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JUST-INFINITE C^*-ALGEBRAS AND THEIR INVARIANTS

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Abstract. Just-infinite C^*-algebras, i.e., infinite dimensional C^*-algebras, whose proper quotients are finite dimensional, were investigated in [2]. One particular example of a just-infinite residually finite dimensional AF-algebras was constructed in [2]. In this paper we extend that construction by showing that each infinite dimensional metrizable Choquet simplex is affinely homeomorphic to the trace simplex of a just-infinite residually finite dimensional C^*-algebra. The trace simplex of any unital residually finite dimensional C^*-algebra is hence realized by a just-infinite one. We determine the trace simplex of the particular residually finite dimensional AF-algebras constructed in [2], and we show that it has precisely one extremal trace of type II_1.

We give a complete description of the Bratteli diagrams corresponding to residually finite dimensional AF-algebras. We show that a modification of any such Bratteli diagram, similar to the modification that makes an arbitrary Bratteli diagram simple, will yield a just-infinite residually finite dimensional AF-algebra.

1. Introduction

Just-infinite C^*-algebras were introduced and studied in [2] to establish an analogue of just infinite groups, which are infinite groups whose proper quotients are finite; and to examine possible connections between the two. It was shown in [2] that a separable C^*-algebra is just-infinite if and only if either it is simple (and infinite dimensional), or it is an essential extension of a simple infinite dimensional C^*-algebra by a finite dimensional one, or it is residually finite dimensional (RFD) and just-infinite. In the latter case its primitive ideal space is the space $Y_\infty = \{0\} \cup \mathbb{N}$ equipped with the (non-Hausdorff) topology making $\{0\}$ dense and all other singletons closed. In other words, the primitive ideal space of a separable RFD just-infinite C^*-algebra $\mathcal{A}$ is $\{0, I_1, I_2, I_3, \dots\}$, where $\mathcal{A}/I_j$ is simple and finite dimensional, hence a full matrix algebra, say M_{k_j}. The sequence $\{k_j\}$ is called the characteristic sequence for $\mathcal{A}$, and it is the set (with multiplicities) of dimensions of irreducible finite dimensional representations of $\mathcal{A}$.

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Its distinguished feature in the just-infinite case is that it is countable and tends to infinity, as shown in [2].

A concrete example of a RFD just-infinite AF-algebra was constructed in [2].

If G is a discrete group, then its universal C^*-algebra $C^*(G)$ is just-infinite if and only if the group algebra $\mathbb{C}[G]$ is just-infinite as a *-algebra and it has a unique C^*-norm. This again implies that the group G is just infinite. The reverse implication does not hold. The group algebra $\mathbb{C}[G]$ has a unique C^*-norm if G is locally finite. It was shown in [1] that there exist residually finite locally finite groups G such that $\mathbb{C}[G]$ is just infinite (as a *-algebra), and consequently, $C^*(G)$ is RFD and just-infinite, thus demonstrating that such C^*-algebras can arise from groups.

We show in Section 3 that each infinite dimensional metrizable Choquet simplex arises as the trace simplex of a RFD just-infinite AF-algebra. We prove this using a result of Lazar and Lindenstrauss, [3], that each such simplex is the inverse limit of finite dimensional simplices with surjective affine connecting mappings. The simplex of tracial states on a unital infinite dimensional RFD C^*-algebra is necessarily infinite dimensional, and so it also arises from a just-infinite RFD C^*-algebra, in fact even an AF-algebra. In Section 4 we give concrete examples of infinite dimensional Choquet simplicies arising in this way, and show that the trace simplex of the RFD just-infinite AF-algebra constructed in [2, Section 4.1] is equal to Δ_∞, the Bauer simplex with extreme boundary equal to the one-point compactification of $\mathbb{N}$. The extreme trace corresponding to the point at infinity has type II_1, and the other extremal traces are of type I.

In Section 5 we give a complete description of all RFD AF-algebras in terms of their Bratteli diagrams, and we describe which of these Bratteli diagrams correspond to RFD just-infinite AF-algebras. Our description suggests that the class of RFD just-infinite C^*-algebras is rather large, and that the inclusion of RFD just-infinite C^*-algebras inside the class of all RFD C^*-algebras is similar to the inclusion of simple C^*-algebras inside the class of all C^*-algebras.

I thank Anatoly Vershik for several useful conversations on topics of this paper, and I thank Stuart White for explaining a construction contained in Proposition 4.2 (ii).

2. Preliminaries

We review here background material needed to prove the main results of our paper.

The simplex of tracial states on a unital C^*-algebra $\mathcal{B}$ is denoted by $T(\mathcal{B})$. To each $\tau \in T(\mathcal{B})$ one has the GNS representation π_τ of $\mathcal{B}$ on a Hilbert space H_τ and its associated von Neumann algebra $\pi_\tau(\mathcal{B})'' \subseteq B(H_\pi)$. The trace τ extends to a (faithful) tracial state on $\pi_\tau(\mathcal{B})''$, which therefore is a finite von Neumann algebra. Moreover, $\pi_\tau(\mathcal{B})''$ is a factor if and only if τ belongs to $\partial_{\mathrm{e}}T(\mathcal{B})$, the set of extreme points in $T(\mathcal{B})$. In this case τ is said to be of type I_n, $1 \leq n < \infty$, respectively, of type II_1, if $\pi_\tau(\mathcal{B})''$ is a factor of type I_n, respectively, of type II_1.

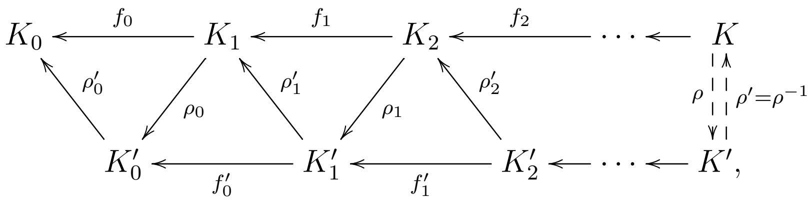
As mentioned in the introduction, a separable RFD just-infinite C^*-algebra $\mathcal{B}$ has countably many non-zero primitive ideals $\{I_j\}_{j=0}^\infty$; and each quotient $\mathcal{B}/I_j$ is isomorphic to a full matrix algebra, say M_{k_j}, for some integer $k_j \geq 1$. Let $\pi_j \colon \mathcal{B} \to \mathcal{B}/I_j$ be the quotient mapping, and let τ_j be the tracial state obtained by composing π_j with the unique normalized tracial state on $\mathcal{B}/I_j \cong M_{k_j}$. Then $\pi_{\tau_j}(\mathcal{B}) = \pi_{\tau_j}(\mathcal{B})'' = M_{k_j}$, which shows that τ_j is an extremal trace on $\mathcal{B}$ of type I_{k_j}.

Proposition 2.1. *Let $\mathcal{B}$ be a separable unital RFD just-infinite C^*-algebra, and let $\{\tau_j\}_{j=0}^\infty$ be the extremal traces on $\mathcal{B}$ defined above. If τ is an extremal trace on $\mathcal{B}$, then either $\tau = \tau_j$, for some $j \geq 0$, in which case τ is of type I_{k_j}, or τ is of type II_1.*

Proof. Let I be the kernel of the GNS-representation π_τ. If $I \neq 0$, then $\mathcal{B}/I$ is finite dimensional, in which case $\pi_\tau(\mathcal{B})'' = \pi_\tau(\mathcal{B})$ is isomorphic to $\mathcal{B}/I$. As τ is extreme, $\mathcal{B}/I$ is a factor, and therefore necessarily a full matrix algebra. Hence I is a primitive ideal, so $I = I_j$, for some $j \geq 0$, which again entails that $\tau = \tau_j$, because the trace on a full matrix algebra is unique. Suppose next that $I = 0$. Then $\pi_\tau(\mathcal{B})''$ is an infinite dimensional finite factor, which entails that it is a factor of type II_1. □

To prove some of our results about the trace simplex of a RFD just-infinite C^*-algebra we need the following standard approximate intertwining result, stated here for compact convex sets. For completeness of the exposition and for the convenience of the reader we include a short sketch of its proof. Equip the set of functions between compact metric spaces K and K' with the uniform metric: $d_\infty(f, g) = \sup\{d_{K'}(f(x), g(x)) : x \in K\}$.

Proposition 2.2. *Suppose we have a system of two inverse limits of compact convex metric spaces:*



where f_j and f'_j are continuous affine maps, where K and K' are the inverse limits of the sequences in the top and bottom rows, respectively, and where ρ_j, ρ'_j are continuous affine maps making the diagram above an approximate intertwining, *i.e.,*

$$\sum_{j=0}^{\infty} d_\infty(\rho'_j \circ \rho_j, f_j) < \infty, \qquad \sum_{j=0}^{\infty} d_\infty(\rho_j \circ \rho'_{j+1}, f'_j) < \infty. \tag{2.1}$$

Then there exists an affine homeomorphism $\rho \colon K \to K'$, with inverse ρ'.

Proof. For $0 \leq i < j$, let $f_{j,i}\colon K_j \to K_i$ and $f'_{j,i}\colon K'_j \to K'_i$ be the continuous affine maps obtained by composing the maps f_n and f'_n, respectively, and let $f_{\infty,j}\colon K \to K_j$ and $f'_{\infty,j}\colon K' \to K'_j$ be the continuous affine maps associated with the two inverse limits. For each $0 \leq i < j$, define $\rho_{i,j}\colon K \to K'_i$ and $\rho'_{i,j}\colon K' \to K_i$ by

$$\rho_{i,j} = f'_{j,i} \circ \rho_j \circ f_{\infty,j+1}, \qquad \rho'_{i,j} = f_{j,i} \circ \rho'_j \circ f'_{\infty,j}.$$

It follows from (2.1) that the sequences $\{\rho_{i,j}\}_{j=i+1}^{\infty}$ and $\{\rho'_{i,j}\}_{j=i+1}^{\infty}$ are Cauchy, and hence convergent, with respect to the uniform metric d_∞. Their limits $\rho_i\colon K \to K'_i$ and $\rho'_i\colon K' \to K_i$ are continuous affine maps satisfying $f'_i \circ \rho_{i+1} = \rho_i$ and $f_i \circ \rho'_{i+1} = \rho'_i$, for all $i \geq 0$. Therefore they factor through continuous affine maps $\rho\colon K \to K'$ and $\rho'\colon K' \to K$, that is, $f'_{\infty,i} \circ \rho = \rho_i$ and $f_{\infty,i} \circ \rho' = \rho'_i$, for all $i \geq 0$. Using these identities, one can now verify that ρ and ρ' are inverses to each other. □

We immediately get the following corollary from the previous proposition.

