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ON THE ORDER OF THE TRIGINOMETRIC DIAMETER OF THE ANISOTROPIC NIKOL'SKII–BESOV CLASS IN THE METRIC OF ANISOTROPIC LORENTZ SPACES

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Abstract. In this paper we estimate the order of the triginometric diameters of the anisotropic Nikol'skii–Besov classes in the anisotropic Lorentz space.

1. Introduction

The order of the trigonometric diameter of the anisotropic Nikol'skii-Besov class $B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n)$ is found in the metric of the anisotropic Lorentz spaces $L_{\mathbf{q}\theta}(\mathbb{T}^n)$ under the condition $\mathbf{1} < \mathbf{p} = (p_1, \dots, p_n) < \mathbf{2} < \mathbf{q} = (q_1, \dots, q_n)$.

Let $V \subset L_1(\mathbb{T}^n)$ be the normed space and $F \subset V$ be some functional class. The trigonometric diameter of the class F in the space V is defined as follows (see [1])

$$d_M^T(F, V) = \inf_{\Omega_M} \sup_{f \in F} \inf_{t(\Omega_M;\mathbf{x})} \|f(\cdot) - t(\Omega_M; \cdot)\|_V ,$$

where $t(\Omega_M; \mathbf{x}) = \sum_{j=1}^{M} c_j \mathrm{e}^{i(\mathbf{k}_j, \mathbf{x})}$, $\Omega_M = \{\mathbf{k}_1, \dots, \mathbf{k}_M\}$ is the set of vectors $\mathbf{k}_j = (k_1^j, \dots, k_n^j)$ from the integer lattice $\mathbb{Z}^n$, c_j is some numbers $(j = 1, \dots, M)$.

The concept of a trigonometric diameter in the one-dimensional case was first introduce by R.S. Ismagilov [1] and he established his estimates for certain classes in the space of continuous functions. For a function of several variables, exact orders of trigonometric diameter of Sobolev class $W_p^{\mathbf{r}}$, Nikol'skii class $H_p^{\mathbf{r}}$ in the space L_q are established by E.S. Belinsky [2], V.E. Mayorov [3], Yu. Makovoz [4], G.G. Magaril-Ilyaev [5], V.N. Temlyakov [6]. This problem for the Besov class was investigated by A.S. Romanyuk [7], D.B.Bazarkhanov [8].

We study the problem of estimating the order of the trigonimetric diameter of the anisotropic Nikol'skii-Besov class $B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n)$ in the metric of anisotropic Lorentz spaces $L_{\mathbf{q}\theta}(\mathbb{T}^n)$.

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We give nessesary definitions and formulate auxiliary assertions.

Let $f(\mathbf{x}) = f(x_1, \dots, x_n)$ be a measurable function defined on $\mathbb{T}^n$. We denote by $f^*(\mathbf{t}) = f^{*_1,\dots,*_n}(t_1, \dots, t_n)$ the function obtained by applying to the first non-icreasing permutation, successively with respect to the variables $x_1, \dots, x_n$ for fixed other variables.

Let multiindexes $\mathbf{p} = (p_1, \dots, p_n)$, $\mathbf{r} = (r_1, \dots, r_n)$ satisfy the conditions: if $0 < p_j < \infty$, then $0 < r_j \le \infty$, if $p_j = \infty$, then $r_j = \infty$ for every $j = 1, \dots, n$. An anisotropic Lorentz space $L_{\mathbf{pr}}(\mathbb{T}^n)$ is the set of functions for which the following quantity is finite

$$\|f\|_{L_{\mathbf{pr}}(\mathbb{T}^n)} = \left(\int_0^{2\pi} \dots \left(\int_0^{2\pi} \left(t_1^{1/p_1} \dots t_n^{1/p_n} f^{*_1,\dots,*_n}(t_1, \dots, t_n)\right)^{r_1} \frac{dt_1}{t_1}\right)^{r_2/r_1} \dots \frac{dt_n}{t_n}\right)^{1/r_n}.$$

Here, the expression $\left(\int_0^{2\pi} (G(s))^r \frac{ds}{s}\right)^{1/r}$ for $r = \infty$ is understood us $\sup_{s>0} G(s)$.

For the functions $f \in L_{\mathbf{pr}}(\mathbb{T}^n)$ we denote by

$$\Delta_{\mathbf{s}}(f, \mathbf{x}) = \sum_{\mathbf{k} \in \rho(\mathbf{s})} a_{\mathbf{k}}(f) e^{i(\mathbf{k}, \mathbf{x})},$$

where $\{a_{\mathbf{k}}(f)\}_{\mathbf{k} \in \mathbb{Z}^n}$ is the Fourier coefficients of the function f with respect to the multiple trigonometric system $\rho(\mathbf{s}) = \{\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n : [2^{s_i - 1}] \le |k_i| < 2^{s_i}$, $i = 1, \dots, n\}$, $(\mathbf{k}, \mathbf{x}) = \sum_{j=1}^n k_j x_j$.

