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NONLINEAR REGULARIZATION OF DISCONTINUOUS VECTOR FIELDS AND SINGULAR PERTURBATION

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ABSTRACT. We consider piecewise smooth vector fields (PSVF) defined in open sets $M \subseteq \mathbb{R}^n$ with switching manifold being a smooth surface Σ. The PSVF are given by pairs $X = (X_+, X_-)$, with $X = X_+$ in Σ_+ and $X = X_-$ in Σ_- where Σ_+ and Σ_- are the regions on M separated by Σ. A regularization of X is an 1-parameter family of smooth vector fields $X^\varepsilon, \varepsilon > 0$, satisfying that $X^\varepsilon \to X$ when $\varepsilon \to 0$. The classical regularization requires that for every $\varepsilon > 0$ the regularized field X^ε is in the convex combination of X_+ and X_-. We relaxed this condition requiring only that X^ε is in a continuous combination of X_+ and X_-. We also prove that the phase portrait of regularized X^ε is determined by a singular perturbation problem with reduced dynamics on the slow manifold equivalent to sliding dynamics on Σ. Normal forms in dimension 2 are analyzed with our techniques.

1. INTRODUCTION

A piecewise-smooth vector field (PSVF) is determined by three elements: a set $M \subset \mathbb{R}^n$, a switching set $\Sigma \subset M$ and a vector field $X: M \to \mathbb{R}^n$. The simplest case is when M is an open set and $\Sigma = h^{-1}(0)$ for some smooth function $h: M \to \mathbb{R}$. Assume that 0 is a regular value of h and Σ divides M in two regions Σ^+, Σ^-. Moreover X^+ and X^- are smooth vector fields on M and

$$X(p) = \begin{cases} X_+(p) & \text{if} \quad p \in \Sigma^+ = h^{-1}[0, +\infty), \\ X_-(p) & \text{if} \quad p \in \Sigma^- = h^{-1}(-\infty, 0]. \end{cases} \tag{1}$$

The problems related with PSVF which have been considered in the literature are: structural stability (using the product topology), normal forms, bifurcations, T–singularity, existence of limit cycles with sewing and with sliding. We are particularly interested in the regularization of PSVF.

A regularization of X is an 1-parameter family of smooth vector fields X^ε: $M \to \mathbb{R}^n, \varepsilon > 0$, satisfying that $X^\varepsilon \to X$, according to C^0–topology, when $\varepsilon \to 0$.

We denote by $[X_-, X_+]$ the *convex combination* of X_- and X_+:

$$[X_-, X_+] = \{(1/2 + \lambda/2)X_+ + (1/2 - \lambda/2)X_- : \lambda \in [-1, 1]\}.$$

A *continuous combination* of X_- and X_+ is an 1–parameter family of smooth vector fields $\widetilde{X}(\lambda, .)$, with $\lambda \in [-1, 1]$ and satisfying that

$$\widetilde{X}(-1, p) = X_-(p), \quad \widetilde{X}(1, p) = X_+(p), \forall p \in M.$$

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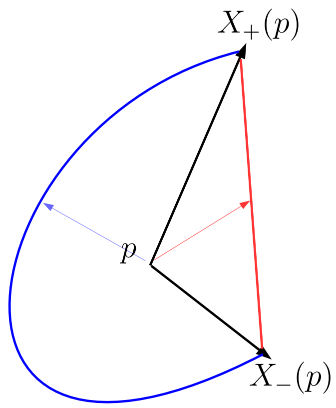

FIGURE 1. Filippov (red) and nonlinear (blue) regularizations.

We say that a regularization $X^\varepsilon\colon M \to \mathbb{R}^n$ of X is of the kind *Filippov* if $X^\varepsilon(p) \in [X_-(p), X_+(p)]$, for any $p \in M$. We say that X^ε is of the kind *nonlinear* if there exists a continuous combination $\widetilde{X}$ such that $X^\varepsilon(p) \in \{\widetilde{X}(\lambda, p), \lambda \in [-1, 1]\}$ for any $p \in M$.

The Filippov regularization was inspired in Filippov definition of flow [4]. The nonlinear regularization was introduced in [13].

The main example of regularization of kind Filippov is the Sotomayor-Teixeira regularization [11, 14]. It is based on the use of a *transition function* $\varphi\colon \mathbb{R} \to \mathbb{R}$, that is, a function satisfying

$$\lim_{x\to\pm\infty} \varphi(x) = \pm 1, \quad \text{and} \quad \varphi'(x) > 0$$

for any $x \in (-1, 1)$. The φ*–Sotomayor-Teixeira regularization* of X_- and X_+ is the one parameter family

$$X^\varepsilon = \big(1/2 + \varphi(h/\varepsilon)/2\big)X_+ + \big(1/2 - \varphi(h/\varepsilon)/2\big)X_-.$$

In the same way we define the φ*–nonlinear regularization* of X_- and X_+ as the 1–parameter family given by $X^\varepsilon(p) = \widetilde{X}(\varphi(h/\varepsilon), p)$. Note that if $h > \varepsilon$ then $\varphi(h/\varepsilon) \sim 1$ and $X^\varepsilon \sim X_+$; and if $h < -\varepsilon$ then $\varphi(h/\varepsilon) \sim 1$ and $X^\varepsilon \sim X_-$.

In [13] the authors prove that the exigence $\varphi'(x) > 0$ in the definition of transition function is not so restrictive. In fact, it is easy to see that if φ and ψ are transition functions but ψ does not satisfies the monotonicity condition then there exists a continuous combination of X_+ and X_- satisfying that the

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ψ–Sotomayor-Teixeira regularization of X_- and X_+ is equal to the φ–nonlinear regularization.