Corollary 2.3. *Suppose we have an approximate intertwining of two inverse limits of compact convex metric spaces:*

$$\begin{array}{ccccccccc}
K_0 & \xleftarrow{f_0} & K_1 & \xleftarrow{f_1} & K_2 & \xleftarrow{f_2} & \cdots & \longleftarrow & K \\
\| & & \| & & \| & & & & \rho \big\downarrow \big\uparrow \rho'=\rho^{-1} \\
K_0 & \xleftarrow[f'_0]{} & K_1 & \xleftarrow[f'_1]{} & K_2 & \xleftarrow[f'_2]{} & \cdots & \longleftarrow & K',
\end{array}$$

where f_j, f'_j are continuous affine maps satisfying $\sum_{j=0}^{\infty} d_\infty(f_j, f'_j) < \infty$. Then there exists an affine homeomorphism $\rho\colon K \to K'$, with inverse ρ'.

We end this section by describing the affine map between the trace simplices of finite dimensional C^*-algebras induced by a *-homomorphism. The proof of the lemma is straightforward and is omitted.

Lemma 2.4. *Let*

$$\mathcal{A} = \bigoplus_{j=0}^{n} M_{k_j}, \qquad \mathcal{B} = \bigoplus_{i=0}^{m} M_{\ell_i}$$

be finite dimensional C^*-algebras, and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a unital *-homomorphism with multiplicity matrix [1] *$A = (A(i,j))$, $i = 0, 1, \dots, m$, $j = 0, 1, \dots, n$, so that $A(i,j)$ is the multiplicity of the partial *-homomorphism $\varphi_{i,j}\colon M_{k_j} \to M_{\ell_i}$ (from the jth summand of $\mathcal{A}$ to the ith summand of $\mathcal{B}$).*

[1] If we identify $K_0(\mathcal{A})$ and $K_0(\mathcal{B})$ with $\mathbb{Z}^{n+1}$ and $\mathbb{Z}^{m+1}$, respectively, then the multiplicity matrix A of φ is the $(m+1) \times (n+1)$ matrix over $\mathbb{Z}$ which represents the group homomorphism $K_0(\varphi)\colon \mathbb{Z}^{n+1} \to \mathbb{Z}^{m+1}$.

Let $\{\tau_{\mathcal{A},j}\}_{j=0}^n$ and $\{\tau_{\mathcal{B},i}\}_{i=0}^m$ denote the extremal tracial states on $\mathcal{A}$ and $\mathcal{B}$ supported on the jth summand M_{k_j} of $\mathcal{A}$ and the ith summand M_{ℓ_i} of $\mathcal{B}$, respectively, and let $T(\varphi)\colon T(\mathcal{B}) \to T(\mathcal{A})$ be the affine homomorphism induced by φ. Then

$$T(\varphi)(\tau_{\mathcal{B},i}) = \tau_{\mathcal{B},i} \circ \varphi = \sum_{j=0}^{n} \frac{A(i,j)k_j}{\ell_i}\, \tau_{\mathcal{A},j}, \tag{2.2}$$

for all $i = 0, 1, \dots, m$.

Observe that $\sum_{j=0}^n A(i,j)k_j = \ell_i$, for all i, by the assumption that φ is unital. This shows that the right-hand side of (2.2) indeed belongs to $T(\mathcal{A})$.

3. The trace simplex of a just-infinite C^*-algebra

For each $n \geq 0$, let Δ_n be the standard n-dimensional simplex with extreme boundary $\partial_e \Delta_n = \{e_0^{(n)}, e_1^{(n)}, \dots, e_n^{(n)}\}$. It was shown by Lazar and Lindenstrauss, [3, Corollary to Theorem 5.2], that each metrizable infinite

$$\Delta_0 \xleftarrow{f_0} \Delta_1 \xleftarrow{f_1} \Delta_2 \xleftarrow{f_2} \Delta_3 \longleftarrow \cdots \longleftarrow \Delta\ , \tag{3.1}$$

where each f_n is an affine surjective map. Since each extreme point of Δ_n lifts to an extreme point of Δ_{n+1} under any surjective affine map $\Delta_{n+1} \to \Delta_n$, we infer that

$$f_n(e_j^{(n+1)}) = e_j^{(n)}, \quad j = 0, 1, \dots, n; \qquad f_n(e_{n+1}^{(n+1)}) = \xi^{(n)}, \tag{3.2}$$

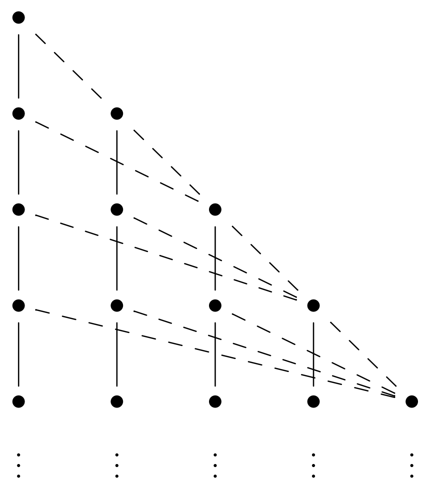
for some $\xi^{(n)} \in \Delta_n$ (possibly after relabelling the extreme points of the simplices Δ_n). The affine maps f_n are determined by (3.2), so the Choquet simplex Δ is determined by the sequence $\{\xi^{(n)}\}_{n=0}^\infty$. Let $f_{\infty,n}\colon \Delta \to \Delta_n$ denote the canonical continuous affine surjection associated with (3.1) satisfying $f_n \circ f_{\infty,n+1} = f_{\infty,n}$, for all $n \geq 0$.

Lemma 3.1. *Let Δ be a Choquet simplex given as in* (3.1) *and* (3.2) *above (for some sequence of elements $\xi^{(n)} \in \Delta_n$, $n \geq 0$). Then, for each $n \geq 0$, there is a (unique) element $e_n \in \Delta$ satisfying $f_{\infty,m}(e_n) = e_n^{(m)}$, whenever $m \geq n$. Moreover, each e_n is an extreme point of Δ, and $\{e_n\}_{n=0}^\infty$ is dense in the extreme boundary of Δ.*

Proof. The existence of $e_n \in \Delta$ follows from (3.2) and standard properties of inverse limits; and its uniqueness from the fact that if $x, y \in \Delta$ are such that $f_{\infty,m}(x) = f_{\infty,m}(y)$, for all sufficiently large m, then $x = y$. Since $f_{\infty,m}(e_n)$ is an extreme point in Δ_m, for all $m \geq n$, e_n must itself be an extreme point in Δ. By Milman's partial converse to the Krein–Milman theorem, to show that $\{e_n\}_{n=0}^\infty$ is dense in the extreme boundary of Δ, it suffices to show that the convex hull, C, of $\{e_n\}_{n=0}^\infty$ is dense in Δ. However, $f_{\infty,m}(C) = \Delta_m$, for all $m \geq 0$ (since $f_{\infty,m}(C)$ is a convex sets that contains all the extreme points of Δ_m), and this shows that C is dense in Δ. □

Consider the AF-algebra $\mathcal{A}$ whose Bratteli diagram is given as follows:

(3.3)



$\mathcal{A}_0 = M_{k_0}$

$\mathcal{A}_1 = M_{k_0} \oplus M_{k_1}$

$\mathcal{A}_2 = M_{k_0} \oplus M_{k_1} \oplus M_{k_2}$

$\mathcal{A}_3 = M_{k_0} \oplus M_{k_1} \oplus M_{k_2} \oplus M_{k_3}$

$\mathcal{A}_4 = M_{k_0} \oplus M_{k_1} \oplus M_{k_2} \oplus M_{k_3} \oplus M_{k_4}$

where the dotted edge from the vertex at position (n, j) to the vertex at position $(n+1, n+1)$ has multiplicity $m_j^{(n)} \geq 0$. The unbroken edges all have multiplicity 1. The *-homomorphisms $\varphi_n \colon \mathcal{A}_n \to \mathcal{A}_{n+1}$ are unital (and hence determined—up to unitary equivalence—by the Bratteli diagram). Let $\varphi_{\infty,n} \colon \mathcal{A}_n \to \mathcal{A}$ denote the canonical inductive limit *-homomorphism, satisfying $\varphi_{\infty,n+1} \circ \varphi_n = \varphi_{\infty,n}$, for all $n \geq 0$. The integer $k_0 \geq 1$ can be chosen arbitrarily, and the remaining integers k_n are determined by the formula

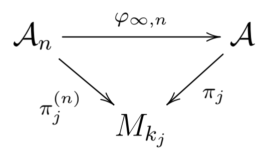
$$k_{n+1} = \sum_{j=0}^{n} m_j^{(n)} k_j. \tag{3.4}$$

The Bratteli diagram (3.3), and hence the AF-algebra $\mathcal{A}$, are thus determined by the choice of the initial integer $k_0 \geq 1$ and of the multiplicity vectors

$$m^{(n)} = \big(m_0^{(n)}, m_1^{(n)}, m_2^{(n)}, \dots, m_n^{(n)}\big), \qquad n \geq 0, \tag{3.5}$$

each of which, moreover, is assumed to be non-zero (to ensure that $k_n > 0$, for each $n \geq 0$).

Lemma 3.2. *The AF-algebra $\mathcal{A}$, described above, is unital and RFD. For each $j \geq 0$, there is a surjective *-homomorphism $\pi_j \colon \mathcal{A} \to M_{k_j}$, making the diagram*



commutative for each $n \geq j$, where $\pi_j^{(n)}$ is the projection onto the jth summand of $\mathcal{A}_n$.

