and Ψ_k

Recall that T_j, $j=1,\dots,d$, are differential-differences Dunkl operators given by (1.2), $T^n=\prod_{j=1}^d T_j^{n_j}$, $n\in\mathbb{Z}_+^d$,

$$\Phi_k = \Big\{f\in\mathcal{S}(\mathbb{R}^d)\colon \int_{\mathbb{R}^d} x^n f(x)\,d\mu_k(x)=0,\quad n\in\mathbb{Z}_+^d\Big\},$$

and

$$\Psi_k = \big\{\mathcal{F}_k(f)\colon f\in\Phi_k\big\}.$$

If $f \in \mathcal{S}(\mathbb{R}^d)$, then from the definition of the generalized exponential function $e_k(x,y)$ and

$$f(x) = \int_{\mathbb{R}^d} e_k(x,y)\mathcal{F}_k(f)(y)\,d\mu_k(y),$$

we obtain that

$$T^n f(x) = i^n \int_{\mathbb{R}^d} y^n e_k(x,y)\mathcal{F}_k(f)(y)\,d\mu_k(y), \quad n \in \mathbb{Z}_+^d,$$

and

$$T^n f(0) = i^n \int_{\mathbb{R}^d} y^n \mathcal{F}_k(f)(y)\,d\mu_k(y).$$

Therefore,

$$\Psi_k = \big\{f \in \mathcal{S}(\mathbb{R}^d) \colon T^n f(0) = 0,\ n \in \mathbb{Z}_+^d\big\}.$$

Note that in the classical case ($k \equiv 0$) we have

$$\Psi = \Psi_0 = \big\{\mathcal{F}(f) \colon f \in \Phi\big\} = \big\{f \in \mathcal{S}(\mathbb{R}^d) \colon D^n f(0) = 0,\ n \in \mathbb{Z}_+^d\big\}.$$

Theorem 6.1. *We have $\Psi_k = \Psi$.*

Proof. Let $f \in \Psi$, $D = (D_1, \dots, D_d)$, and let $\partial_a f(x) = \langle Df(x), \frac{a}{|a|}\rangle$ be the directional derivative with respect to a vector a. Taking into consideration

$$\frac{f(x) - f(\sigma_a x)}{\langle a, x\rangle} = \frac{2}{|a|}\int_0^1 \partial_a f\Big(x - \frac{2t\langle a, x\rangle}{|a|^2}a\Big)\,dt$$

and

$$T_j f(x) = D_j f(x) + \sum_{a \in R_+} \frac{2k(a)\langle a, e_j\rangle}{|a|}\int_0^1 \partial_a f\Big(x - \frac{2t\langle a, x\rangle}{|a|^2}a\Big)\,dt \tag{6.1}$$

we obtain that $T_j f(0) = 0$, $j = 1, \dots, d$. By (6.1), we derive that $\Psi \subset \Psi_k$. In addition, if $D^n f(0) = 0$ for $|n| = \sum_{j=1}^d n_j \le m$, then $T^n f(0) = 0$ for $|n| \le m$.

Let $m \in \mathbb{Z}_+$ and $f \in \Psi_k$ be a real function. Using the Taylor formula, we write

$$f(x) = p(x) + r(x),$$

where $p(x)$ is a polynomial of degree $\deg p \le m$, and $D^n r(0) = 0$ for $|n| \le m$. Since $T^n f(0) = T^n r(0) = 0$ for $|n| \le m$, it follows that $T^n p(0) = 0$ for $|n| \le m$ and, in particular, $p(0) = 0$. By [21],

$$0 = p(T)p(0) = \int_{\mathbb{R}^d} \big(e^{-\Delta_k/2}p(x)\big)^2 e^{-|x|^2/2}\,d\mu_k(x)$$

and $e^{-\Delta_k/2}p(x) = 0$. Since $e^{-\Delta_k/2}$ is a bijective operator on the set of all polynomials [21], we obtain that $p(x) \equiv 0$, and $D^n f(0) = D^n p(0) = 0$ for $|n| \le m$. Thus, $\Psi_k \subset \Psi$. □

Theorem 6.1 immediately implies the following

Corollary 6.2. *We have* $I_\alpha^k(\Phi_k)=\Phi_k$ *and* $\mathcal{F}_k(I_\alpha^k)(\Psi_k)=\Psi_k$.