The paper is organized as follows. In Section (2) we recall the usual definitions of sewing and sliding regions and generalize them for the nonlinear case. We compare the classical and the new sliding regions. We also define the nonlinear sliding vector field. Finally we state the main theorem which establishes a connection between the nonlinear sliding vector fields and singular perturbation problems. In Section (3) we prove our main result and present an illustrative example. In Sections (4) and (6) we apply our techniques in the description of the nonlinear regularization of normal forms of PSVF in $\mathbb{R}^2$.

2. Preliminaries and Main Result

Let X be a PSVF like (1). We use the notation $X(p) = (X_+(p), X_-(p))$ or simply $X = (X_+, X_-)$ and assume that X is bi-valuated on the *switching manifold* $\Sigma = h^{-1}(0)$. We may not have unicity of trajectories by points $p \in \Sigma$.

The points $p \in \Sigma$ are classifyed as *regular* if $(X_-.h)(p) \neq 0$ and $(X_+.h)(p) \neq 0$; and as *singular* if $(X_-.h)(p) = 0$ or $(X_+.h)(p) = 0$. There are two kinds of singular points: the equilibrium points of X_- or X_+ on Σ and the points where the trajectories of X_- or X_+ are tangent to Σ.

The regular points are classified as *sewing* if $(X_-.h).(X_+.h) > 0$ or *sliding* if $(X_-.h).(X_+.h) < 0$. The sets of sewing and sliding points are denoted by Σ^{sw} and Σ^{sl} respectively.

Consider the PSVF given by (1), a continuous combination $\widetilde{X}(\lambda, p)$ of X_- and X_+ and a regular point $p \in \Sigma$.

(a) We say that p is a *nonlinear sewing* point and denote $p \in \Sigma_n^{sw}$ if $(\widetilde{X}.h)(\lambda, p) \neq 0$ for all $\lambda \in [-1, 1]$.
(b) We say that p is a *nonlinear slidding* point and denote $p \in \Sigma_n^{sl}$ if there exists $\lambda \in [-1, 1]$, such that $(\widetilde{X}.h)(\lambda, p) = 0$.

Proposition 2.1. *Consider a PSVF given by* (1). *We have* $\Sigma_n^{sw} \subseteq \Sigma^{sw}$ *and* $\Sigma^{sl} \subseteq \Sigma_n^{sl}$.

Proof. Suppose that $p \in \Sigma_n^{sw}$. Since $(\widetilde{X}.h)(p) \neq 0$ for all $\lambda \in [-1, 1]$ then both $X_+(p)$ and $X_-(p)$ point toward either Σ^+ or to Σ^-. Thus $(X_-.h).(X_+.h) > 0$ in p. It follows that $p \in \Sigma^{sw}$. Suppose now that $p \in \Sigma^{sl}$. It implies that $X_+(p)$ and $X_-(p)$ are directed to opposite sides. Thus all continuous path connecting them intersects Σ. It means that $p \in \Sigma_n^{sl}$. □

Consider the PSVF given by (1) and a continuous combination of X_- and X_+. For each $p \in \Sigma_n^{sl}$ there exists $\lambda(p) \in [-1, 1]$ such that $(\widetilde{X}.h)(p) = 0$. We say that $\widetilde{X}(\lambda(p), p)$ is a *nonlinear sliding vector field*.

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Example 1. Let $X=(X_+,X_-)$ be a PSVF defined on $\mathbb{R}^2$ with $h(x,y)=y$, $X_+=(1,1-x)$ and $X_-=(1,3-x)$. Consider the continuous combination of X_- and X_+ given by $\widetilde{X}=(-1+2\lambda^2,-\lambda+2\lambda^2-x)$. Solving $-\lambda+2\lambda^2-x=0$ in the variable λ we determine two possible sliding vector fields

$$X^{\lambda_1}=\big((-3+4x+\sqrt{1+8x})/4,0\big),\quad X^{\lambda_2}=\big((-3+4x-\sqrt{1+8x})/4,0\big).$$

Note that $x\geq -1/8$ is a necessary condition. Moreover, we also require $|\lambda_1|<1$ and $|\lambda_2|<1$. It implies that $\Sigma^{sl}=(1,3)$ and $\Sigma^{sl}_n=[-1/8,1)\cup(1,3)$. In $[-1/8,1)$ are defined two nonlinear sliding vector fields (X^{λ_1} and X^{λ_2}) and on $(1,3)$ is defined only X^{λ_2}.

Consider a PSVF as given by (1), a transition function φ and the $\varphi-$ nonlinear regularization of X_- and X_+. Our main result is the following.

Theorem 2.1. *Consider a PSVF as given by* (1) *with a continuous combination $\widetilde{X}$ of X_+ and X_- and nonlinear sliding region Σ^{sl}_n. There exists a singular perturbation problem*

$$r\dot{\theta}=\alpha(r,\theta,p)=-\sin\theta\alpha_0(r,\theta,p),\quad \dot{p}=\beta(r,\theta,p), \tag{2}$$

with $r\geq 0,\theta\in[0,\pi],p\in\Sigma^{sl}_n$ and critical manifold $\mathcal{S}$ satisfying the following.

(a) *For any normally hyperbolic $(\theta,p)\in\mathcal{S}$ there exist homeomorphic neighborhoods $p\in I_p\subset\Sigma^{sl}_n$ and $\mathcal{S}_p\subset\mathcal{S}$ and a sliding vector field $\widetilde{X}(\lambda(p),p)$ defined in I_p which is conjugated to the slow flow of* (2) *on $\mathcal{S}_p\subset\mathcal{S}$.*

(b) *For any $p\in int(\Sigma^{sl}_n)$ consider $\ell=\#\{\theta\in(0,\pi):(\theta,p)\in\mathcal{S}\}$. There exist ℓ choices of sliding vector fields defined in p.*

(c) *If all points on $\mathcal{S}$ are normally hyperbolic then there exists only one choice for the sliding vector field in Σ^{sl}_n.*

The zero set $\mathcal{S}=\{(\theta,p):\alpha_0(0,\theta,p)=0\}$ is called *slow manifold* and a point $q=(\theta,p)\in\mathcal{S}$ is *normally hyperbolic* is $\dfrac{\partial\alpha_0}{\partial\theta}(0,q)\neq 0$.