If each multiplicity $m_j^{(n)}$ *is non-zero, then* $\mathcal{A}$ *is just-infinite, and the primitive ideal space of* $\mathcal{A}$ *consists of* 0 *and of the ideals* $\ker(\pi_j)$, $j \geq 0$. *In particular, the characteristic sequence for* $\mathcal{A}$ *is precisely the sequence* $\{k_j\}_{j=0}^{\infty}$ *defined in (and above)* (3.4).

Proof. By the assumptions on the connecting maps φ_n, we see that $\pi_j^{(n+1)} \circ \varphi_n = \pi_j^{(n)}$, for all $n \geq j$, so there exists a *-homomorphism $\pi_j \colon \mathcal{A} \to M_{k_j}$, making the diagram in the lemma commutative. As $\bigoplus_{j=0}^{n} \pi_j^{(n)}$ obviously is injective on $\mathcal{A}_n$, for each $n \geq 0$, it follows that $\bigoplus_{j=0}^{\infty} \pi_j$ is injective, and hence isometric, on $\bigcup_{n=0}^{\infty} \varphi_{\infty,n}(\mathcal{A}_n)$, and hence isometric (and therefore injective) on $\mathcal{A}$. This shows that $\mathcal{A}$ is RFD.

The proof that $\mathcal{A}$ is just-infinite with the stipulated primitive ideal space is analogous to the proof that the C^*-algebra constructed in [2, Section 4.1] has these property. (In fact, the C^*-algebra constructed therein is precisely our C^*-algebra $\mathcal{A}$ in the case where $m_j^{(n)} = 1$, for all n, j.) These facts are also stated and proved explicitly in the more general result, Theorem 5.4, proven later in this article. The last claim about $\{k_j\}$ follows by the definition of the characteristic sequence given above. ☐

The trace simplex $T(\mathcal{A}_n)$ has extremal points $\{\tau_j^{(n)}\}_{j=0}^{n}$, where $\tau_j^{(n)}$ is the normalized trace on the jth summand, M_{k_j}, of $\mathcal{A}_n$. We therefore have affine isomorphisms $\chi_n \colon \Delta_n \to T(\mathcal{A}_n)$ such that $\chi_n(e_j^{(n)}) = \tau_j^{(n)}$, for all j. Under this identification, we obtain surjective affine maps $f_n \colon \Delta_{n+1} \to \Delta_n$, $n \geq 0$, and an affine homeomorphism χ making the diagram

$$
\text{(3.6)} \qquad
\begin{array}{ccccccccc}
\Delta_0 & \xleftarrow{f_0} & \Delta_1 & \xleftarrow{f_1} & \Delta_2 & \xleftarrow{f_2} & \Delta_3 & \longleftarrow \cdots \longleftarrow & \Delta \\
\downarrow{\scriptstyle \chi_0} & & \downarrow{\scriptstyle \chi_1} & & \downarrow{\scriptstyle \chi_2} & & \downarrow{\scriptstyle \chi_3} & & \downarrow{\scriptstyle \chi} \\
T(\mathcal{A}_0) & \xleftarrow[T(\varphi_0)]{} & T(\mathcal{A}_1) & \xleftarrow[T(\varphi_1)]{} & T(\mathcal{A}_2) & \xleftarrow[T(\varphi_2)]{} & T(\mathcal{A}_3) & \longleftarrow \cdots \longleftarrow & T(\mathcal{A})
\end{array}
$$

commutative. The lemma below, which states that the sequence in the upper row of (3.6) is of the kind described in (3.1) and (3.2), is a restatement of Lemma 2.4 in the case of the particular connecting mappings under consideration.

Lemma 3.3. *In the notation of* (3.3) *and* (3.6), *for each* $n \geq 0$, *we have*

$$f_n(e_j^{(n+1)}) = e_j^{(n)}, \quad j = 0, 1, \dots, n; \qquad f_n(e_{n+1}^{(n+1)}) = \zeta^{(n)},$$

where

$$\zeta^{(n)} = \left(\frac{m_0^{(n)} k_0}{k_{n+1}}, \frac{m_1^{(n)} k_1}{k_{n+1}}, \cdots, \frac{m_n^{(n)} k_n}{k_{n+1}} \right) \in \Delta_n.$$

With e_j the extreme points of Δ defined in Lemma 3.1, we have $\chi(e_j) = \tau_j$, where τ_j is the extremal trace on $\mathcal{A}$ obtained by composing the *-homomorphism

$\pi_j \colon \mathcal{A} \to M_{k_j}$ with the normalized trace on M_{k_j}. In particular, τ_j is of type I_{k_j}. It follows from Lemma 3.1 and (3.6) that $\{\tau_j\}_{j=0}^{\infty}$ is dense in the set of extremal traces on $T(\mathcal{A})$, and that all other extremal traces are of type II_1.

Theorem 3.4. *The following four statements are equivalent for each metrizable Choquet simplex* Δ*:*

(i) Δ *is infinite dimensional (i.e.,* $\Delta \neq \Delta_n$ *for all* $n \geq 0$*).*

(ii) *There is a unital separable infinite dimensional RFD* C^**-algebra whose trace simplex is affinely homeomorphic to* Δ*.*

(iii) *There is a unital separable RFD just-infinite* C^**-algebra whose trace simplex is affinely homeomorphic to* Δ*.*

(iv) *There is a unital separable RFD just-infinite AF-algebra, arising from a Bratteli diagram of the type described in* (3.3)*, with* $m_j^{(n)} \geq 1$*, for all* n, j*, whose trace simplex is affinely homeomorphic to* Δ*.*

Proof. (i) $\Rightarrow$ (iv). If (i) holds, then by the theorem of Lazar and Lindenstrauss, mentioned in Section 2, we may realize Δ as an inverse limit as described in (3.1) and (3.2), with respect to some sequence of elements $\xi^{(n)} = (\xi_0^{(n)}, \xi_1^{(n)}, \dots, \xi_n^{(n)}) \in \Delta_n$, $n \geq 0$.

Let $m_j^{(n)} \geq 1$, $0 \leq j \leq n$, be a system of multiplicities giving a RFD just-infinite AF-algebra $\mathcal{A}$, as defined in (3.3), cf. Lemma 3.2, with $k_0 = 1$ (or any other value of k_0), and with k_{n+1} given as in (3.4), for $n \geq 0$. We show that $m_j^{(n)}$ can be chosen such that $T(\mathcal{A})$ is affinely homeomorphic to Δ. Let

$$\zeta^{(n)} = \left(\frac{m_0^{(n)} k_0}{k_{n+1}}, \frac{m_1^{(n)} k_1}{k_{n+1}}, \cdots, \frac{m_n^{(n)} k_n}{k_{n+1}} \right) \in \Delta_n, \quad n \geq 0,$$

and let $f_n' \colon \Delta_{n+1} \to \Delta_n$ be given by $f_n'(e_j^{(n+1)}) = e_j^{(n)}$, for $0 \leq j \leq n$, and $f_n'(e_{n+1}^{(n+1)}) = \zeta^{(n)}$. By (3.6) and Lemma 3.3, $T(\mathcal{A})$ is affinely homeomorphic to the inverse limit of the sequence

$$\Delta_0 \xleftarrow{f_0'} \Delta_1 \xleftarrow{f_1'} \Delta_2 \xleftarrow{f_2'} \Delta_3 \longleftarrow \cdots .$$

The simplices Δ and $T(\mathcal{A})$ are affinely homeomorphic if $\sum_{n=0}^{\infty} d_\infty(f_n, f_n') < \infty$, by Corollary 2.3, where d_∞ is the uniform metric (as in Proposition 2.2). It is easily seen that $d_\infty(f_n, f_n') = d(\xi^{(n)}, \zeta^{(n)})$, where d is the Euclidian metric on $\Delta_n \subseteq \mathbb{R}^{n+1}$.

We determine $m_j^{(n)} \geq 1$ inductively (after $n \geq 0$) such that $d(\xi^{(n)}, \zeta^{(n)}) \leq 2^{-n}$, for all $n \geq 0$. For $n = 0$, we have $\xi^{(0)} = \zeta^{(0)}$ for any choice of $m_0^{(0)}$. Let

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$n \geq 1$ and suppose we have chosen $m_j^{(r)}$, for all $0 \leq j \leq r < n$. Find integers $\ell_0, \ell_1, \dots, \ell_n \geq 1$ such that

$$\left| \frac{\ell_j}{\sum_{i=0}^n \ell_i} - \xi_j^{(n)} \right| < \frac{1}{2^n \sqrt{n}}, \quad j = 0, 1, \dots, n. \tag{3.7}$$

Let $1 = k_0, k_1, \dots, k_n$ be given as in (3.4) (with respect to the already made choices of $m_j^{(r)}$). Put $K = \prod_{j=0}^n k_j$, and set $m_j^{(n)} = K\ell_j/k_j$, for $j = 0, 1, \dots, n$. Then $k_{n+1} = \sum_{j=0}^n m_j^{(n)} k_j = K \sum_{j=0}^n \ell_j$, so

$$\zeta_j^{(n)} = \frac{m_j^{(n)} k_j}{k_{n+1}} = \frac{\ell_j}{\sum_{i=0}^n \ell_i}.$$

Hence, by (3.7), we obtain that $d(\xi^{(n)}, \zeta^{(n)}) < 2^{-n}$ as desired.

The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are trivial.