Let $\mathbf{0} < \alpha = (\alpha_1, \dots, \alpha_n) < \infty$, $\mathbf{0} < \tau = (\tau_1, \dots, \tau_n) \le \infty$. The anisotropic class of Nikol'skii-Besov $B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n)$ ([9, 10]) is the set of functions f from $L_{\mathbf{pr}}(\mathbb{T}^n)$ for which the inequality holds

$$\|f\|_{B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n)} = \left\| \left\{ \mathbf{2}^{(\alpha, \mathbf{s})} \|\Delta_{\mathbf{s}}(f, \cdot)\|_{L_{\mathbf{pr}}(\mathbb{T}^n)} \right\}_{\mathbf{s} \in \mathbb{Z}_+^n} \right\|_{l_\tau} \le 1,$$

where $\|\cdot\|_{l_\tau}$ is the norm of discrete Lebesgue space l_τ with a mixed metric.

2. Auxiliary results

Let Ω_M be set containing at most M vectors $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$.

Lemma 1 ([11])**.** *Let $2 \le q < \infty$. Then for any trigonometric polinomial*

$$P(\Omega_M, \mathbf{x}) = \sum_{j=1}^M e^{i(\mathbf{k}^j, \mathbf{x})}$$

and any number $N \le M$ there exist a trigonometric polinomial $P(\Omega_N, \mathbf{x})$ containing at most N harmonics and such that

$$\|P(\Omega_M, \cdot) - P(\Omega_N, \cdot)\|_{L_q(\mathbb{T}^n)} \le C M N^{-1/2},$$

moreover $\Omega_N \subset \Omega_M$*, all coefficients* $P(\Omega_N, \mathbf{x})$ *are the same and do not exceed* MN^{-1}.

Corollary 1 ([12])**.** *Let* $\mathbf{2} < \mathbf{q} = (q_1, \dots, q_n)$*,* $\mathbf{0} < \theta = (\theta_1, \dots, \theta_n) \leq \infty$*. Then for any trigonometric polinomial*

$$P(\Omega_M, \mathbf{x}) = \sum_{j=1}^{M} e^{i(\mathbf{k}^j, \mathbf{x})}$$

and any number $N \leq M$ *there exist a trigonometric polinomial* $P(\Omega_N, \mathbf{x})$ *containing at most* N *harmonics and such that*

$$\|P(\Omega_M, \cdot) - P(\Omega_N, \cdot)\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \leq CMN^{-1/2},$$

moreover $\Omega_N \subset \Omega_M$*, all coefficients* $P(\Omega_N, \mathbf{x})$ *are the same and do not exceed* MN^{-1}.

For any $\mathbf{s} \in \mathbb{Z}_+^n$ we consider a linear operator

$$\left(T_{N_\mathbf{s}} f\right)(\mathbf{x}) = f(\mathbf{x}) * \Bigg(\sum_{\mathbf{k} \in \rho(\mathbf{s})} e^{i(\mathbf{k}, \mathbf{x})} - t(\Omega_{N_\mathbf{s}}, \mathbf{x}) \Bigg),$$

where $t(\Omega_{N_\mathbf{s}}, \mathbf{x})$ is trigonometric polinomial from corollary 1, which is approaching the "block" $t_\mathbf{s}(\mathbf{x}) = \sum_{\mathbf{k} \in \rho(\mathbf{s})} e^{i(\mathbf{k}, \mathbf{x})}$.

Lemma 2. *Let* $1 < p < 2$*, the multiindex* $\mathbf{q} = (q_1, \dots, q_n)$ *such that* $2 < q_j < p'$ *for all* $j = 1, \dots, n$ *and* $\mathbf{0} < \theta = (\theta_1, \dots, \theta_n) \leq \infty$*. Then the norm operator* $T_{N_\mathbf{s}}$ *acting from* $L_p(\mathbb{T}^n)$ *to* $L_{\mathbf{q}\theta}(\mathbb{T}^n)$ *satisfies the inequality*

$$\|T_{N_\mathbf{s}}\|_{L_p(\mathbb{T}^n) \to L_{\mathbf{q}\theta}(\mathbb{T}^n)} \leq C 2^{(\mathbf{1}, \mathbf{s})} N_\mathbf{s}^{-(1/2 + 1/p')}.$$

Proof. Taking into account that the coefficients of the polinomial $t(\Omega_{N_\mathbf{s}}, \mathbf{x})$ are the same and do not exceed $2^{(\mathbf{1}, \mathbf{s})} N_\mathbf{s}^{-1}$ by Parseval's equality we have

$$\|T_{N_\mathbf{s}}\|_{L_2(\mathbb{T}^n) \to L_2(\mathbb{T}^n)} \leq C_1 2^{(\mathbf{1}, \mathbf{s})} N_\mathbf{s}^{-1}. \tag{1}$$