If we consider a time-rescaling, system (2) becomes

$$\theta'=\alpha(r,\theta,p),\quad p'=r\beta(r,\theta,p) \tag{3}$$

In general we refer to systems (2) and (3) as *slow* and *fast* systems respectively. If $r=0$ in (2) we have the *reduced* system and if $r=0$ in (3) we have the *layer* system. Note that $\mathcal{S}$ is a set of equilibrium points of the layer system which has a vertical flow. Slow and fast systems are equivalent for $\varepsilon>0$, that is, they have the same phase portrait. It means that the phase portrait of system (2) for $\varepsilon\sim 0$ approaches two limit situations: the phase portrait of reduced and layer systems.

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3. Proof of the Main Theorem

In this section we prove our main result and consider an illustrative example.

Proof of the main Theorem. Take a PSVF X and a continuous combination $\widetilde{X}$ as in the statement. Choose local coordinates $(x, y) = (x_1, ..., x_{n-1}, y)$ such that $h(x, y) = y$ and $\widetilde{X}(\lambda, x, y) = (s_1(\lambda, x, y), ..., s_n(\lambda, x, y))$. Thus $\Sigma = \{p = (x, 0)\}$ and

$$\Sigma_n^{sl} = \{p = (x, 0) \in \Sigma : \exists\lambda \in (-1, 1), \widetilde{X}(\lambda, x, 0)(0, ..., 0, 1) = s_n(\lambda, x, 0) = 0\}.$$

Consider a strictly increasing transition function φ, that is, it tends to 1 and -1 without actually achieve these values, for instance $\varphi(s) = \tanh(s)$. The $\varphi-$ nonlinear regularization is given by

$$\widetilde{X}(\varphi(y/\varepsilon), x, y) = (s_1(\varphi(y/\varepsilon), x, y), ..., s_n(\varphi(y/\varepsilon), x, y)).$$

Its trajectories are determined by the system

$$x_i' = s_i(\varphi(y/\varepsilon), x, y) \quad y' = s_n(\varphi(y/\varepsilon), x, y)., \quad i = 1, ..., n-1. \tag{4}$$

The nonlinear sliding vector field is $\widetilde{X}(\lambda, x, y)$ with λ satisfying that $s_n(\lambda, x, y) = 0$.

Consider the blow up $y = r\cos\theta$ and $\varepsilon = r\sin\theta$ with $r \geq 0$ and $\theta \in [0, \pi]$. Denote $\psi(\theta) = \varphi(\cot\theta)$ which is an injective decreasing function with $\psi(0) = 1, \psi(\pi) = -1$. Thus the system (4) become

$$r\dot{\theta} = -\sin\theta s_n(\psi(\theta), x, r\cos\theta), \quad \dot{x}_i = s_i(\psi(\theta), x, r\cos\theta), \quad i = 1, ..., n-1.$$

Then

$$\alpha_0(r, \theta, x) = s_n(\psi(\theta), x, r\cos\theta)$$

and

$$\beta(r, \theta, x) = (s_1(\psi(\theta), x, r\cos\theta), ..., s_{n-1}(\psi(\theta), x, r\cos\theta))$$

determine the singular perturbation desired.

(a). Note that slow manifold and nonlinear sliding region are defined by the same equation : $s_n(\psi(\theta), x, 0) = 0$ for the slow manifold and $s_n(\lambda, x, 0) = 0$ for the sliding. Thus the local difeomorphism is immediate because $\psi(\theta)$ is decreasing in $(-1, 1)$. Moreover the slow flow is determined by the reduced system $\dot{x}_i = s_i(\psi(\theta), x, 0)$ and the nonlinear sliding vector field by $\dot{x}_i = s_i(\lambda, x, 0)$. Since $\psi(\theta)$ and λ have the same expression it concludes (a).

(b). The number of possible choices of nonlinear sliding vector fields is the number of possible choises of λ such that $s_n(\lambda, x, 0) = 0$. Since the number of choices of λ is exactly the same that the number of $\theta = \theta(x, 0)$ defined implicitly by $s_n(\psi(\theta), x, 0) = 0$, the statement (b) is proved.

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(c). The normal hyperbolicity of the points on $\mathcal{S}$ implies that the graphic implicitly defined by $s_n(\psi(\theta), x, 0) = 0$ is the graphic of only one $\theta = \theta(x)$. It means that $s_n(\lambda, x, 0) = 0$ defines uniquely $\lambda = \lambda(x)$. So the statement (c) is proved. $\square$

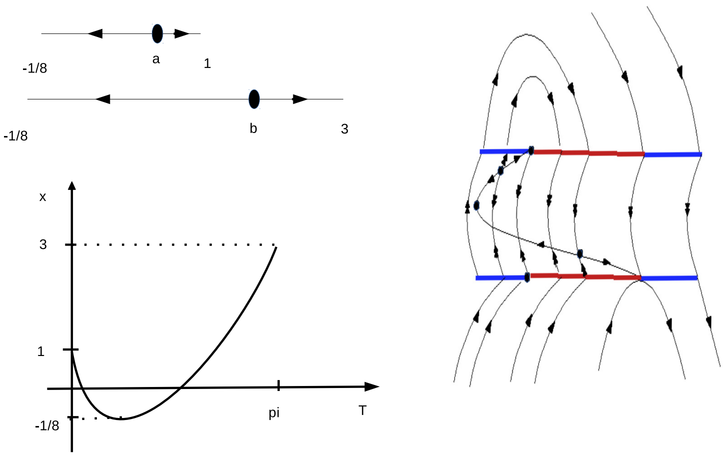

FIGURE 2. The nonlinear sliding vector field X^{λ_1} with $a = 1 - \sqrt{2}/2$ and the nonlinear sliding vector field X^{λ_2} with $b = 1 + \sqrt{2}/2$. Sewing (blue) and sliding (red) regions and the blow up of a nonlinear regularization of X_- and X_+ producing a nonlinear sliding.