(ii) $\Rightarrow$ (i). If $\mathcal{A}$ is infinite dimensional and RFD, then there is an infinite family $\{\pi_\alpha\}$ of pairwise inequivalent finite dimensional irreducible representations of $\mathcal{A}$. Let τ_α be the tracial state on $\mathcal{A}$ obtained by composing π_α with the unique tracial state on $\pi_\alpha(\mathcal{A})$ (which is a full matrix algebra). Then each τ_α is extremal (because $\pi_\alpha(\mathcal{A}) = \pi_\alpha(\mathcal{A})''$ is a factor), and they are mutually distinct, because the π_α's are inequivalent, and hence have mutually distinct kernels. We have thus exhibited an infinite subset $\{\tau_\alpha\}$ of the extreme boundary of $T(\mathcal{A})$, showing that (i) holds. □

Remark 3.5 (Other invariants)**.** Suppose that $\mathcal{A}$ is a unital just-infinite AF-algebra arising from a Bratteli diagram of the type described in (3.3). Modifying slightly the argument from [2, Section 4.1] one can show that the dimension group (G, G^+, u), as an ordered abelian group (G, G^+) with a distinguished order unit u, can be described as follows: Let $\prod_{n=0}^\infty \mathbb{Z}$ be the ordered abelian group equipped with the standard order relation, whereby an element is positive when each coordinate is non-negative. Then G is the subgroup of $\prod_{n=0}^\infty \mathbb{Z}$ consisting of all $x = (x_n)_{n=0}^\infty$ for which there exists $n_0 \geq 0$ such that

$$x_{n+1} = \sum_{j=0}^n m_j^{(n)} x_j,$$

for all $n \geq n_0$. The order unit u is the element $(k_0, k_1, k_2, \dots) \in G^+$ determined in (3.4). The order on G is the one inherited from $\prod_{j=0}^\infty \mathbb{Z}$.

The group G has the additional (non-degeneracy) property that whenever F is a finite subset of $\{0, 1, 2, \dots\}$ and $\rho_F \colon \prod_{n=0}^\infty \mathbb{Z} \to \prod_{n \in F} \mathbb{Z}$ is the canonical projection map, then $\rho_F(G) = \prod_{n \in F} \mathbb{Z}$. If $F = \{j\}$, then ρ_F is the homomorphism of K_0-groups induced by the irreducible representation $\pi_j \colon \mathcal{A} \to M_{k_j}$.

The group (G, G^+, u) contains all information about $\mathcal{A}$, but our picture of G is not sufficiently explicit to reveal this information easily.

Two other invariants of RFD just-infinite C^*-algebras, already mentioned, are their trace simplex, discussed in the previous theorem, and their characteristic sequence $\{k_n\}_{n=0}^\infty$, which by the discussion above coincides with the order unit $u \in G$. Where the trace simplex can be any infinite dimensional metrizable Choquet simplex Δ by Theorem 3.4, it is less obvious what are the possible values of the characteristic sequence. It was shown in [2, Proposition 3.18], that $k_n \to \infty$, as $n \to \infty$.

For each fixed Choquet simples Δ as above, the set of possible characteristic sequences $\{k_n\}_{n=0}^\infty$, of a RFD just-infinite C^*-algebra $\mathcal{A}$ with $T(\mathcal{A}) = \Delta$, is uncountable. Indeed, inspecting the proof of (i) $\Rightarrow$ (iv) in Theorem 3.4, we see that, for each $n \geq 0$, there are (countably) infinitely many $(n+1)$-tuples $(\ell_0, \ell_1, \dots, \ell_n)$ satisfying (3.7). These, in turn, give rise to infinitely many choices for the multiplicity vector $(m_0^{(n)}, m_1^{(n)}, \dots, m_n^{(n)})$, and hence infinitely many values of k_{n+1}. This algorithm produces uncountably many possibilities for the characteristic sequence $\{k_n\}$ of a RFD just-infinite AF-algebra with trace simplex Δ.

4. An example

In this section we make explicit computations of the Choquet simplex Δ described in (3.1) and (3.2), and in particular of the trace simplex of the RFD just-infinite C^*-algebra constructed in [2, Section 4.1] (and in (3.3) with $m_j^{(n)} = 1$, for all j, n).

Recall that Δ is determined by the sequence $\xi^{(n)} \in \Delta_n$, $n \geq 0$, cf. (3.2). The first class of examples we shall consider will be referred to as *stationary case*, i.e., the case where the sequence of points $\xi^{(n)} \in \Delta_n$ eventually satisfies $f_n(\xi^{(n+1)}) = \xi^{(n)}$.

Lemma 4.1. *Let $t = \{t_n\}_{n=0}^\infty$ be a non-zero sequence of non-negative numbers. Let $n_0 \geq 0$ be the smallest integer such that $t_{n_0} \neq 0$. For each $n \geq n_0$, define*

$$\xi^{(n)} = \left(\sum_{j=0}^{n} t_j \right)^{-1} (t_0, t_1, \dots, t_n) \in \Delta_n, \tag{4.1}$$

and let $\xi^{(n)} \in \Delta_n$ be arbitrary, for $0 \leq n < n_0$ (if $n_0 > 0$). Then $\{\xi^{(n)}\}_{n=0}^\infty$ gives rise to a stationary sequence. Conversely, each stationary sequence $\{\xi^{(n)}\}_{n=0}^\infty$ arises in this way.

Proof. Let $t = \{t_n\}_{n=0}^\infty$ be given as in the lemma, and let $\{\xi^{(n)}\}_{n=0}^\infty$ be given as in (4.1). We show that $f_n(\xi^{(n+1)}) = \xi^{(n)}$ for all $n \geq n_0$. Let $n \geq n_0$ and put $\alpha = \sum_{j=0}^n t_j$ and $\beta = \sum_{j=0}^{n+1} t_j$. Then

$$f_n(\xi^{(n+1)}) = \sum_{j=0}^{n} \beta^{-1} t_j f_n(e_j^{(n+1)}) + \beta^{-1} t_{n+1} f_n(e_{n+1}^{(n+1)})$$

$$= \sum_{j=0}^{n} \beta^{-1} t_j e_j^{(n)} + \beta^{-1} t_{n+1} \xi^{(n)} \;=\; \big(\beta^{-1}\alpha + \beta^{-1} t_{n+1}\big)\xi^{(n)} = \xi^{(n)}.$$

To prove the converse claim, it suffices to show that if $f_n(\xi^{(n+1)}) = \xi^{(n)}$, then $\xi^{(n)}$ and $\big(\xi_0^{(n+1)}, \xi_1^{(n+1)}, \dots, \xi_n^{(n+1)}\big)$ are proportional. This, however, follows from the identity:

$$\begin{aligned} \xi^{(n)} = f_n(\xi^{(n+1)}) &= \sum_{j=0}^{n} \xi_j^{(n+1)} \cdot f_n(e_j^{(n+1)}) + \xi_{n+1}^{(n+1)} \cdot f_n(e_{n+1}^{(n+1)}) \\ &= \sum_{j=0}^{n} \xi_j^{(n+1)} \cdot e_j^{(n)} + \xi_{n+1}^{(n+1)} \cdot \xi^{(n)} \\ &= \big(\xi_0^{(n+1}, \xi_1^{(n+1)}, \dots, \xi_n^{(n+1)}\big) + \xi_{n+1}^{(n+1)} \cdot \xi^{(n)}. \end{aligned}$$ □

Two non-zero sequences $t = \{t_n\}_{n=0}^\infty$ and $t' = \{t'_n\}_{n=0}^\infty$ of non-negative numbers give rise to the same stationary sequence $\{\xi^{(n)}\}_{n=0}^\infty$, as in Lemma 4.1, if and only if they are propotional. Hence, if $\sum_{n=0}^\infty t_n < \infty$, we may without loss of generality assume that $\sum_{n=0}^\infty t_n = 1$.

Let Δ_∞ denote the infinite dimensional Bauer simplex with extreme boundary $\partial_e \Delta_\infty = \{e_j^{(\infty)} : 0 \le j \le \infty\}$ equipped with the topology making it homeomorphic to $\mathbb{N}_0 \cup \{\infty\}$, the one-point compactification of the discrete space $\mathbb{N}_0$. In particular, $e_j^{(\infty)} \to e_\infty^{(\infty)}$ as $j \to \infty$. Each point x in Δ_∞ is a unique infinite convex combination $x = \sum_j \lambda_j e_j^{(\infty)}$.

Proposition 4.2. *Let Δ be the Choquet simplex defined in* (3.1) *and* (3.2) *with respect to a stationary sequence $\xi^{(n)} \in \Delta_n$, $n \ge 0$. Let $\{t_n\}_{n=0}^\infty$ be a sequence of non-negative numbers generating $\{\xi^{(n)}\}$ as in* (4.1)*, and let $n_0 \ge 0$ be the smallest integer such that $t_{n_0} \neq 0$.*

Let $e_\infty \in \Delta$ be determined by $f_{\infty,n}(e_\infty) = \xi^{(n)}$, for all $n \ge n_0$, and let $\{e_n : 0 \le n < \infty\}$ be the dense subset of the extreme boundary of Δ defined in Lemma 3.1*.*

- (i) *If $\sum_{n=0}^\infty t_j = \infty$, then Δ is affinely homeomorphic to Δ_∞, and the extreme boundary of Δ is equal to $\{e_n : 0 \le n \le \infty\}$. Moreover, $e_n \to e_\infty$ as $n \to \infty$.*
- (ii) *If $\sum_{n=0}^\infty t_j = 1$, then the extreme boundary of Δ is equal to the set $\{e_n : 0 \le n < \infty\}$, and*

$$\lim_{n\to\infty} e_n = e_\infty = \sum_{j=0}^\infty t_j e_j.$$

In particular, Δ is not a Bauer simplex.

Proof. For each $n \geq n_0$, let $g_n \colon \Delta_\infty \to \Delta_n$ be the (unique) continuous affine surjective map satisfying

$$g_n(e_j^{(\infty)}) = \begin{cases} e_j^{(n)}, & 0 \leq j \leq n, \\ \xi^{(n)}, & n < j \leq \infty. \end{cases} \tag{4.2}$$

One can easily verify that $(f_n \circ g_{n+1})(e_j^{(\infty)}) = g_n(e_j^{(\infty)})$, for all $0 \leq j \leq \infty$ and all $n \geq n_0$, so $f_n \circ g_{n+1} = g_n$, for all $n \geq n_0$. We therefore obtain a well-defined continuous affine surjective map $g \colon \Delta_\infty \to \Delta$ satisfying $f_{\infty,n} \circ g = g_n$, for all $n \geq 0$. Moreover, $g(e_j^{(\infty)}) = e_j$, for all $0 \leq j \leq \infty$. This follows from Lemma 3.1 and (4.2), when $0 \leq j < \infty$; and from the identity

$$f_{\infty,n}(e_\infty) = \xi^{(n)} = g_n(e_\infty^{(\infty)}) = f_{\infty,n}\big(g(e_\infty^{(\infty)})\big),$$

which holds for all $n \geq n_0$, in the case when $j = \infty$. In particular, $e_j \to e_\infty$ as $j \to \infty$.