Further, using the generalized Minkowski's inequalities and the corollary 1 we can write

$$\|T_{N_\mathbf{s}} f\|_{L_{\mathbf{q}^*\theta^*}(\mathbb{T}^n)} \leq \|f\|_{L_1(\mathbb{T}^n)} \Bigg\| \sum_{\mathbf{k} \in \rho(\mathbf{s})} e^{i(\mathbf{k}, \cdot)} - t(\Omega_{N_\mathbf{s}}, \cdot) \Bigg\|_{L_{\mathbf{q}^*\theta^*}(\mathbb{T}^n)}$$

$$\leq C_2 2^{(\mathbf{1}, \mathbf{s})} N_\mathbf{s}^{-1/2} \|f\|_{L_1(\mathbb{T}^n)}.$$

From this, by definition, $\|T_{N_\mathbf{s}}\|_{L_1(\mathbb{T}^n) \to L_{\mathbf{q}^*\theta^*}(\mathbb{T}^n)}$ we find

$$\|T_{N_\mathbf{s}}\|_{L_1(\mathbb{T}^n) \to L_{\mathbf{q}^*\theta^*}(\mathbb{T}^n)} \leq C_2 2^{(\mathbf{1}, \mathbf{s})} N_\mathbf{s}^{-1/2}. \tag{2}$$

Further, using the interpolation theorem of Riesz-Torin for Lebegue spaces and anisotropic Lorentz spaces

(3) $$\|T_{N_{\mathbf{s}}}\|_{L_p(\mathbb{T}^n)\to L_{\mathbf{q}\theta}(\mathbb{T}^n)} \leq \|T_{N_{\mathbf{s}}}\|^{1-\lambda}_{L_2(\mathbb{T}^n)\to L_2(\mathbb{T}^n)}\|T_{N_{\mathbf{s}}}\|^{\lambda}_{L_1(\mathbb{T}^n)\to L_{\mathbf{q}^*\theta^*}(\mathbb{T}^n)},$$

where $0<\lambda<1$ $1/p=(1-\lambda)/2+\lambda/1$, $\mathbf{1/q}=(1-\lambda)/\mathbf{2}+\lambda/\mathbf{q}^*$ $\mathbf{1}/\theta=(1-\lambda)/\mathbf{2}+\lambda/\theta^*$.

By substuting (1) and (2) to (3) and performing elementary transformations, we arrive at the required estimate with the additional condition $\mathbf{0}<\theta=(\theta_1,\dots,\theta_n)<\mathbf{p}'=(p',\dots,p')$. For the remaining values of the parameters $\theta=(\theta_1,\dots,\theta_n)$ the validity of the assertion follows from the embedding $L_{\mathbf{q}\theta_1}(\mathbb{T}^n)\hookrightarrow L_{\mathbf{q}\theta_2}(\mathbb{T}^n)$ for $\mathbf{0}<\theta_1=(\theta_1^1,\dots,\theta_n^1)\leq\theta_2=(\theta_1^2,\dots,\theta_n^2)\leq\infty$.
The proof is satisfied.

We formulate in the form of a lemma a special case of the embedding theorem from the paper E.D. Nursultanov ([9]).

Lemma 3 ([9])**.** *Let* $\mathbf{1}\leq\mathbf{p}=(p_1,\dots,p_n)<\mathbf{q}=(q_1,\dots,q_n)<\infty$, $\mathbf{0}<\theta=(\theta_1,\dots,\theta_n),\mathbf{r}=(r_1,\dots,r_n)\leq\infty$ *and* $\alpha=\mathbf{1/p}-\mathbf{1/q}$, *then*

$$B^{\alpha\theta}_{\mathbf{pr}}(\mathbb{T}^n)\hookrightarrow L_{\mathbf{q}\theta}(\mathbb{T}^n).$$

Further we need the following sets

$$Y^n(N,\gamma)=\left\{\mathbf{s}=(s_1,\dots,s_n)\in\mathbb{Z}^n_+:\sum_{j=1}^n\gamma_js_j\geq N\right\},$$

$$\aleph^n(N,\gamma)=\left\{\mathbf{s}=(s_1,\dots,s_n)\in\mathbb{Z}^n_+:\sum_{j=1}^n\gamma_js_j=N\right\}.$$

Lemma 4 ([13])**.** *Let* $n\in\mathbb{N}$, $n\geq2$, $\mathbf{0}<\gamma'=(\gamma_1',\dots,\gamma_n')\leq\gamma=(\gamma_1,\dots,\gamma_n)<\infty$, $\delta>0$ *and* $\mathbf{0}<\varepsilon=(\varepsilon_1,\dots,\varepsilon_n)\leq\infty$. *Then*

$$\left\|\{2^{-\delta(\gamma,\mathbf{s})}\}_{\mathbf{s}\in Y^n(N,\gamma')}\right\|_{l_\varepsilon(\mathbb{Z}^n_+)}\leq C2^{-\delta\eta N}N^{\sum_{j\in A\setminus\{j_1\}}1/\varepsilon_j},$$

where $\eta=\min\{\gamma_j/\gamma_j':j=1,\dots,n\}$, $A=\{j:\gamma_j/\gamma_j'=\eta,j=1,\dots,n\}$, $j_1=\min\{j:j\in A\}$.