We could apply the direcional blow up in the proof of the main theorem. The direcional blow up consists in the following change of coordinates:

$$\Gamma\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, \quad (x, y, \varepsilon) = \Gamma(x, \overline{y}, \overline{\varepsilon}) = (x, \overline{\varepsilon}.\overline{y}, \overline{\varepsilon}).$$

This blow up was considered by the authors in [1] and in [13]. However, geometrically speaking, it could be more convenient to consider the polar blow up coordinates

$$\Lambda\colon [0, +\infty) \times [0, \pi] \times \mathbb{R}^{n-1} \to \mathbb{R}^{n+1}, \quad (x, y, \varepsilon) = \Lambda(r, \theta, x) = (x, r\cos\theta, r\sin\theta).$$

The map Λ induces the vector field on $[0, +\infty) \times [0, \pi] \times \mathbb{R}^{n-1}$. The parameter value $\varepsilon = 0$ is now represented by $r = 0$. We observe that the directional blow up and the polar blow up are essentially the same. In fact, if we consider the map $G\colon C = [0, +\infty) \times [0, \pi] \times \mathbb{R}^{n-1} \to \mathbb{R}^{n+1}$ given by $G(r, \theta, \pi) = (\cot\theta, y, r\sin\theta)$ then $\Gamma \circ G = \Lambda$.

$$\begin{array}{ccc} C & \xrightarrow{\ \Lambda\ } & \mathbb{R}^{n+1} \\ {\scriptstyle G}\downarrow & \nearrow{\scriptstyle \Gamma} & \\ \mathbb{R}^{n+1} & & \end{array}$$

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Corollary 3.1. *The regularization of the kind nonlinear does not depend of the transition function considered.*

Example 2. Let $X = (X_+, X_-)$ as in Example 1 We consider the nonlinear regularization given by $X^\varepsilon = \widetilde{X}(\varphi(y/\varepsilon), x, y)$. The trajectories of X^ε satisfies the system

$$x' = -1 + 2\varphi(y/\varepsilon)^2, \quad y' = -\varphi(y/\varepsilon) + 2\varphi(y/\varepsilon)^2 - x.$$

Consider the blow up $y = r\cos\theta$ and $\varepsilon = r\sin\theta$ with $r \geq 0$ and $\theta \in [0, \pi]$. Thus, denoting $\psi(\theta) = \varphi(\cot\theta)$, the system becomes

$$r\dot{\theta} = -\psi(\theta) + 2\psi(\theta)^2 - x, \quad \dot{x} = -1 + 2\psi(\theta)^2.$$

The slow manifold is given by $x = -\psi(\theta)(2\psi(\theta) - 1)$. It is easy to see that x is a smooth curve connecting $(\theta, x) = (0, 1)$ and $(\theta, x) = (\pi, 3)$. Moreover the derivative of x is $x' = \psi'(4\psi - 1)$ and it is zero if $\psi = \dfrac{1}{4}$. In this case $x = -1/8$. The slow flow is given by $x' = (-3 + 4x \pm \sqrt{1 + 8x})/4$ which is exactly the same expression of the sliding X^{λ_1} and X^{λ_2}. See figure (2).

4. Nonlinear regularization of Generic PSVF's on $\mathbb{R}^2$

In this section normal forms of generic PSVF (sewing, sliding, saddle and fold regular) on $\mathbb{R}^2$ are discussed. We use singular perturbation techniques to analyze the dynamics of their φ–nonlinear regularizations. From now on we are considering the continuous combination

$$\widetilde{X} = \frac{1+\lambda}{2} X_+(x, y) + \frac{1-\lambda}{2} X_-(x, y) + (P(\lambda), Q(\lambda)) \tag{5}$$

with P, Q real polynomials of degree m and n respectively, satisfying the condition $P(\pm 1) = 0$ and $Q(\pm 1) = 0$.

We recall that φ denotes a strictly increasing transition function and $\psi(\theta) = \varphi(\cot\theta)$.

4.1. **Sewing.** We say that $X_N = (X_+, X_-)$ is the normal form of the sewing PSVF defined on $\mathbb{R}^2$ if $X_+(x, y) = (a, b)$, $X_-(x, y) = (c, d)$, a, b, c, $d \in \mathbb{R} \setminus \{0\}$, $bd > 0$, $h(x, y) = y$ and $\Sigma = h^{-1}(0)$. In this case we have $\Sigma = \Sigma^{sw}$.

Proposition 4.1. *Let $X_N = (X_+, X_-)$ be the normal form of the sewing PSVF defined on $\mathbb{R}^2$ and consider the continuous combination*

$$\widetilde{X} = \big(a/2 + c/2 + (a/2 - c/2)\lambda + P(\lambda), b/2 + d/2 + (b/2 - d/2)\lambda + Q(\lambda)\big).$$

(a) *There exists a real continuous function $\Delta(\lambda)$ such that: If Δ has $k \leq n$ real roots $\lambda_i \in [-1, 1]$, $i = 1, \ldots, k$ then $\Sigma = \Sigma^{sl}_n$; if $\Delta(\lambda) \neq 0$, for all $\lambda \in [-1, 1]$ then $\Sigma^{sw}_n = \Sigma$.*

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(b) *For the case that* $\Sigma_n^{sl} = \Sigma$ *the singular perturbation problem* (2) *in the Theorem 2.1 is such that the slow manifold is the union of* $k \leq n$ *lines* $\theta = \theta_i$, $i = 1, \ldots, k$. *Moreover the slow flow on* $\theta = \theta_i$ *is determined by* $\dot{x} = \mathrm{sgn}((a+c)/2 + \psi(\theta_i)(a-c)/2 + P(\psi(\theta_i)))$.