(i). Suppose now that $\sum_{n=0}^\infty t_j = \infty$. We show that $g \colon \Delta_\infty \to \Delta$ is injective (and hence an affine homeomorphism) in this case. Let

$$y = \sum_{0 \leq j \leq \infty} y_j e_j^{(\infty)} \in \Delta_\infty, \qquad z = \sum_{0 \leq j \leq \infty} z_j e_j^{(\infty)} \in \Delta_\infty,$$

be given and suppose that $g(y) = g(z)$. Then $g_n(y) = g_n(z)$, for all integers $n \geq n_0$, so

$$(y_0, y_1, \dots, y_n) + \Big(1 - \sum_{j=0}^n y_j\Big)\xi^{(n)} = (z_0, z_1, \dots, z_n) + \Big(1 - \sum_{j=0}^n z_j\Big)\xi^{(n)},$$

for all $n \geq 0$. The assumption on the sequence $\{t_j\}$ implies that the ith entry of $\xi^{(n)}$ converges to 0 as $n \to \infty$, for each $i \geq 0$. Hence, if we fix attention to the ith coordinate in the equation above, and let $n \to \infty$, then we obtain that $y_i = z_i$, for all $0 \leq i < \infty$. This entails that $y_\infty = z_\infty$, so $y = z$, as desired.

(ii). Since g is surjective it maps $\partial_{\mathrm{e}}\Delta_\infty$ onto $\partial_{\mathrm{e}}\Delta$, so $\partial_{\mathrm{e}}\Delta$ is contained in $g(\partial_{\mathrm{e}}\Delta_\infty) = \{e_j : 0 \leq j \leq \infty\}$. By Lemma 3.1 we know that $e_j \in \partial_{\mathrm{e}}\Delta$ for all $0 \leq j < \infty$. Now,

$$f_{\infty,n}\Big(\sum_{j=0}^\infty t_j e_j\Big) = \sum_{j=0}^\infty t_j f_{\infty,n}(e_j) = \sum_{j=0}^n t_j e_j^{(n)} + \Big(1 - \sum_{j=0}^n t_j\Big)\xi^{(n)} = \xi^{(n)} = f_{\infty,n}(e_\infty),$$

for all $n \geq n_0$, which implies that $e_\infty = \sum_{j=0}^\infty t_j e_j$. In particular, g is not injective and e_∞ is not an extreme point of Δ. We already observed that $e_j \to e_\infty$, which proves the last claim in (ii). □

Example 4.3. Let $\mathcal{A}$ be the just-infinite RFD AF-algebra constructed in [2, Section 4.1], and in (3.3) with multiplicities $m_j^{(n)} = 1$, for all $0 \leq j \leq n$. It has characteristic sequence $k_0 = 1$, and $k_j = 2^{j-1}$, for $j \geq 1$, cf. (3.6). Let τ_j,

$0 \le j < \infty$, be the extremal traces on $\mathcal{A}$ of type I_{k_j}, obtained by composing the irreducible representation $\pi_j \colon \mathcal{A} \to M_{k_j}$ from Lemma 3.2 with the unique tracial state on M_{k_j}, cf. the comments below Lemma 3.1. Then $\chi(e_j) = \tau_j$, for $0 \le j < \infty$, where $\chi \colon \Delta \to T(\mathcal{A})$ the affine homeomorphism defined in (3.6). We conclude from Lemma 3.1 that $\{\tau_j\}_{j=0}^{\infty}$ is dense in $\partial_{\mathrm{e}} T(\mathcal{A})$.

In the notation of Lemma 3.3 (and under the identification of $T(\mathcal{A})$ given in (3.6)), we have

$$\zeta^{(n)} = \Big(\frac{k_0}{k_{n+1}}, \frac{k_1}{k_{n+1}}, \dots, \frac{k_n}{k_{n+1}} \Big) = \Big(\sum_{j=0}^{n} t_j \Big)^{-1} \big(t_0, t_1, t_2, \dots, t_n\big),$$

when $t_j = k_j$, for all $j \ge 0$. In other words, we are in the stationary case covered in Proposition 4.2 (i). Hence $\Delta = \Delta_\infty$ and $\partial_{\mathrm{e}}\Delta = \{e_j : 0 \le j \le \infty\}$ (as in Proposition 4.2). It follows that $\partial_{\mathrm{e}} T(\mathcal{A}) = \{\tau_j : 0 \le j \le \infty\}$, where $\tau_\infty = \chi(e_\infty)$ is an extremal trace of type II_1, by Proposition 2.1, and $\tau_j \to \tau_\infty$ as $j \to \infty$.

The extremal type II_1 trace τ_∞ can be described in a little more detail as follows using Proposition 2.1. Since $f_{\infty,n}(e_\infty) = \zeta^{(n)}$, for all $n \ge n_0$ ($n_0 = 0$ in our case), the restriction $\tau_\infty^{(n)}$ of τ_∞ to the subalgebra $\mathcal{A}_n$ is given as

$$\tau_\infty^{(n)} = 2^{-n}\big(\tau_0^{(n)} + \tau_1^{(n)} + 2\tau_2^{(n)} + 4\tau_3^{(n)} + \cdots + 2^{n-1}\tau_n^{(n)}\big),$$

for all $n \ge 1$, where $\tau_j^{(n)}$ is the jth extremal trace (cf. the comments above (3.6)) on $\mathcal{A}_n$.

In conclusion, we have shown that the trace simplex of $\mathcal{A}$ is the Bauer simplex Δ_∞, and we have identified each extremal trace of $\mathcal{A}$. In particular, $\mathcal{A}$ has precisely one extremal trace of type II_1, which is identified by the equation above.

Example 4.4. Let Δ be the Choquet simplex arising as in Proposition 4.2 (ii) with respect to a sequence $\{t_n\}_{n=0}^{\infty}$ of positive numbers adding up to 1. Then $\partial_{\mathrm{e}}\Delta = \{e_j : 0 \le j < \infty\}$, where e_j is as defined in Lemma 3.1, and $e_n \to \sum_{j=0}^{\infty} t_j e_j$ as $n \to \infty$.

Let $\mathcal{A}$ be a RFD just-infinite AF-algebra with Bratteli diagram (3.3), for suitable multiplicities $m_j^{(n)} \ge 1$, for each $0 \le j \le n$, so that $T(\mathcal{A})$ is affinely homeomorphic to Δ via the affine homeomorphism $\chi \colon \Delta \to T(\mathcal{A})$ defined in (3.6), cf. Theorem 3.4 (and its proof). As in Example 4.3 above, $\chi(e_j) = \tau_j$, for $0 \le j < \infty$, where τ_j is an extremal trace on $\mathcal{A}$ of type I_{k_j} The set of extremal traces on $\mathcal{A}$ is therefore equal to $\{\tau_j\}_{j=0}^{\infty}$,

$$\lim_{n\to\infty} \tau_n = \sum_{j=0}^{\infty} t_j \tau_j,$$

in the weak* topoology, and $\mathcal{A}$ has no extremal trace of type II_1. In particular, $\mathcal{A}$ has no representation of type II_1, so the bidual $\mathcal{A}^{**}$ has no central portion of type II_1.

The Bratteli diagram for $\mathcal{A}$ is determined by the multiplicities $m_j^{(n)}$, which again can be derived (although not uniquely) from the given sequence $\{t_n\}_{n=0}^\infty$, as in the proof of Theorem 3.4. In the cases where all t_n are rational numbers, the proof of Theorem 3.4 yields a recipe for finding the multiplicities $m_j^{(n)}$ such that $\zeta^{(n)} = \xi^{(n)}$, for all $n \geq 0$, (we can choose the integers ℓ_j such that the quantity on the left-hand side of (3.7) is identically zero). In this case we obtain the affine homeomorphism $\chi \colon \Delta \to T(\mathcal{A})$ without using the approximate intertwining of Corollary 2.3.

Example 4.5. It follows from the theorem of Lazar and Lindstrauss, mentioned at the beginning of Section 3, that any infinite dimensional metrizable Choquet simplex is an inverse limit as in (3.1) and (3.2). In particular, each Bauer simplex $\mathcal{P}(X)$ of probability measures on an infinite metrizable compact Hausdorff space X arises in this way.

We indicate here a direct way to see this. For each integer $n \geq 0$, choose $x_n \in X$, an open cover $\{U_j^{(n)}\}_{j=0}^n$ of X, and a partition $\{\varphi_j^{(n)}\}_{j=0}^n \subseteq C(X)$ of the unit subordinate to $\{U_j^{(n)}\}_{j=0}^n$ such that:

(i) $\varphi_j^{(n)}(x_j) = 1$, for each $n \geq 0$ and for each $0 \leq j \leq n$,

(ii) $\mathrm{span}\{\varphi_j^{(n)} : 0 \leq j \leq n\}$ is dense in $C(X)$.