Lemma 5 ([13])**.** *Let* $n\in\mathbb{N}$, $n\geq2$, $\mathbf{0}<\gamma=(\gamma_1,\dots,\gamma_n)<\infty$, $\delta\in\mathbb{R}$ *and* $\mathbf{0}<\varepsilon=(\varepsilon_1,\dots,\varepsilon_n)\leq\infty$. *Then*

$$\left\|\left\{2^{-\delta(\gamma,\mathbf{s})}\right\}_{\mathbf{s}\in\aleph^n(N,\gamma)}\right\|_{l_\varepsilon(\mathbb{Z}^n_+)}\asymp2^{-\delta N}N^{\sum_{j=2}^n\frac{1}{\varepsilon_j}}.$$

3. Main result

The main result of this paper is the following assertion.

Theorem 1. *Let* $\mathbf{1} < \mathbf{p} = (p_1, \dots, p_n) < \mathbf{2} < \mathbf{q} = (q_1, \dots, q_n) < \mathbf{p_0'} = (p_0', \dots, p_0')$, $p_0 = \max\{p_j : j = 1, \dots, n\}$, $\mathbf{1} \le \tau = (\tau_1, \dots, \tau_n), \theta = (\theta_1, \dots, \theta_n)$, $\mathbf{r} = (r_1, \dots, r_n) \le \infty$ *and* $\alpha = (\alpha_1, \dots, \alpha_n)$ *such that* $\alpha_j > 1 + 1/p_j - 1/p_0$ *for all* $j = 1, \dots, n$. *Let* $\zeta = \min\{\alpha_j - 1/p_j + 1/q_j : j = 1, \dots, n\}$, $D = \big\{j = 1, \dots, n : \alpha_j - 1/p_j + 1/q_j = \zeta\big\}$, $j_1 = \min\{j : j \in D\}$, $q_j = q_{j_1}$ *for all* $j \in D$ $\ q_j \ge q_{j_1}$ *for all* $j \notin D$.

Then the following relation holds

$$(4) \quad d_M^T(B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n), L_{\mathbf{q}\theta}(\mathbb{T}^n)) \asymp M^{-(\alpha_{j_1} - 1/p_{j_1} + 1/2)} (\log M)^{(|D|-1)(\alpha_{j_1} - 1/p_{j_1} + 1/2) + \sum_{j \in D \setminus \{j_1\}} (1/2 - 1/\tau_j)_+},$$

where $|D|$ *is the number of elements of set* D, $a_+ = \max(a; 0)$.

Proof. Let $f \in B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n)$. For any natural number M there exists the natural number m such that $M \asymp 2^m m^{(|D|-1)}$. We will seek an approximating polinomial $P(\Omega_M, \mathbf{x})$ in the following form

$$(5) \qquad P(\Omega_M, \mathbf{x}) = \sum_{(\gamma', \mathbf{s}) < m} \Delta_{\mathbf{s}}(f, \mathbf{x}) + \sum_{m \le (\gamma', \mathbf{s}) < \beta m} t(\Omega_{N_{\mathbf{s}}}, \mathbf{x}) * \Delta_{\mathbf{s}}(f, \mathbf{x}),$$

where

$$\beta = \Bigg(\alpha_{j_1} - 1/p_{j_1} + 1/2 - \frac{\log m}{m} \sum_{j \in D \setminus \{j_1\}} \big((1/2 - 1/\tau_j)_+ - (1/\theta_j - 1/\tau_j)_+\big)\Bigg) / (\alpha_{j_1} - 1/p_{j_1} + 1/q_{j_1}),$$

$\gamma_j = (\alpha_j - 1/p_j + 1/q_j)/(\alpha_{j_1} - 1/p_{j_1} + 1/q_{j_1})$, $j = 1, \dots, n$, $\gamma_j' = \gamma_j$ at $j \in D$ and $1 < \gamma_j' < \gamma_j$ at $j \notin D$. The polinomials $t(\Omega_{N_{\mathbf{s}}}, \mathbf{x})$ are choose for every "block" $t_{\mathbf{s}}(\mathbf{x}) = \sum_{\mathbf{k} \in \rho(\mathbf{s})} e^{i(\mathbf{k}, \mathbf{x})}$ according Corollary 1 and numbers $N_{\mathbf{s}} = \big[2^{(\alpha_{j_1} - 1/p_{j_1} + 1/p_0)m} 2^{-(\alpha - \mathbf{1}/\mathbf{p} + \mathbf{1}/\mathbf{p_0} - \mathbf{1}, \mathbf{s})}\big]$.