Proof. Let $\widetilde{X}$ be the continuous combination of X given by the statement of the theorem. Let Δ be the real continuous function given by

$$\Delta = \Delta(\lambda) := (b/2 + d/2) + (b/2 - d/2)\lambda + Q(\lambda). \tag{6}$$

According with the definition $p \in \Sigma_n^{sl}$ if there exists $\lambda \in [-1, 1]$, satisfying $(\widetilde{X}.h)(p) = \Delta = 0$. Thus if exists $k \leq n$ real roots of Δ, $\lambda_i \in [-1, 1]$, $i = 1, \ldots, k$, the nonlinear sliding region $\Sigma_n^{sl} \neq \emptyset$, in fact, $\Sigma_n^{sl} = \Sigma$. Otherwise, $\Sigma_n^{sw} = \Sigma$ and the statement (a) is proved.

By the Theorem 2.1, there exists a singular perturbation problem, $r \geq 0$, $x \in \Sigma_n^{sl}$, given by

$$\begin{aligned} r\dot{\theta} &= -\sin\theta\left((b/2 + d/2) + (b/2 - d/2)\psi(\theta) + Q(\psi(\theta))\right), \\ \dot{x} &= (a+c)/2 + \psi(\theta)(a-c)/2 + P(\psi(\theta)). \end{aligned}$$

The slow manifold for $\theta \in (0, \pi)$ is determined by the equation $(b/2 + d/2) + (b/2 - d/2)\psi(\theta) + Q(\psi(\theta)) = 0$, then it is the union of $k \leq n$ lines $\theta = \theta_i$, $i = 1, \ldots, k$. The slow flow is given by the solutions of the reduced problem represented by

$$0 = -\sin\theta\left((b+d) + (b-d)\psi(\theta) + Q(\psi(\theta))\right) \tag{7}$$

$$\dot{x} = (a+c)/2 + \psi(\theta)(a-c)/2 + P(\psi(\theta)). \tag{8}$$

So, for $r = 0$ the dynamics on each connected component of the slow manifold is given by $\mathrm{sgn}(\dot{x})$ and (b) is proved. ☐

Example 3. Consider the PSVF $X_N = (X_+, X_-)$ with $X_+ = (1, -1)$, $X_- = (2, -1)$ and the continuous combination

$$\widetilde{X} = (1/2 - \lambda/2 + \lambda^2, 1 - \lambda - 2\lambda^2 + \lambda^3).$$

The corresponding slow–fast system is

$$r\dot{\theta} = -\sin\theta(1 - \lambda - 2\lambda^2 + \lambda^3), \quad \dot{x} = 1/2 - \lambda/2 + \lambda^2,$$

with $\lambda = \varphi(\cot\theta)$. Since $1 - \lambda - 2\lambda^2 + \lambda^3 = 0$ has two solutions $\lambda_1 \in (-1, 0)$ and $\lambda_2 \in (0, 1)$ the slow manifold has two components $\theta = \theta_1 \in (0, \pi/2)$ and $\theta = \theta_2 \in (\pi/2, \pi)$. Finally $1/2 - \lambda/2 + \lambda^2 > 0$ for $\lambda \in [-1, 1]$ implies that the slow flow is equivalent to $\dot{x} = 1$. See figure (3).

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4.2. **Saddle.** We say that $X_N = (X_+, X_-)$ is the normal form of the saddle PSVF defined on $\mathbb{R}^2$ if $X_+(x,y) = (x, -1-y)$, $X_-(x,y) = (x, 1-y)$, $h(x,y) = y$ and $\Sigma = h^{-1}(0)$. In this case we have $\Sigma = \Sigma^s$.

Proposition 4.2. *Let $X_N = (X_+, X_-)$ be the normal form of the saddle PSVF and the continuous combination*

$$\widetilde{X} = \big(P(\lambda) + x, -\lambda + Q(\lambda) - y\big).$$

(a) $\Sigma = \Sigma_n^{sl}$ *and there exists a real continuous function* $\Delta = \Delta(\lambda)$ *such that if* Δ *has* $k \leq n$ *real roots* $\lambda_i \in [-1,1]$, $i = 1, \dots, k$ *then the singular perturbation problem* (2) *in the Theorem 2.1 is such that the slow manifold is the union of* k *lines* $\theta = \theta_i$, $i = 1, \dots, k$.

(b) *The slow flow on* $\theta = \theta_i$ *follows the positive direction of* x*-axis if* $x > -P(\psi(\theta_i))$ *and follows the negative direction of* x*-axis if* $x < -P(\psi(\theta_i))$.

Proof. Let $\widetilde{X}$ be the continuous combination of X_N given by the statement of the proposition. Then $\widetilde{X} = \big(P(\lambda) + x, -\lambda + Q(\lambda) - y\big)$. Using our definition, the nonlinear sliding region Σ_n^{sl} in Σ is given by the solutions of $(\widetilde{X}.h)(p) = 0$, $\lambda \in [-1,1]$. So, let us consider the continuous function $\Delta = \Delta(\lambda) = -\lambda + Q(\lambda)$. Since $\Delta(1) = -1$ and $\Delta(-1) = 1$, the intermediate value theorem guarantees that $\Sigma = \Sigma_n^{sl}$. Now, we consider the polar blow up coordinates given by $y = r\cos\theta$ and $\varepsilon = r\sin\theta$, with $r \geq 0$ and $\theta \in [0,\pi]$. Using these coordinates the parameter value $\varepsilon = 0$ is represented by $r = 0$ and the blow up induces the vector field given by

$$r\dot{\theta} = -\sin\theta\left(-\psi(\theta) + Q(\psi(\theta)) - r\cos\theta\right), \quad \dot{x} = P(\psi(\theta)) + x.$$