Let $f\in\mathcal{S}(\mathbb{R}^d)$. Using the positive L_p-bounded generalized translation operator

$$\mathcal{T}^t f(x)=\int_{\mathbb{S}^{d-1}}\tau^{ty'}f(x)\,d\sigma_k(y')$$

[11], we can write the D-Riesz potential and the convolution with a radial function $g_0(|y|)$ as follows

$$I_\alpha^k f(x)=(\gamma_\alpha^k)^{-1}\int_0^\infty \mathcal{T}^t f(x)\,t^{\alpha-d_k}\,d\nu_{\lambda_k}(t) \tag{6.2}$$

and

$$\int_{\mathbb{R}^d}\tau^{-y}f(x)\,g_0(|y|)\,d\mu_k(y)=\int_0^\infty \mathcal{T}^t f(x)g_0(t)\,d\nu_{\lambda_k}(t). \tag{6.3}$$

The proof of the following result is based on Theorem 1.3.

Theorem 6.3. *If* $1<p<\infty$, $-\frac{d_k}{p}<\beta<\frac{d_k}{p'}$, *then* Φ_k *is dense in* $L^p(\mathbb{R}^d,|x|^{\beta p}\,d\mu_k)$.

Proof. Let $\eta\in\mathcal{S}(\mathbb{R}^d)$ be such that $\eta(x)=1$ if $|x|\le 1$, $\eta(x)>0$ if $|x|<2$, and $\eta(x)=0$ if $|x|\ge 2$. We can assume that $f\in\mathcal{S}(\mathbb{R}^d)$.

Set

$$\psi_0(|y|)=\mathcal{F}_k(\eta)(y),\quad \psi_{0N}(|y|)=\frac{1}{N^{d_k}}\psi_0\Big(\frac{|y|}{N}\Big)=\mathcal{F}_k(\eta(N\cdot))(y),$$

$$\varphi_N(x)=f(x)-\int_{\mathbb{R}^d}\tau^{-y}f(x)\psi_{0N}(|y|)\,d\mu_k(y).$$

Since (see [11])

$$\mathcal{F}_k(\varphi_N)(z)=(1-\eta(Nz))\mathcal{F}_k(f)(z)\in\Psi_k,$$

it follows that $\varphi_N\in\Phi_k$ and, by (6.3),

$$\big\||x|^\beta(f(x)-\varphi_N(x))\big\|_{p,d\mu_k}=\Big\||x|^\beta\int_0^\infty \mathcal{T}^t f(x)\psi_{0N}(t)\,d\nu_{\lambda_k}(t)\Big\|_{p,d\mu_k}. \tag{6.4}$$

For any $\alpha\in(0,d_k)$, we have

$$|\psi_0(t)|\lesssim t^{\alpha-d_k}\quad\text{and}\quad |\psi_{0N}(t)|\lesssim N^{-\alpha}t^{\alpha-d_k}.$$

Hence, by positivity of the operator T^t and (6.2),

$$\Big|\int_0^\infty \mathcal{T}^t f(x)\psi_{0N}(t)\,d\nu_{\lambda_k}(t)\Big|\le\int_0^\infty \mathcal{T}^t|f|(x)\,|\psi_{0N}(t)|\,d\nu_{\lambda_k}(t)$$
$$\lesssim N^{-\alpha}\int_0^\infty \mathcal{T}^t|f(x)|\,t^{\alpha-d_k}\,d\nu_{\lambda_k}(t)=N^{-\alpha}I_\alpha^k|f|(x).$$

This, (6.4), and Theorem 1.3 imply

$$\big\||x|^{\beta}(f(x)-\varphi_N(x))\big\|_{p,d\mu_k}\lesssim N^{-\alpha}\big\||x|^{\beta}I_{\alpha}^{k}|f|(x)\big\|_{p,d\mu_k}$$
$$\lesssim N^{-\alpha}\big\||x|^{\delta}f(x)\big\|_{p,d\mu_k}\lesssim N^{-\alpha},$$

where $\alpha>0$ is chosen so that $\delta=\alpha+\beta<\frac{d_k}{p'}$. □

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