Note that according to lemma 4

$$\sum_{m\le(\gamma',\mathbf{s})<\beta m} N_{\mathbf{s}} = 2^{(\alpha_{j_1}-1/p_{j_1}+1/p_0)m} \sum_{m\le(\gamma',\mathbf{s})<\beta m} 2^{-(\alpha-\mathbf{1/p}+\mathbf{1/p_0}-\mathbf{1},\mathbf{s})}$$

$$\le 2^{(\alpha_{j_1}-1/p_{j_1}+1/p_0)m} \left\| \left\{ 2^{-(\alpha-\mathbf{1/p}+\mathbf{1/p_0}-\mathbf{1},\mathbf{s})} \right\}_{\mathbf{s}\in Y^n(m,\gamma')} \right\|_{l_1}$$

$$\le 2^{(\alpha_{j_1}-1/p_{j_1}+1/p_0)m} \cdot 2^{-(\alpha_{j_1}-1/p_{j_1}+1/p_0-1)m} m^{(|D|-1)}$$

$$= 2^m m^{(|D|-1)} \asymp M,$$

so that $(\alpha_j - 1/p_j + 1/p_0 - 1)/(\alpha_{j_1} - 1/p_{j_1} + 1/p_0 - 1) > \gamma'_j$ at $j \notin D$.

By equality (5) and Minkowski's inequality we have

$$\|f(\cdot) - P(\Omega_M,\cdot)\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)}$$

$$\le C_1 \left(\left\| \sum_{m\le(\gamma',\mathbf{s})<\beta m} \Big(\Delta_{\mathbf{s}}(f,\cdot) - \Delta_{\mathbf{s}}(f,\cdot) * t(\Omega_{N_{\mathbf{s}}},\cdot) \Big) \right\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} + \left\| \sum_{(\gamma',\mathbf{s})\ge\beta m} \Delta_{\mathbf{s}}(f,\cdot) \right\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \right)$$

$$= C_1(I_1(f) + I_2(f)). \tag{6}$$

At first we estimate $I_2(f)$. By lemma 3 we have

$$I_2(f) \le C_2 \left\| \left\{ 2^{(\mathbf{1/p}-\mathbf{1/q},\mathbf{s})} \|\Delta_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{pr}}(\mathbb{T}^n)} \right\}_{\mathbf{s}\in Y^n(\beta m,\gamma')} \right\|_{l_\theta} = C_2 I_3(f). \tag{7}$$

By Hölder's inequality with parameters $1/\theta = 1/\tau + 1/\varepsilon$, where $1/\varepsilon = (1/\theta - 1/\tau)_+$ and lemma 4, taking into account that $\gamma' \le \gamma$ we find

$$I_3(f) = \left\| \left\{ 2^{(\alpha,\mathbf{s})} \|\Delta_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{pr}}(\mathbb{T}^n)} \cdot 2^{-(\alpha_{j_1}-1/p_{j_1}+1/q_{j_1})(\gamma,\mathbf{s})} \right\}_{\mathbf{s}\in Y^n(\beta m,\gamma')} \right\|_{l_\theta}$$

$$\le \left\| \left\{ 2^{(\alpha,\mathbf{s})} \|\Delta_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{pr}}(\mathbb{T}^n)} \right\}_{\mathbf{s}\in Y^n(\beta m,\gamma')} \right\|_{l_\tau} \cdot \left\| \left\{ 2^{-(\alpha_{j_1}-1/p_{j_1}+1/q_{j_1})(\gamma,\mathbf{s})} \right\}_{\mathbf{s}\in Y^n(\beta m,\gamma')} \right\|_{l_\varepsilon}$$

$$\le C_3 \|f\|_{B^{\alpha\tau}_{\mathbf{pr}}(\mathbb{T}^n)} \cdot 2^{-(\alpha_{j_1}-1/p_{j_1}+1/q_{j_1})\beta m} m^{\sum_{j\in D\setminus\{j_1\}} 1/\varepsilon_j}$$

$$\le C_3 2^{-(\alpha_{j_1}-1/p_{j_1}+1/q_{j_1})\beta m} m^{\sum_{j\in D\setminus\{j_1\}} (1/\theta_j - 1/\tau_j)_+}. \tag{8}$$

By replacing (8) to (7) we have

$$I_2(f) \le C_4 2^{-(\alpha_{j_1}-1/p_{j_1}+1/q_{j_1})\beta m} m^{\sum_{j\in D\setminus\{j_1\}} (1/\theta_j - 1/\tau_j)_+}.$$

Next by using β we obtain

$$2^{-(\alpha_{j_1}-1/p_{j_1}+1/q_{j_1})\beta m} = 2^{-(\alpha_{j_1}-1/p_{j_1}+1/2)m} m^{\sum_{j\in D\setminus\{j_1\}}\left((1/2-1/\tau_j)_+-(1/\theta_j-1/\tau_j)_+\right)},$$

and consequently

$$I_2(f) \le C_4 2^{-(\alpha_{j_1}-1/p_{j_1}+1/2)m} m^{\sum_{j\in D\setminus\{j_1\}}(1/2-1/\tau_j)_+}.$$

Taking to account that $M \asymp 2^m m^{(|D|-1)}$ we have

$$(9)\quad I_2(f) \le C_5 M^{-(\alpha_{j_1}-1/p_{j_1}+1/2)} (\log M)^{(|D|-1)(\alpha_{j_1}-1/p_{j_1}+1/2)+\sum_{j\in D\setminus\{j_1\}}(1/2-1/\tau_j)_+}.$$