The slow manifold is $\mathcal{S} = \{(\theta_i, x) \in (0,\pi) \times \mathbb{R} : -\psi(\theta_i) + Q(\psi(\theta_i)) = 0\}$. The slow flow is given by the solutions of the reduced problem represented by

$$0 = -\psi(\theta) + Q(\psi(\theta)), \quad \dot{x} = P(\psi(\theta)) + x. \tag{9}$$

Then the reduced flow on $\theta = \theta_i$ follows the positive direction of x-axis if $x > -P(\psi(\theta_i))$ and follows the negative direction of x-axis if $x < -P(\psi(\theta_i))$. □

Corollary 4.1. *In the conditions of the Proposition 4.2, let* q *be a normally hyperbolic singular point of the reduced problem* (9) *with* $r = 0$. *Then there exists* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$, X_ε *has a saddle point or a repelling node point near* q.

Proof. We observe that any point q on the slow manifold $\mathcal{S}$ is normally hyperbolic if

$$(\partial\alpha/\partial\theta)(q,0) = \frac{\partial}{\partial\theta}\big(-\sin\theta(-\psi(\theta) + Q(\psi(\theta)))\big) \neq 0.$$

Let us assume that, for every normally hyperbolic $q \in \mathcal{S}$, $(\partial\alpha/\partial\theta)(q,0)$ has k^s eigenvalues with negative real part and k^u eigenvalues with positive real part for the fast system. Lemma 14 in [1] implies that X_ε has a singular point q_ε with approaches q when ε is near to zero and q_ε has a $(j^s + k^s)$-dimensional local stable

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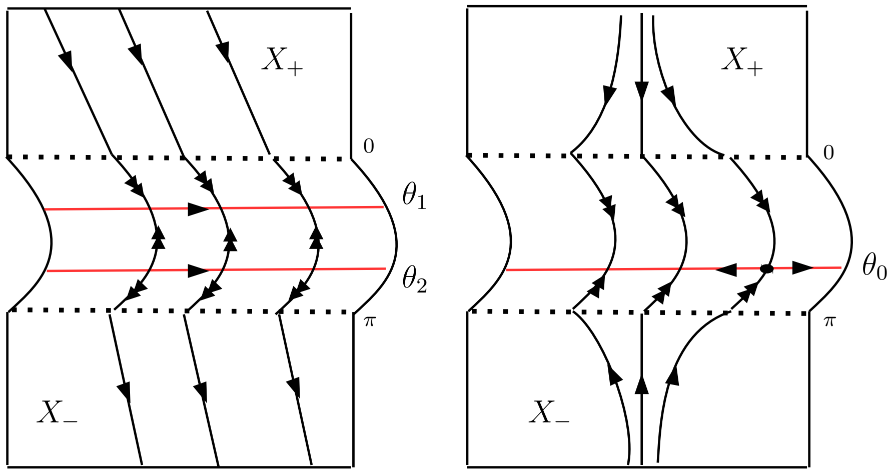

FIGURE 3. Slow-fast system obtained blowing up the nonlinear regularization of the normal forms of sewing and saddle PSVF

manifold W^s_ε and (j^u+k^u)-dimensional local unstable manifold W^u_ε, where j^s and j^u are the dimensions of the local stable and unstable manifold, respectively. In this case, $j^u = 1$ and $j^s = 0$. If $k^u = 0$ and $k^s = 1$, then q_ε is a saddle or if $k^u = 1$ and $k^s = 0$, then q_ε is a repelling node. □

Example 4. Consider the PSVF $X_N = (X_+, X_-)$ with $X_+ = (x, -1-y)$, $X_- = (x, 1-y)$ and the continuous combination

$$\widetilde{X} = \big(-1+\lambda^2+x, -1-\lambda+\lambda^2-y\big).$$

The corresponding slow–fast system is

$$r\dot{\theta} = -\sin\theta\big(-1-\lambda+\lambda^2-r\cos\theta\big), \quad \dot{x} = -1+\lambda^2+x,$$

with $\lambda = \varphi(\cot\theta)$. Since $\lambda^2-\lambda-1=0$ has two solutions $\lambda_0 \in (-1,0)$ and $\lambda_1 > 1$ the slow manifold has only one component $\theta = \theta_0 \in (\pi/2, \pi)$. The slow flow $\dot{x} = x + {\lambda_0}^2 - 1$ has one repelling equilibrium point $x_0 \approx 0.6...$ See figure (3).

4.3. **Fold.** We say that $X_N = (X_+, X_-)$ is the normal form of the fold PSVF defined on $\mathbb{R}^2$ if $X_+(x,y) = (1,x)$, $X_-(x,y) = (1,1)$, $h(x,y) = y$ and $\Sigma = h^{-1}(0)$. In this case $\Sigma^s = \{(x,0) \in \Sigma : x < 0\}$.

Proposition 4.3. *Let $X_N = (X_1, X_2)$ be the normal form of the fold PSVF defined on $\mathbb{R}^2$ and the continuous combination*

$$\widetilde{X} = \big(1+P(\lambda), 1/2-\lambda/2+Q(\lambda)+(1/2+\lambda/2)x\big).$$

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(a) *The singular perturbation problem* (2) *in the Theorem 2.1 for this case is such that the slow manifold is a graphic* $x = x(\theta)$ *which is zero for* $\theta = 0$ *and goes to* $-\infty$ *when* θ *goes to* π.

(b) *The slow flow is determined by* $\dot{x} = 1 + P(\psi(\theta))$.

