CENTRE DE RECERCA MATEMÀTICA

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Large Cardinals and Strong Logics

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Large Cardinals and Strong Logics

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Foreword

The main purpose of mathematical logic is to devise formal languages in which one can reason about properties of mathematical structures, analyze them, classify them, and also establish relationships between different classes of structures. The most successful logical language is first order logic, which has a very streamlined and coherent model theory as well as a well-developed proof theory. Unfortunately, many natural mathematical concepts cannot be expressed in first order logic but need stronger logics, by which we mean extensions of first order logic by generalized quantifiers or infinitary operations. Examples of concepts that go beyond first order logic are some fundamental mathematical notions such as freeness of a group, separability of a space, completeness of an order, etc. When we extend first order logic we run immediately into set theoretical questions. On the one hand, the set of validities, and even the truth definition for those logics, model theoretic properties like the Löwenheim-Skolem theorems, questions of compactness, interpolation theorems, etc., depend essentially on set theoretical assumptions such as strong infinitary combinatorial principles and the existence of large cardinals. On the other hand, some properties of natural strong logics are a source of set theoretical problems, well-known examples being Chang's Conjecture and Stationary Reflection. Moreover, strong logics are relevant in the foundations of set theory. For instance they can be used to find interesting canonical inner models. There are some exciting new results in this direction.

The connection between strong logics and large cardinals takes place usually through a fundamental property of the universe of sets known as reflection. Roughly speaking reflection means that every property that holds of the entire universe of sets is permitted already by a set-sized sub-universe. By qualifying what "property" means one can relate reflection closely to large cardinal properties. In model theory the analogue of reflection is the Löwenheim-Skolem Theorem which in its various variants says, roughly speaking, that if a sentence of a logic has a model then the sentence has a small (sub)model, e.g., a countable model. All this reveals a tight correspondence between expressibility in a strong logic extending first-order logic and definability in set theory. We call this tight correspondence symbiosis. The main reasons for studying symbiosis between model-theoretic and set-theoretic definability are the following. First of all, results on strong logics are dependent on set-theoretical assumptions such as $V = L$ (the Axiom of Constructibility), the Continuum Hypothesis, Jensen's Diamond principle, and the existence of large cardinals. It has become therefore instrumental to uncover exactly what is the nature of the dependence on set-theoretical hypotheses in each case. Symbiosis pinpoints the position of a given logic in the set-theoretical definability hierarchy and thereby helps us understand better the set-theoretical nature of the logic. Secondly, strong logics give rise to natural set-theoretical principles; for example, Completeness Theorems of various logics based on uncountability (e.g., the quantifier Q_1, Magidor-Malitz quantifiers, stationary logic, etc.) can be used as set-theoretical principles which unify certain constructions and give rise to absoluteness results. Moreover, Löwenheim-Skolem-type results for strong logics give new types of reflection principles in set theory, hence new large cardinal notions. In some fortunate cases the correspondence yields an exact equivalence. A good early indication of this is Magidor's characterization of supercompactness in terms of a strong Löwenheim-Skolem theorem for second order logic. More recent examples are the numerous characterizations of Vopenka's Principle in terms of category-theoretical reflection principles.

The week long tutorial on large cardinals and strong logics will form an introduction to the basic ideas of the connection, and will be run as two parallel courses: one on large cardinals and the other on stronger logics, but there will be a strong emphases on the connections.

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Tutorial on Strong Logics and Large Cardinals: The Large Cardinals Section

by

Menachem Magidor and Jouko Väänänen

1. Summary

Large cardinals notions, also known as axioms of strong infinity, play a very important role in the current research in Set Theory. Many times they provide a natural extensions of ZFC which decides many set theoretic statements, which otherwise independent of ZFC mainly in descriptive Set Theory. The consistency of many statements is equivalent to the consistency of appropriate large cardinal, and since typically the large cardinals notions form a linear hierarchy, this give a scale for gauging the consistency strength of different principles.

The present tutorial is dedicated to the interface of large cardinals and strong logics. In many cases, a property of a given strong logic is equivalent to the existence of a certain large cardinal. Typical examples are compactness properties and Löwenheim-Skolem properties ("LS properties"). In these cases the regularity property of the strong logic provides a philosophical motivation for the existence of the large cardinal. In other cases we need the large cardinal to get the consistency of a large cardinal, in order to get the consistency of a certain regularity property of a strong logic. A recent study [**4**] uses strong logics to construct various natural inner models of Set Theory.

In the the present section of the tutorial we shall survey some of large cardinals notions which are relevant to the study strong logics and we shall present some of the connections. On the other side we shall survey some combinatorial principles (like "squares - $\square_\kappa$") which can be considered to be "anti regularity" properties of strong logics.

Most of the large cardinals notions we shall discuss are introduced in [**3**], but we shall present many more recent results.

We shall only assume basic familiarity with axiomatic Set Theory, but sometimes there will be parts, whose full understanding will require some familiarity with more advanced material, like advanced forcing techniques or combinatorial Set Theory. We plan some accompanying recitation sections in which we shall try to bridge the gaps of the prerequisites required for the full understanding of the tutorial.

2. Schemes for generating strong axioms of infinity

- Reflection principles
- Compactness principles
- Elementary embeddings.

3. Examples

We shall introduce the main large cardinals notions that will play a major part the the subsequent exposition. Each of these notions can be introduced by several of the above schemes.