Condition (i) implies that $x_j \in U_j^{(n)}$, for all $n \geq 0$, and that $\varphi_j^{(n)}(x_i) = \delta_{i,j}$, for all $n \geq 0$ and for all $i, j = 0, 1, \dots, n$. Put $X_n = \{x_0, x_1, \dots, x_n\}$. Define ucp maps $G_n \colon C(X_n) \to C(X)$, $n \geq 0$, by

$$G_n(h) = \sum_{j=0}^n h(x_j)\varphi_j^{(n)}, \qquad h \in C(X_n),$$

and define ucp maps $F_n \colon C(X_n) \to C(X_{n+1})$ by $F_n(h) = G_n(h)|_{X_{n+1}}$, for $n \geq 0$. Let $f_n \colon \mathcal{P}(X_{n+1}) \to \mathcal{P}(X_n)$ and $g_n \colon \mathcal{P}(X) \to \mathcal{P}(X_n)$ be the continuous affine maps induced by F_n and G_n, respectively, and let Δ be the inverse limit of the sequence

$$\mathcal{P}(X_0) \xleftarrow{f_0} \mathcal{P}(X_1) \xleftarrow{f_1} \mathcal{P}(X_2) \xleftarrow{f_2} \mathcal{P}(X_3) \longleftarrow \cdots,$$

with associated affine continuous maps $f_{\infty,n} \colon \Delta \to \mathcal{P}(X_n)$. The extreme boundary of $\mathcal{P}(X_n)$ is equal to $\{\delta_{x_j}\}_{j=0}^n$, where δ_x denotes the Dirac measure in $x \in X$. One can now verify that $f_n(\delta_{x_j}) = \delta_{x_j}$, for $j = 0, 1, \dots, n$, and that

$$\xi^{(n)} := f_n(\delta_{x_{n+1}}) = \sum_{j=0}^n \varphi_j^{(n)}(x_{n+1})\, \delta_{x_j}.$$

In other words, Δ arises as in (3.1) and (3.2) with $\xi^{(n)}$ given as above. Since each f_n is surjective, and since $f_{n+1} \circ g_{n+1} = g_n$, for all $n \geq 0$, there is a continuous affine surjective map $g \colon \mathcal{P}(X) \to \Delta$ such that $f_{\infty,n} \circ g = g_n$, for all $n \geq 0$.

To see that g is injective, if $\mu, \nu \in \mathcal{P}(X)$ are such that $g(\mu) = g(\nu)$, then $g_n(\mu) = g_n(\nu)$, for all $n \geq 0$, which implies that $\mu(\varphi_j^{(n)}) = \nu(\varphi_j^{(n)})$, for all $0 \leq j \leq n$. Hence $\mu = \nu$ by (ii).

5. The Bratteli diagrams of general RFD AF-algebras and of RFD just-infinite AF-algebras

In this section we shall give a complete description of which Bratteli diagram give rise to RFD AF-algebras and among those, which give rise to RFD just-infinite AF-algebras (cf. Theorems 5.3 and 5.4 below). Our results show that passing from an arbitrary Bratteli diagram of a RFD AF-algebra to one of a just-infinite RFD AF-algebra, is much like passing from a general Bratteli diagram (of an arbitrary AF-algebra) to a simple one.

It is notationally more convenient to formulate these results in terms of direct limits of simplical groups, which carries the same information as a Bratteli diagram. Moreover, we index by integers ≥ 1 (rather than integers ≥ 0 as in the previous sections). In other words, we consider a direct limit of the form

$$(\mathbb{Z}^{m_1}, u_1) \xrightarrow{A_1} (\mathbb{Z}^{m_2}, u_2) \xrightarrow{A_2} (\mathbb{Z}^{m_3}, u_3) \xrightarrow{A_3} \cdots, \tag{5.1}$$

where $m_n \geq 1$ are integers, and where each A_n is an $m_{n+1} \times m_n$ matrix with non-negative integer coefficients, where all rows and columns in A_n are non-zero, and where u_n is an order unit for $\mathbb{Z}^{m_n}$, satisfying $A_n u_n \leq u_{n+1}$. (The order of $\mathbb{Z}^m$ is the usual one given by $x \leq y$ if $x_j \leq y_j$, for all $1 \leq j \leq m$.)

The AF-algebra associated with the sequence (5.1) is the inductive limit of the sequence

$$\mathcal{A}_1 \xrightarrow{\varphi_1} \mathcal{A}_2 \xrightarrow{\varphi_2} \mathcal{A}_3 \xrightarrow{\varphi_3} \cdots, \tag{5.2}$$

of finite dimensional C^*-algebras $\mathcal{A}_n = \bigoplus_{i=1}^{m_n} \mathcal{A}_{n,i}$, where each $\mathcal{A}_{n,i}$ is (isomorphic to) the full matrix algebra $M_{u_n(i)}$. The connecting mapping $\varphi_n \colon \mathcal{A}_n \to \mathcal{A}_{n+1}$ is determined, up to unitary equivalence, by the property the partial map $\mathcal{A}_{n,k} \to \mathcal{A}_{n+1,\ell}$ have multiplicity $A_n(\ell, k)$. If $A_n u_n = u_{n+1}$, then φ_n is unital.

We shall say that the sequence (5.1) has property *(RFD)* if there exists a strictly increasing sequence $\{r_n\}_{n=1}^{\infty}$ of integers $1 \leq r_n \leq m_n$, such that $u_{n+1}(j) = u_n(j)$, whenever $0 \leq j \leq r_n$, and when A_n is written as a block matrix,

$$A_n = \begin{pmatrix} I_{r_n} & 0 \\ A_n^{(2,1)} & A_n^{(2,2)} \\ A_n^{(3,1)} & A_n^{(3,2)} \end{pmatrix}, \tag{5.3}$$

with respect to the decomposition

$$m_{n+1} = r_n + (r_{n+1} - r_n) + (m_{n+1} - r_{n+1}), \qquad m_n = r_n + (m_n - r_n),$$

then each column of the $(r_{n+1} - r_n) \times (m_n - r_n)$ matrix $A_n^{(2,2)}$ is non-zero. For each such sequence of simplicial groups there is a sequence $\{k_j\}_{j=1}^\infty$ of positive integers such that $u_n(j) = k_j$, whenever $r_n \geq j$. If $r_n = m_n$, then the second column in (5.3) disappears, and so does the condition on $A_n^{(2,2)}$. If $r_{n+1} = m_{n+1}$, then the third row in (5.3) will disappear. We show in Example 5.6 below that in general one cannot assume that $r_n = m_n$, for all n.

Lemma 5.1. *Let $\mathcal{A}$ be the inductive limit of the sequence* (5.2) *associated with a sequence of simplicial groups* (5.1) *with property (RFD). Then, for each $j \geq 1$, there a surjective *-homomorphism $\pi_j \colon \mathcal{A} \to M_{k_j}$ so that, for all $n \geq 1$ for which $r_n \geq j$, the diagram*

$$\begin{array}{ccc} \mathcal{A}_n & \xrightarrow{\pi_j^{(n)}} & \mathcal{A}_{n,j} \\ {\scriptstyle \varphi_{\infty,n}}\big\downarrow & & \big\downarrow{\scriptstyle \cong} \\ \mathcal{A} & \xrightarrow[\pi_j]{} & M_{k_j} \end{array}$$

is commutative, where $\pi_j^{(n)}$ is the projection onto the jth summand of $\mathcal{A}_n$.

Proof. Recall that $\mathcal{A}_{n,j} \cong M_{k_j}$ whenever $r_n \geq j$; and $A_n(j,j) = 1$, when $1 \leq j \leq r_n$, by (5.3). Choosing a suitable isomorphism between $\mathcal{A}_{n,j}$ and M_{k_j} we can therefore assume that $\pi_j^{(n+1)} \circ \varphi_n = \pi_j^{(n)}$, for all n for which $r_n \geq j$. We now obtain the existence of π_j from standard properties of inductive limits. □

Lemma 5.2. *Let $\mathcal{A}$ be a separable infinite dimensional RFD C^*-algebra. Then there exists a separating sequence $\{\nu_j\}_{j=1}^\infty$ of pairwise inequivalent irreducible finite dimensional representations of $\mathcal{A}$.*

Proof. Let $\{a_j\}_{j=1}^\infty$ be a dense subset of the set elements in $\mathcal{A}$ of norm one. Let $\mathcal{P}$ be the set of all irreducible finite dimensional representations of $\mathcal{A}$. By the assumption that $\mathcal{A}$ is RFD, the direct sum of all the representations from $\mathcal{P}$ is faithful and therefore isometric. Hence $\|a\| = \sup\{\|\nu(a)\| : \nu \in \mathcal{P}\}$ for all $a \in \mathcal{A}$. In particular, for each $j \geq 1$, we can find $\nu_j \in \mathcal{P}$ such that $\|\pi_j(a_j)\| \geq 1/2$. If $a \in \mathcal{A}$ is an arbitrary element of norm one, then $\|a - a_j\| < 1/2$ for some $j \geq 1$, whence $\|\nu_j(a)\| > \|\nu_j(a_j)\| - 1/2 > 0$, so $\nu_j(a) \neq 0$. This proves that $\{\nu_j\}_{j=1}^\infty$ is separating. Finally, by passing to a subset of the sequence $\{\nu_j\}_{j=1}^\infty$, we can arrange that the representations ν_j are pairwise inequivalent. □

Theorem 5.3. *Any AF-algebra associated with a sequence of simplicial groups* (5.1) *with property (RFD) is itself RFD. Conversely, any RFD infinite dimensional separable AF-algebra is realized by a sequence of simplical groups which is (RFD).*

Proof. Assume first that $\mathcal{A}$ is an AF-algebra, which is the direct limit of the sequence (5.2) arising from a RFD sequence of simplicial groups (5.1). We must show that $\mathcal{A}$ is RFD. For this it suffices to show that the sequence $\{\pi_j\}_{j=1}^\infty$ of irreducible finite dimensional representations of $\mathcal{A}$, found in Lemma 5.1 above, is separating. This will follow if we can show that the sequence $\{\pi_j \circ \varphi_{\infty,n}\}_{j=1}^\infty$ is separating for $\mathcal{A}_n$, for each $n \geq 1$. By the assumptions on the multiplicity matrix A_n, that each column in $A_n^{(2,2)}$ is non-zero and that $A_n(j,j) = 1$, for $1 \leq j \leq r_n$, we obtain that the *-homomorphism

$$\bigoplus_{j=1}^{r_{n+1}} \pi_j^{(n+1)} \circ \varphi_n \colon \mathcal{A}_n \longrightarrow \bigoplus_{j=1}^{r_{n+1}} \mathcal{A}_{n+1,j},$$

is injective. As $\pi_j \circ \varphi_{\infty,n} = \pi_j^{(n+1)} \circ \varphi_n$, it follows that $\{\pi_j \circ \varphi_{\infty,n}\}_{j=1}^{r_{n+1}}$ is separating for $\mathcal{A}_n$, as desired.