Then we estimate the value $I_1(f)$. By using the theorem of Littlewood-Paley (see [14]), we obtain

$$\begin{aligned} I_1(f) &= \left\| \sum_{m\le(\gamma',\mathbf{s})<\beta m} \Big(\Delta_{\mathbf{s}}(f,\cdot) - \Delta_{\mathbf{s}}(f,\cdot) * t(\Omega_{N_{\mathbf{s}}},\cdot)\Big) \right\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \\ &\le C_6 \left(\sum_{m\le(\gamma',\mathbf{s})<\beta m} \left\| \Big(\Delta_{\mathbf{s}}(f,\cdot) - \Delta_{\mathbf{s}}(f,\cdot) * t(\Omega_{N_{\mathbf{s}}},\cdot)\Big) \right\|^2_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \right)^{1/2} \\ &= C_6 \left\| \left\{ \left\| \Delta_{\mathbf{s}}(f,\cdot) * \Big(\sum_{\mathbf{k}\in\rho(\mathbf{s})} e^{i(\mathbf{k},\cdot)} - t(\Omega_{N_{\mathbf{s}}},\cdot)\Big) \right\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \right\}_{\mathbf{s}\in\varkappa^n(m,\beta m,\gamma')} \right\|_{l_2} \\ (10)\qquad &= C_6 \left\| \left\{ \|T_{N_{\mathbf{s}}}\Delta_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \right\}_{\mathbf{s}\in\varkappa^n(m,\beta m,\gamma')} \right\|_{l_2} \end{aligned}$$

where $\varkappa^n(m,\beta m,\gamma') = \{\mathbf{s}\in\mathbb{Z}^n_+ : m \le (\gamma',\mathbf{s}) < \beta m\}$.

By using lemma 2 and inequality of different metrics for trigonometric polinomials in the anisotropic Lorentz spaces (see [15]) for $1 < p_j < p_0$ $(j = 1,\dots,n)$, from (10) we have

$$\begin{aligned} I_1(f) &\le C_7 \left\| \left\{ 2^{(\mathbf{1},\mathbf{s})} N_{\mathbf{s}}^{-(1/2+1/p_0')} \|\Delta_{\mathbf{s}}(f,\cdot)\|_{L_{p_0}(\mathbb{T}^n)} \right\}_{\mathbf{s}\in\varkappa^n(m,\beta m,\gamma')} \right\|_{l_2} \\ &\le C_8 \left\| \left\{ 2^{(\mathbf{1},\mathbf{s})} N_{\mathbf{s}}^{-(1/2+1/p_0')} 2^{(\mathbf{1/p}-\mathbf{1/p_0},\mathbf{s})} \|\Delta_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{pr}}(\mathbb{T}^n)} \right\}_{\mathbf{s}\in\varkappa^n(m,\beta m,\gamma')} \right\|_{l_2} \\ (11)\qquad &= C_8 \left\| \left\{ N_{\mathbf{s}}^{-(1/2+1/p_0')} 2^{-(\alpha-\mathbf{1/p}+\mathbf{1/p_0}-\mathbf{1},\mathbf{s})} \cdot 2^{(\alpha,\mathbf{s})} \|\Delta_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{pr}}(\mathbb{T}^n)} \right\}_{\mathbf{s}\in\varkappa^n(m,\beta m,\gamma')} \right\|_{l_2}. \end{aligned}$$

By using Hölder's inequality with parameters $1/2 = 1/\tau + 1/\varepsilon$, where $1/\varepsilon = (1/2 - 1/\tau)_+$, then from (11) we find

$$I_1(f) \le C_8 \left\| \left\{ 2^{(\alpha,\mathbf{s})} \|\Delta_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{pr}}(\mathbb{T}^n)} \right\}_{\mathbf{s}\in\varkappa^n(m,\beta m,\gamma')} \right\|_{l_\tau} \cdot$$

$$\cdot \left\| \left\{ N_{\mathbf{s}}^{-(1/2+1/p_0')} 2^{-(\alpha - \mathbf{1/p} + \mathbf{1/p_0} - \mathbf{1},\mathbf{s})} \right\}_{\mathbf{s}\in\varkappa^n(m,\beta m,\gamma')} \right\|_{l_\varepsilon}$$

$$\le C_8 \|f\|_{B^{\alpha\tau}_{\mathbf{pr}}(\mathbb{T}^n)} \cdot \left\| \left\{ N_{\mathbf{s}}^{-(1/2+1/p_0')} 2^{-(\alpha - \mathbf{1/p} + \mathbf{1/p_0} - \mathbf{1},\mathbf{s})} \right\}_{\mathbf{s}\in\varkappa^n(m,\beta m,\gamma')} \right\|_{l_\varepsilon}$$

$$\le C_8 \left\| \left\{ N_{\mathbf{s}}^{-(1/2+1/p_0')} 2^{-(\alpha - \mathbf{1/p} + \mathbf{1/p_0} - \mathbf{1},\mathbf{s})} \right\}_{\mathbf{s}\in\varkappa^n(m,\beta m,\gamma')} \right\|_{l_\varepsilon} \tag{12}$$

for any function $f \in B^{\alpha\tau}_{\mathbf{pr}}(\mathbb{T}^n)$.