- Weakly compact cardinals
- Measurable cardinals
- Strong cardinals.
- Strongly compact cardinals.
- Supercompact cardinals.
- Extendible cardinals
- Vopenka's principle

4. Second order logic and large cardinals

Supercompacts are equivalent to Löwenheim-Skolem-Tarski property ("LST" property) for second order logic. Similarly extendible cardinal is equivalent to compactness properties for second order logics. We shall explore the problem of finding the minimal fragment of second order logic for which this connection holds and show that the Π^1_1 fragment of second order logic is sufficient for establishing the connection between the logic and the large cardinal.

5. Vopenka's principles

Vopenka's principle is essentially a super regularity principle for every abstract logic.

6. Anti compactness principles

There are several combinatorial principles which imply that many strong logics lack compactness or LST property.

- Square like principles
- Non reflecting stationary sets
- The existence of good scale

7. ω_1-strongly compact cardinals and small fragment of second order logic.

The ω_1-strongly compacts were introduced in [**1**] and [**2**] and are equivalent to compactness properties for several versions of strong logics.

8. For some interesting logics, relatively small cardinals can be a LST cardinal

In some cases (like the Härtig's quantifier) a relatively small cardinal (like the first inaccessible cardinal.) can be a LST cardinal for the given strong logic. But the consistency strength of this situation is very large [**5**].

9. Inner models from strong logics

This section is based on [**4**].

- Gödel's L.
- Constructibility using strong logics.
- The cof-model.
- The aa-model.

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Tutorial on Strong Logics and Large Cardinals: The Strong Logics Section

by

Menachem Magidor and Jouko Väänänen

1. Summary

This tutorial will give an introduction to strong logics, by which we mean extensions of first order logic. The most typical strong logics are infinitary logics, logics with generalized quantifiers, and higher order logics. We consider model theoretic properties of strong logics, such as compactness properties, Löwenheim-Skolem Theorem type properties, Hanf-numbers, and axiomatizability properties. It is very common that such properties of strong logics depend in an essential way on purely set theoretic concepts such as large cardinals, combinatorial principles and cardinal arithmetic. On the other hand, properties of the set-theoretical universe can be expressed in terms of model theoretic properties of strong logics. The connection between strong logics and set theory is thus a two-way one. An exact formulation of this connection is provided by the concept of symbiosis. A general concept covering all our examples of strong logics is the concept of an abstract logic. The study of strong logics is therefore sometimes called abstract model theory.

2. Course material

I am going to follow to a large extent my own book "Models and Games" [**12**]. For a free excerpt of the book (with some pages deleted for copyright reasons), see

`http://www.math.helsinki.fi/logic/people/jouko.vaananen/gam.pdf`.

A good source for almost everything in this tutorial is the classic "Model-theoretic Logics" book [**2**] which is freely available through ProjectEuclid at

`https://projecteuclid.org/euclid.pl/1235417263#toc`.

3. Basic concepts and examples

3.1. Basic examples. We give an overview of the main examples of strong logics, such as first order logic $L_{\omega\omega}$, logic with the generalized quantifier Q_0, the infinitary logic $L_{\omega_1\omega}$, logic with the generalized quantifier Q_1, and second order logic L^2.

3.2. Abstract model theory. As a general concept covering the strong logics above—as well as many others—we introduce the concept of an abstract logic. We define what it means for an abstract (i.e. strong) logic to satisfy the LS-theorem, have a Hanf-number and a pinning down number [**2**], have a compactness property, satisfy interpolation, and have a Δ-extension.

Finally we discuss the so-called Lindström's Theorem which characterizes first order logic among all abstract logics. A good reference to these matters is [**2**].

3.3. Set theoretic preliminaries. To go deeper into properties of strong logics we need to review some preliminaries on formulas of set theory, absoluteness of such formulas, and Levy Reflection. More details on these can be found in [**5, 8**].

3.4. Model theoretic preliminaries. We review some fundamental constructions in model theory such as consistency properties [**7**] and ultraproducts [**4**] with an eye on generalizing them to strong logics.

4. Strong logics

In this part of the tutorial we investigate the most important strong logics in more detail.

4.1. Generalized quantifiers. Generalized quantifiers are a corner stone of the area of strong logics. We review the basic properties of the most important generalized quantifiers:

- Q_0: LS-theorem, incompactness, Hanf-number.
- Q_1: LS-theorem, countable compactness, Hanf-number [**6**].
- Magidor-Malitz quantifier Q_1^{MM}: LS-theorem, countable compactness assuming $\diamondsuit$, consistency of the failure of countable compactness [**9**].
- Cofinality quantifier Q_ω^{cf}: LS-theorem, compactness [**10**].
- Stationary logic $L(\mathtt{aa})$: LS-theorem, axiomatization, countable compactness [**3**].
- Härtig-quantifier, Henkin-quantifier.

4.2. Infinitary logic. Infinitary logics constitute a different kind of approach to strong logics than generalized quantifiers. The main examples of infinitary logics are:

- $L_{\omega_1\omega}$, LS-theorem, incompactness, Hanf-number, Barwise-compactness [**7**].
- $L_{\infty\omega}$, absoluteness, undefinability of well-order.
- $L_{\kappa\lambda}$, undefinability of well-orderings of the type $\alpha+\alpha$.
- Game quantifiers.

4.3. Higher order logic. The oldest strong logics are the higher order logics. They manifest deep entanglement with set theory [**13**].

(1) L^2, failure of typical model-theoretic properties.
(2) Higher order logic and set theory.

5. Symbiosis

A technical concept that tries to capture the entanglement between strong logics and set theory is the concept of symbiosis [**1, 11**].

- Definition and examples of symbiosis.
- Symbiosis and LS-properties.

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