Suppose now that $\mathcal{A}$ is a separable infinite dimensional RFD AF-algebra. Choose an increasing sequence $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \cdots$ of finite dimensional sub-C^*-algebras of $\mathcal{A}$ with dense union, and choose a separating countably infinite family $\{\nu_j\}_{j=1}^\infty$ of irreducible pairwise inequivalent finite dimensional representations of $\mathcal{A}$, cf. Lemma 5.2. Let k_j denote the dimension of the representation ν_j, so that $\nu_j(\mathcal{A}) = M_{k_j}$.

We claim that there are increasing sequences $1 \leq n_1 < n_2 < n_3 < \cdots$ and $1 = r_1 < r_2 < r_3 < \cdots$ such that:

(i) $\big(\bigoplus_{j=1}^{r_k} \nu_j\big)(\mathcal{A}_{n_k}) = \big(\bigoplus_{j=1}^{r_k} \nu_j\big)(\mathcal{A})$, for each $k \geq 1$,

(ii) the restriction of $\bigoplus_{j=1}^{r_k} \nu_j$ to $\mathcal{A}_{n_{k-1}}$ is faithful, for each $k \geq 2$.

To see this note that for each $r \geq 1$ there exists $n \geq 1$ such that $\big(\bigoplus_{j=1}^{r} \nu_j\big)(\mathcal{A}_n) = \big(\bigoplus_{j=1}^{r} \nu_j\big)(\mathcal{A})$; and for each $n \geq 1$ there exists $r \geq 1$ such that the restriction of $\bigoplus_{j=1}^{r} \nu_j$ to $\mathcal{A}_n$ is faithful. We can therefore find $n_1 \geq 1$ such that (i) holds with $r_1 = 1$. Next, we find $r_2 > r_1$ such that (ii) holds for $k = 2$. Proceed like this to construct the desired sequences.

Upon passing to a subsequence of $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots$, we may assume that $n_k = k$, for all $k \geq 1$, so that $\big(\bigoplus_{j=1}^{r_n} \nu_j\big)(\mathcal{A}_n) = \big(\bigoplus_{j=1}^{r_n} \nu_j\big)(\mathcal{A})$, for all $n \geq 1$, and the restriction of $\bigoplus_{j=1}^{r_{n+1}} \nu_j$ to $\mathcal{A}_n$ is faithful, for all $n \geq 1$.

Since the ν_j's are pairwise inequivalent, we obtain that

$$\Big(\bigoplus_{j=1}^{r_n} \nu_j\Big)(\mathcal{A}_n) = \Big(\bigoplus_{j=1}^{r_n} \nu_j\Big)(\mathcal{A}) \cong \bigoplus_{j=1}^{r_n} \nu_j(\mathcal{A}) \cong \bigoplus_{j=1}^{r_n} M_{k_j},$$

for each $n \geq 1$. Being a finite dimensional C^*-algebra, $\mathcal{A}_n$ is the direct sum of the image and of the kernel of the *-homomorphism $\bigoplus_{j=1}^{r_n} \nu_j$. If the kernel is non-zero, then it is equal to $\bigoplus_{j=r_n+1}^{m_n} \mathcal{A}_{n,j}$, for some $m_n > r_n$ and some full matrix algebras $\mathcal{A}_{n,j}$. We therefore have $\mathcal{A}_n = \bigoplus_{j=1}^{m_n} \mathcal{A}_{n,j}$ (with $m_n = r_n$ if the kernel of $\bigoplus_{j=1}^{r_n} \nu_j$ is zero) where the *-homomorphism

$$\mathcal{A}_{n,j} \longrightarrow \mathcal{A}_n \xrightarrow{\nu_i} M_{k_i} \ ,$$

is an isomorphism if $1 \le i = j \le r_n$, and zero otherwise.

It remains to verify that the inclusion mapping $\varphi_n \colon \mathcal{A}_n \to \mathcal{A}_{n+1}$ has multiplicity matrix A_n which satisfies the (RFD) conditions given in (and below) (5.3). For $1 \le j \le r_n$ and $1 \le i \le r_{n+1}$, consider the commuting diagram

$$\begin{array}{ccccc} \mathcal{A}_{n,j} & \longrightarrow & \mathcal{A}_n & \xrightarrow{\nu_j} & M_{k_j} \\ \downarrow & & \downarrow{\scriptstyle \varphi_n} & & \| \\ \mathcal{A}_{n+1,i} & \longrightarrow & \mathcal{A}_{n+1} & \xrightarrow{\nu_j} & M_{k_j} \end{array}$$

The multiplicity of the dotted arrow is $A_n(i,j)$. The composed map in the upper row is an isomorphism, and the map of the lower row is zero, when $i \neq j$, and an isomorphism when $i = j$. This shows that $A_n(i,j) = \delta_{i,j}$. By (ii) we know that the map $\rho = \bigoplus_{j=1}^{r_{n+1}} \nu_j$ is injective on $\mathcal{A}_n$, so

$$\mathcal{A}_n \xrightarrow{\varphi_n} \mathcal{A}_{n+1} \xrightarrow{\rho} \bigoplus_{j=1}^{r_{n+1}} M_{k_j}$$

is injective. The multiplicity matrix of $\rho \circ \varphi_n$ is given by the submatrix

$$\begin{pmatrix} I_{r_n} & 0 \\ A_n^{(2,1)} & A_n^{(2,2)} \end{pmatrix},$$

of A_n. By injectivity of $\rho \circ \varphi_n$, this submatrix has non-zero columns, which again tells us that $A_n^{(2,2)}$ has non-zero columns.

Finally, $u(n+1,j) = k_j = u(n,j)$, for each $n \ge 1$ and each $j \le r_n$. □

A sequence of simplical groups, as in (5.1), is said to have *property (RFD-JI)* if it has property (RFD) and if each entry of each of the block matrices matrices $A_n^{(i,j)}$, $n \ge 1$, $i = 2,3$, $j = 1,2$, from (5.3), is non-zero.

Every AF-algebra arises from a sequence of simplicial groups as in (5.1), and it is well-known that this AF-algebra is simple if each entry of each of the matrices A_n, $n \ge 1$, is non-zero. We can in this way interpret the theorem below as saying that the class of RFD just-infinite AF-algebras inside the class of all RFD AF-algebras is similar to the class of simple AF-algebras inside the class of all AF-algebras.

Theorem 5.4. *Any AF-algebra $\mathcal{A}$ associated with a sequence of simplicial groups* (5.1) *with property (RFD-JI) is RFD and just-infinite. Moreover,*

$$\mathrm{Prim}(\mathcal{A}) = \big\{0, \ker(\pi_1), \ker(\pi_2), \ker(\pi_3), \dots \big\},$$

where π_j, $j \ge 1$, are the irreducible representations determined in Lemma 5.1.

Proof. Let $\mathcal{A}$ be an AF-algebra obtained from a sequence of simplicial groups as in (5.1) with property (RFD-JI). Write $\mathcal{A}$ as the inductive limit of the sequence

$$\mathcal{A}_1 \xrightarrow{\varphi_1} \mathcal{A}_2 \xrightarrow{\varphi_2} \mathcal{A}_3 \xrightarrow{\varphi_3} \cdots \longrightarrow \mathcal{A} ,$$

with $\mathcal{A}_n = \sum_{i=1}^{m_n} \mathcal{A}_{n,i}$, where $\mathcal{A}_{n,i} = M_{u_n(i)}$, and where $u_n(i) = k_i$, when $r_n \geq i$. Then $\mathcal{A}$ is RFD by Theorem 5.3. We must show that $\mathcal{A}$ also is just-infinite. For this purpose, let I be a non-zero closed two-sided ideal of $\mathcal{A}$, and set $I_j = \varphi_{\infty,j}^{-1}(I) \subseteq \mathcal{A}_j$, for all $j \geq 1$. Then $I_j = \bigoplus_{k \in T_j} \mathcal{A}_{j,k}$ for some subset T_j of $\{1, 2, \dots, m_j\}$. The quotient $\mathcal{A}/I$ is isomorphic to the inductive limit of the sequence

$$\bigoplus_{k\in F_1} \mathcal{A}_{1,k} \xrightarrow{\varphi_1'} \bigoplus_{k\in F_2} \mathcal{A}_{2,k} \xrightarrow{\varphi_2'} \bigoplus_{k\in F_3} \mathcal{A}_{3,k} \xrightarrow{\varphi_3'} \cdots , \tag{5.4}$$

where $F_j = \{1, 2, \dots, m_j\}\setminus T_j$, and where φ_j' is the restriction of φ_j to $\bigoplus_{k\in F_j} \mathcal{A}_{j,k}$.

Choose $j_0 \geq 0$ such that $I_{j_0} \neq 0$ (as we can by standard properties of inductive limit C^*-algebras). Set $F = \{1, 2, \dots, r_{j_0}\} \setminus T_{j_0}$. Then

$$T_j = \{1, 2, \dots, m_j\} \setminus F, \tag{5.5}$$

for all $j > j_0$. To see this, we use the following general facts about ideals: Let $j \geq 1$. If $k \in T_j$ and $A_j(\ell, k) \neq 0$, then $\ell \in T_{j+1}$. Moreover, if $\ell \in T_{j+1}$, for all ℓ for which $A_j(\ell, k) \neq 0$, then $k \in T_j$. Property (RFD-JI) implies that $A_j(\ell, k) \neq 0$, for all $1 \leq k \leq m_j$ and $r_j < \ell \leq m_{j+1}$, and that $A_j(\ell, k) = \delta_{k,\ell}$, when $1 \leq k \leq m_j$ and $1 \leq \ell \leq r_j$. With this information, we can conclude that $\{1, 2, \dots, m_j\} \setminus F \subseteq T_j$, for all $j > j_0$. Conversely, if $j \geq j_0$ and $k \in \{1, 2, \dots, r_j\}$, then $A_j(\ell, k) \neq 0$ if and only if $\ell \in \{k, r_j+1, r_j+2, \dots, m_{j+1}\}$, and since $\{r_j+1, r_j+2, \dots, m_{j+1}\} \subseteq T_{j+1}$, we conclude that $k \in T_j$ if and only if $k \in T_{j+1}$. This proves that (5.5) holds, for all $j > j_0$.