By continuing (12), according to the lemma 4 we have

$$I_1(f) \le C_8 2^{-(1/2+1/p_0')(\alpha_{j_1} - 1/p_{j_1} + 1/p_0)m}$$

$$\left\| \left\{ 2^{(1/2+1/p_0')(\alpha - \mathbf{1/p} + \mathbf{1/p_0} - \mathbf{1},\mathbf{s})} \cdot 2^{-(\alpha - \mathbf{1/p} + \mathbf{1/p_0} - \mathbf{1},\mathbf{s})} \right\}_{\mathbf{s}\in\varkappa^n(m,\beta m,\gamma')} \right\|_{l_\varepsilon}$$

$$= C_8 2^{-(1/2+1/p_0')(\alpha_{j_1} - 1/p_{j_1} + 1/p_0)m} \left\| \left\{ 2^{-(1/2-1/p_0')(\alpha - \mathbf{1/p} + \mathbf{1/p_0} - \mathbf{1},\mathbf{s})} \right\}_{\mathbf{s}\in\varkappa^n(m,\beta m,\gamma')} \right\|_{l_\varepsilon}$$

$$\le C_8 2^{-(1/2+1/p_0')(\alpha_{j_1} - 1/p_{j_1} + 1/p_0)m} \left\| \left\{ 2^{-(1/2-1/p_0')(\alpha - \mathbf{1/p} + \mathbf{1/p_0} - \mathbf{1},\mathbf{s})} \right\}_{\mathbf{s}\in Y^n(m,\gamma')} \right\|_{l_\varepsilon}$$

$$\le C_9 2^{-(1/2+1/p_0')(\alpha_{j_1} - 1/p_{j_1} + 1/p_0)m} \cdot 2^{-(1/2-1/p_0')(\alpha_{j_1} - 1/p_{j_1} + 1/p_0 - 1)m} m^{\sum_{j\in D\setminus\{j_1\}} 1/\varepsilon_j}$$

$$= C_9 2^{-(\alpha_{j_1} - 1/p_{j_1} + 1/2)m} m^{\sum_{j\in D\setminus\{j_1\}} (1/2 - 1/\tau_j)_+},$$

as $(\alpha_j - 1/p_j + 1/p_0 - 1)/(\alpha_{j_1} - 1/p_{j_1} + 1/p_0 - 1) > \gamma_j'$ at $j \notin D$.

Taking account that $M \asymp 2^m m^{(|D|-1)}$ we find

$$I_2(f) \le C_{10} M^{-(\alpha_{j_1} - 1/p_{j_1} + 1/2)} (\log M)^{(|D|-1)(\alpha_{j_1} - 1/p_{j_1} + 1/2) + \sum_{j\in D\setminus\{j_1\}} (1/2 - 1/\tau_j)_+}. \tag{13}$$

By replacing (9) and (13) to (6) we obtain the inequality, which gives the upper estimate in (4).

For the proof of the lower estimate we consider the following value

$$e_M(F)_V = \sup_{f\in F} e_M(f)_V = \sup_{f\in F} \inf_{\{b_j,\mathbf{k}_j\}_{j=1}^M} \left\| f - \sum_{j=1}^{M} b_j e^{i(\mathbf{k}_j,\mathbf{x})} \right\|_V$$

this value is called a best M-term appoximation of the class F.

On the basis of the definition, the following inequality holds

$$e_M(F)_V \le d_M^T(F,V).$$

Because the lower estimates for $e_M(F)_V$ are the lower estimates also for $d_M^T(F,V)$.

By using the condition $2 < q_j$ $(j = 1,\dots,n)$ we have

$$e_M(f)_{L_2(\mathbb{T}^n)} \le C_{11} e_M(f)_{L_{\mathbf{q}\theta}(\mathbb{T}^n)}.$$

Next for the proof the lower estimate we will use double relation, which follows from the general results of S.M. Nikol'skii (see [16]). By according for this relation for any function $f \in L_2(\mathbb{T}^n)$ the following equality holds

$$e_M(f)_{L_2(\mathbb{T}^n)} = \inf_{\Omega_M} \sup_{P\in\mathcal{L}^\perp, \|P\|_{L_2(\mathbb{T}^n)}\le 1} \left| \int_{\mathbb{T}^n} f(\mathbf{x}) P(\mathbf{x}) d\mathbf{x} \right|, \tag{14}$$

where $\mathcal{L}$ is a linear span of a system of functions $\{e^{i(\mathbf{k},\mathbf{x})}\}_{\mathbf{k}\in\Omega_M}$.