Proof. Let $\widetilde{X}$ be the continuous combination of X given by the statement of the proposition. Then $p = (x, 0) \in \Sigma_n^{sl}$ if there exists $\lambda \in [-1, 1]$ such that $(\widetilde{X}.h)(p) = 0$, so solving this we obtain

$$1/2 - \lambda/2 + Q(\lambda) + (1/2 + \lambda/2)x = 0. \tag{10}$$

The singular perturbation problem (2) in the Theorem 2.1 for this case is

$$\begin{aligned} r\dot{\theta} &= -\sin\theta\,(1/2 - \psi(\theta)/2 + Q(\psi(\theta)) + (1/2 + \psi(\theta)/2)x)\,, \\ \dot{x} &= 1 + P(\psi(\theta)). \end{aligned}$$

The slow manifold is the graphic of a function which is 0 for $\theta = 0$ and goes to $-\infty$ when θ goes to π. The reduced flow is determined by $\dot{x} > 0$. The fast flow satisfies that that $\dot{\theta} > 0$, for $0 < \theta < M(x)$ and $\dot{\theta} < 0$, for $M(x) < \theta < \pi$, with $M(x)$ given implicitly by $1/2 - \psi(\theta)/2 + Q(\psi(\theta)) + (1/2 + \psi(\theta)/2)x = 0$. □

Example 5. Consider the PSVF $X_N = (X_+, X_-)$ with $X_+ = (1, x)$, $X_- = (1, 1)$ and the continuous combination

$$\widetilde{X} = (\lambda^2, -1/2 - \lambda/2 + \lambda^2 + (\lambda/2 + 1/2)x).$$

The corresponding slow–fast system is

$$r\dot{\theta} = -\sin\theta\big(-1/2 - \lambda/2 + \lambda^2 + (\lambda/2 + 1/2)x\big), \quad \dot{x} = \lambda^2,$$

with $\lambda = \varphi(\cot\theta)$. The equation $-1/2 - \lambda/2 + \lambda^2 + (\lambda/2 + 1/2)x = 0$ defines the slow manifold implicitly $x = x(\theta)$ as a smooth function satisfying that $x(0) = 0$ and $\lim_{x\to\pi} x(\theta) = -\infty$. The slow flow is locally equivalent to $\dot{x} = 1$. Moreover there exists one non normally hyperbolic point on the slow manifold. See figure (4).

5. Nonlinear regularization of Codimension One PSVF on $\mathbb{R}^2$

In what follows we discuss about the normal forms X_N of codimension one PSVF defined on $\mathbb{R}^2$. Saddle-node, elliptical fold, hyperbolic fold and parabolic fold are considered. For each one of these normal forms we consider a continuous combination $\widetilde{X}$ given by (5) and analyze the dynamics of their nonlinear regularizations using singular perturbation techniques.

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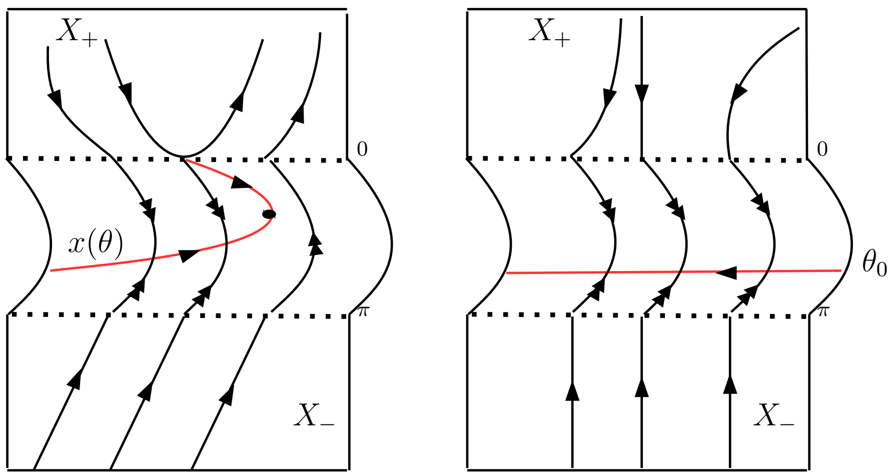

FIGURE 4. Slow-fast system obtained blowing up the nonlinear regularization of the normal forms of fold and saddle-node PSVF.

5.1. **Saddle-node.** We say that $X_N = (X_1, X_2)$ is the normal form of the saddle-node PSVF defined on $\mathbb{R}^2$ if $X_+(x,y) = (-x^2, -1)$, $X_-(x,y) = (0,1)$, $h(x,y) = y$ and $\Sigma = h^{-1}(0)$. In this case $\Sigma^s = \Sigma$.

Proposition 5.1. *Let $X_N = (X_+, X_-)$ be the normal form of the saddle-node PSVF and the continuous combination*

$$\widetilde{X} = \big(P(\lambda) + (-1/2 - \lambda/2)x^2, -\lambda + Q(\lambda)\big).$$

(a) *There exists a real continuous function $\Delta = \Delta(\lambda)$ such that if Δ has $k \leq m$ real roots $\lambda_i \in [-1,1]$, $i = 1, \dots, k$ then $\Sigma = \Sigma_n^{sl}$.*
(b) *The singular perturbation problem* (2) *in the Theorem 2.1 is such that the slow manifold is the union of k lines $\theta = \theta_i$, $i = 1, \dots, k \leq m$. Moreover the slow flow on each $\theta = \theta_i$ is determined by $\dot{x} = \operatorname{sgn}\left(P(\psi(\theta)) - (x^2/2)(1 + \psi(\theta))\right)$.*
(c) *For each $i = 1, \dots, k \leq m$, the number of singular points of the slow flow on $\theta = \theta_i$ is at most two whenever $P(\psi(\theta_i)) \geq 0$.*

Proof. The items (a) and (b) follow using the same ideas as in the proofs of the previous theorems. For the item (c), we consider the singular perturbation problem (2)

$$r\dot{\theta} = -\sin\theta\left(-\psi(\theta) + Q(\psi(\theta))\right), \quad \dot{x} = P(\psi(\theta)) + (-1/2 - \psi(\theta)/2)x^2. \tag{11}$$