Hence $F_j = F$, for all $j > j_0$. By the properties of the multiplicity matrix A_j, we see that

$$\varphi_j' \colon \bigoplus_{k\in F} \mathcal{A}_{j,k} \longrightarrow \bigoplus_{k\in F} \mathcal{A}_{j+1,k}$$

is an isomorphism, whenever $j > j_0$. It therefore follows from (5.4) that $\mathcal{A}/I$ is isomorphic to the finite dimensional C^*-algebra $\bigoplus_{k\in F} \mathcal{A}_{j_0+1,k}$. This proves that $\mathcal{A}$ is just-infinite.

To prove the last claim, recall first that 0 is a prime ideal in any just-infinite C^*-algebra, cf. [2, Lemma 3.2], (and hence primitive, since $\mathcal{A}$ is separable). The representations π_j, $j \geq 1$, are irreducible, so their kernels are primitive ideals. Conversely, suppose that I is a non-zero primitive ideal of $\mathcal{A}$. Then $\mathcal{A}/I$ is isomorphic to $\bigoplus_{k\in F} \mathcal{A}_{j,k}$, for some $j \geq 1$ and some finite subset F of $\{1, 2, \dots, r_j\}$, by the argument above. As $A/\mathcal{I}$ is primitive, and hence prime, F must be a singleton, viz. $F = \{j\}$, for some $j \geq 1$. However, in that case $I = \ker(\pi_j)$, which proves that $\mathrm{Prim}(\mathcal{A})$ is as claimed. □

We end our paper by showing that not all RFD AF-algebras arise in the way described in Section 3, i.e., from a Bratteli diagram of the form given in (3.3), nor, more generally, as in (5.1), (5.2) and (5.3) with $r_n = m_n$, for all $n \geq 0$. Hence, in general, we cannot collapse the multiplicity matrix A_n in (5.3) to a matrix of the form

$$A_n = \begin{pmatrix} I_{r_n} \\ B_n \end{pmatrix}, \tag{5.6}$$

for some $(r_{n+1} - r_n) \times r_n$ matrix B_n over $\mathbb{Z}^+$.

Lemma 5.5. *Let $\mathcal{A}$ be an AF-algebra which contains a closed two-sided ideal I generated by a single projection such that $\mathcal{A}/I$ has no finite dimensional quotients. Then $\mathcal{A}$ cannot be realized as in* (5.1) *and* (5.2) *with multiplicity matrices A_n given as in* (5.6) *above.*

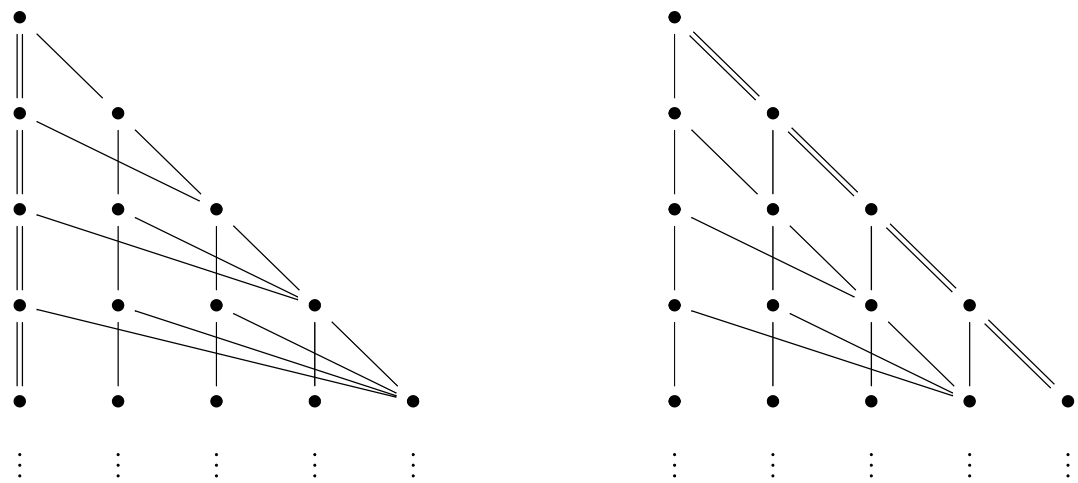
Proof. Suppose, to reach a contradiction, that $\mathcal{A}$ is realized as a limit as in (5.2) with $\mathcal{A}_n = \bigoplus_{i=1}^{r_n} \mathcal{A}_{n,i}$ and with connecting unital maps φ_n with multiplicity matrix A_n in the form (5.6). We may assume that the projection, which generates I as a closed two-sided ideal, belongs to $\mathcal{A}_{n_0}$, for some $n_0 \geq 1$. In the notation of the proof of Theorem 5.4, set $I_n = \varphi_{\infty,n}^{-1}(I) = \bigoplus_{j \in T_n} \mathcal{A}_{n,j}$, for all $n \geq 1$. Then I is the closed two-sided ideal in $\mathcal{A}$ generated by $\varphi_{\infty,n_0}(I_{n_0})$, and I_{n+1} is the closed two-sided ideal of $\mathcal{A}_{n+1}$ generated by $\varphi_n(I_n)$, for all $n \geq n_0$. In particular, $i \in T_{n+1}$ if and only if $A_n(i,j) \neq 0$ for some $j \in T_n$, when $n \geq n_0$. Set $F_n = \{1, 2, \dots, r_n\} \setminus T_n$, for all $n \geq n_0$. Then F_{n_0} is non-empty, because $I \neq \mathcal{A}$, and $F_{n_0} \subset F_n$, for all $n \geq n_0$ (by the argument above). As in the proof of Theorem 5.4, $\mathcal{A}/I$ is the inductive limit of the sequence

$$\bigoplus_{j \in F_1} \mathcal{A}_{1,j} \xrightarrow{\varphi_1'} \bigoplus_{j \in F_2} \mathcal{A}_{2,j} \xrightarrow{\varphi_2'} \bigoplus_{j \in F_3} \mathcal{A}_{3,j} \xrightarrow{\varphi_3'} \cdots, \tag{5.7}$$

where φ_n' is the restriction of φ_n to $\bigoplus_{j \in F_n} \mathcal{A}_{n,j}$. Inspecting the Bratteli diagram for the sequence (5.7), we see that there is a surjection from $\mathcal{A}/I$ onto $\bigoplus_{j \in F_{n_0}} \mathcal{A}_{n_0,j}$. However, this contradicts the assumption on $\mathcal{A}$ that it has no finite dimensional quotients. □

Example 5.6. We give here an example of a RFD AF-algebra satisfying the assumptions of Lemma 5.5, which means that it cannot be realized as in (5.1), (5.2) and (5.3) with A_n given as in (5.6), with $r_n = m_n$, (or from the construction described in Section 3).

Consider the following Bratteli diagram, written in two different ways:



and let $\mathcal{A}$ be the unital AF-algebra arising from this diagram (where we assume the connecting mappings all are unital and that the full matrix algebra corresponding to the vertex in the first row is $\mathbb{C}$). Arguing as in Lemma 5.1, from the point of view of the left-hand diagram, we have surjective *-homomorphisms

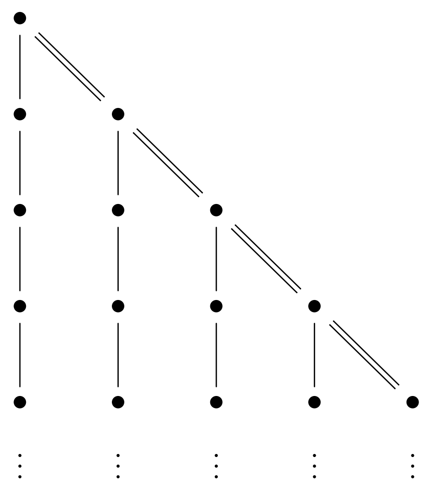
$$\pi_1 \colon \mathcal{A} \to M_{2^\infty}, \qquad \pi_j \colon \mathcal{A} \to M_{k_j}, \quad j = 2, 3, \dots,$$

for suitable integers k_j, $j \geq 2$, and where M_{2^∞} denotes the CAR-algebra (or the UHF-algebra of type 2^∞). As π_1 is not injective (because $\mathcal{A}$ cannot be simple), it follows in particular that $\mathcal{A}$ is not just-infinite. The right-hand diagram complies with the recipe given in (5.1), (5.2) and (5.3) for a RFD AF-algebra, with $r_n = n$, $m_n = n+1$, and with

$$A_n = \left(\begin{array}{c|c} I_{n-1} & 0 \\ \hline 1 \cdots 1 & 1 \\ \hline 0 \cdots 0 & 2 \end{array} \right).$$

Since $A_n^{(2,2)} = (1)$ is a non-zero 1×1 matrix, Theorem 5.3 yields that $\mathcal{A}$ is RFD. This conclusion can also be derived more directly from the easily verfied fact, that $\bigoplus_{j=2}^\infty \pi_j$ is injective. As $\ker(\pi_1)$ is generated by a single projection (eg., a non-zero projection in the summand corresponding to the vertex at position $(2,1)$ in the right-hand side Bratteli diagram), it follows from Lemma 5.5 that $\mathcal{A}$ is as desired.

Let us finally note that the AF-algebra $\mathcal{B}$ given by the Bratteli diagam



also has a closed two-sided ideal I with $\mathcal{B}/I$ being isomorphic to the CAR-algebra M_{2^∞}. It is of the form given in (5.1), (5.2) with

$$A_n = \left(\frac{I_n}{0 \cdots 0 \; 2} \right),$$

i.e, with $r_n = m_n = n+1$ and A_n as in (5.6). In particular, $\mathcal{B}$ is RFD (but not just-infinite). However, I is not generated by a single projection, so the conditions of Lemma 5.5 are not satisfied.

The C^*-algebra $\mathcal{B}$ admits the following concrete description as an extension:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{n=1}^\infty M_{2^n} & \longrightarrow & \prod_{n=1}^\infty M_{2^n} & \longrightarrow & \frac{\prod_{n=1}^\infty M_{2^n}}{\bigoplus_{n=1}^\infty M_{2^n}} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \bigoplus_{n=1}^\infty M_{2^n} & \longrightarrow & \mathcal{B} & \longrightarrow & M_{2^\infty} & \longrightarrow & 0
\end{array}$$

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