We consider the function

$$f(\mathbf{x}) = m^{-\sum_{j\in D'\setminus\{j_1\}} 1/\tau_j} \sum_{m\le(\gamma',\mathbf{s_0})\le m+n} \prod_{j=1}^n 2^{-(\alpha_j+1-1/p_j)s_j^0} \sum_{\mathbf{k}\in\rho(\mathbf{s_0})} e^{i(\mathbf{k},\mathbf{x})},$$

where $D' = \{j \in D : 2 < \tau_j\} \cup \{j_1\}$, $\mathbf{s_0} = (s_1^0,\dots,s_n^0)$, $s_j^0 = s_j$ at $j \in D'$ and $s_j^0 = 0$ at $j \notin D'$.

In the paper [13] is showed that the function $C_{12} f(\mathbf{x})$ belongs to the class $B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n)$.

Then we construct the function $P(\mathbf{x})$ satisfying the condition from (14).

Let

$$u(\mathbf{x}) = \sum_{(\gamma',\mathbf{s_0})\le m} \sum_{\mathbf{k}\in\rho(\mathbf{s_0})} e^{i(\mathbf{k},\mathbf{x})}, \tag{15}$$

and Ω_M is arbitrary collection of integer vectors $\mathbf{k} = (k_1,\dots,k_n)$.

Denote by

$$v(\mathbf{x}) = \sum_{(\gamma',\mathbf{s_0})\le m} \sum_{\mathbf{k}\in\rho(\mathbf{s_0})\cap\Omega_M} e^{i(\mathbf{k},\mathbf{x})}$$

the function, containing only those terms of (15), for which $\mathbf{k} \in \Omega_M$. From Minkowski's inequality and Parseval's equality for function $w(\mathbf{x}) = u(\mathbf{x}) - v(\mathbf{x})$ we have

$$\|w\|_{L_2(\mathbb{T}^n)} \le C_{13} M^{1/2}.$$

We consider the function $P(\mathbf{x}) = C_{13}^{-1} M^{-1/2} w(\mathbf{x})$, then $\|P\|_{L_2(\mathbb{T}^n)} \le 1$. And sine the function $w(\mathbf{x}) = u(\mathbf{x}) - v(\mathbf{x})$ does not contain the harmonics from Ω_M, then function $P \in \mathcal{L}^\perp$. Thus, the function $P(\mathbf{x})$ satisfies the conditions from (14).

From (14) and the lemma 5 we obtain

$$e_M(f)_{L_2(\mathbb{T}^n)} \geq C_{14} M^{-1/2} \left| \int_{\mathbb{T}^n} f(\mathbf{x}) w(\mathbf{x}) d\mathbf{x} \right|$$

$$\geq C_{14} M^{-1/2} m^{-\sum_{j\in D'\setminus\{j_1\}} 1/\tau_j} \sum_{(\gamma', \mathbf{s_0})=m} \prod_{j=1}^{n} 2^{-(\alpha_j+1-1/p_j)s_j^0} \sum_{\mathbf{k}\in\rho(\mathbf{s_0})} 1$$

$$= C_{14} M^{-1/2} m^{-\sum_{j\in D'\setminus\{j_1\}} 1/\tau_j} \sum_{(\gamma', \mathbf{s_0})=m} \prod_{j=1}^{n} 2^{-(\alpha_j-1/p_j)s_j^0}$$

$$= C_{14} M^{-1/2} m^{-\sum_{j\in D'\setminus\{j_1\}} 1/\tau_j} \left\| \left\{ 2^{-(\alpha_{j_1}-1/p_{j_1})(\bar{\mathbf{1}}, \bar{\mathbf{s}})} \right\}_{\aleph^{|D|}(\bar{\mathbf{s}}, \bar{\mathbf{1}})} \right\|_{l_1}$$

$$\asymp M^{-1/2} m^{-\sum_{j\in D'\setminus\{j_1\}} 1/\tau_j} 2^{-(\alpha_{j_1}-1/p_{j_1})m} m^{(|D|-1)}, \tag{16}$$

where $\bar{\mathbf{s}} = (s_{j_1}, \ldots, s_{j_{|D|}})$.

Take account that $M \asymp 2^m m^{(|D|-1)}$ from (16) we have

$$e_M(f)_{L_2(\mathbb{T}^n)} \geq C_{15} 2^{-(\alpha_{j_1}-1/p_{j_1}+1/2)m} m^{\sum_{j\in D\setminus\{j_1\}} (1/2-1/\tau_j)_+} =$$

$$= C_{16} M^{-(\alpha_{j_1}-1/p_{j_1}+1/2)} (\log M)^{(|D|-1)(\alpha_{j_1}-1/p_{j_1}+1/2)+\sum_{j\in D\setminus\{j_1\}} (1/2-1/\tau_j)_+}. \tag{17}$$

From (17) follows lower estimate in (4).
The proof is satisfied.

Remark 1. *Note, that for* $\mathbf{p} = \mathbf{r} = (p, \ldots, p)$, $\tau = (\tau, \ldots, \tau)$ $\mathbf{q} = \theta = (q, \ldots, q)$ *the assertion of the theorem just proved coincides with the corresponding result of A.S. Romanyuk* [7].

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