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The there are two singular points at $(\theta, x) = \big(\theta_i, \pm\sqrt{2P(\psi(\theta_i))/(1+\psi(\theta_i))}\,\big)$, if $P(\psi(\theta_i)) \geq 0$, for each $i = 1, \dots, k$. $\square$

Example 6. Consider the PSVF $X_N = (X_+, X_-)$ with $X_+ = (-x^2, -1)$, $X_- = (0, 1)$ and the continuous combination

$$\widetilde{X} = \big(-1 + \lambda^2 + (-1/2 - \lambda/2)x^2, -1 - \lambda + \lambda^2\big).$$

The corresponding slow–fast system is

$$r\dot{\theta} = -\sin\theta\big(-1 - \lambda + \lambda^2\big), \quad \dot{x} = -1 + \lambda^2 + \big(-1/2 - \lambda/2\big)x^2,$$

with $\lambda = \varphi(\cot\theta)$. The equation $-1 - \lambda + \lambda^2 = 0$ has only one solution $\lambda_0 \in (-1, 0)$ and consequently the slow manifold is $\theta = \theta_0$ with $\theta_0 \in (\pi/2, \pi)$. Since $-1 + {\lambda_0}^2 + (-1/2 - \lambda_0/2)x^2 < 0$ the slow flow is locally equivalent to $\dot{x} = -1$. See figure (4).

5.2. **Elliptical fold.** We say that $X_N = (X_+, X_-)$ is the normal form of the elliptical fold PSVF defined on $\mathbb{R}^2$ if $X_+(x, y) = (1, -x)$, $X_-(x, y) = (-1, -x)$, $h(x, y) = y$ and $\Sigma = h^{-1}(0)$.

Proposition 5.2. *Let $X_N = (X_+ X_-)$ be the normal form of the elliptical fold PSVF and the continuous combination*

$$\widetilde{X} = \big(\lambda + P(\lambda), Q(\lambda) - x\big).$$

(a) *The singular perturbation problem* (2) *in the Theorem 2.1, for this case, is such that the slow manifold is a graphic $x = Q(\psi(\theta)$ which joins $(0, 0)$ with $(\pi, 0)$. Moreover, it is possible that the curve $x = Q(\psi(\theta)$ has points $(\theta_0, 0)$ such that $Q(\psi(\theta_0) = 0$, $\theta_0 \in (0, \pi)$.*

(b) *The reduced flow is determined by $\dot{x} = \mathrm{sgn}(\psi(\theta_0) + P(\psi(\theta_0)))$, $\theta_0 \in (0, \pi)$.*

Example 7. Consider the PSVF $X_N = (X_+, X_-)$ with $X_+ = (1, -x)$, $X_- = (-1, -x)$ and the continuous combination

$$\widetilde{X} = \big(-1 + \lambda + \lambda^2, -1 - x + \lambda^2\big).$$

The corresponding slow–fast system is

$$r\dot{\theta} = -\sin\theta\big(-1 - x + \lambda^2\big), \quad \dot{x} = -1 + \lambda + \lambda^2,$$

with $\lambda = \varphi(\cot\theta)$. The equation $-1 - x + \lambda^2 = 0$ defines implicitly the slow manifold as a smooth curve connecting $\theta = 0$ and $\theta = \pi$. The slow flow is determined by $\dot{x} = -1 + \lambda + \lambda^2$. Thus $\dot{x} > 0$ if $\lambda \in (\lambda_0, 1)$ and $\dot{x} < 0$ if $\lambda \in (-1, \lambda_0)$, where $\lambda_0 = \varphi(\cot\theta_0), \theta_0 \in (0, \pi/2)$. See figure (5).

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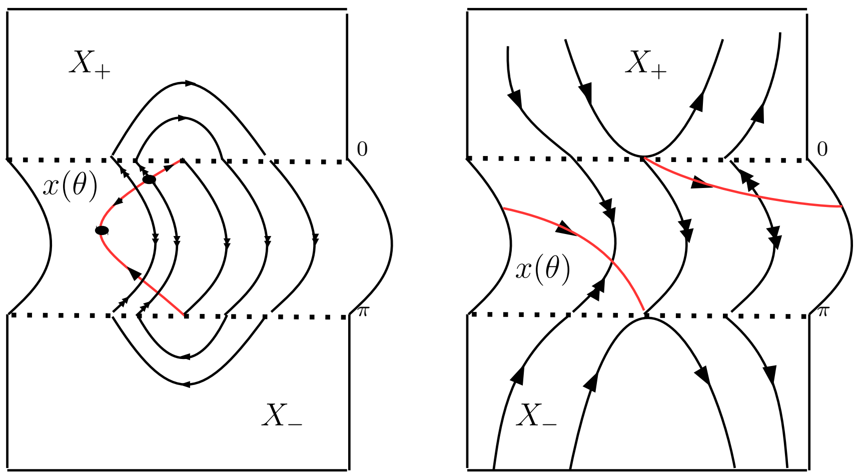

FIGURE 5. Slow-fast system obtained blowing up the nonlinear regularization of the normal forms of elliptical fold and hyperbolic fold PSVF.

5.3. **Hyperbolic fold.** We say that $X_N = (X_+, X_-)$ is the normal form of the hyperbolic fold PSVF defined on $\mathbb{R}^2$ if $X_+(x,y) = (1,x)$, $X_-(x,y) = (1,-2x)$, $h(x,y) = y$ and $\Sigma = h^{-1}(0)$.

Proposition 5.3. *Let X_N^- be the normal form of the hyperbolic fold and the continuous combination*

$$\widetilde{X} = (1 + P(\lambda), Q(\lambda) + (-1/2 + 3\lambda/2)x).$$

The singular perturbation problem (2) *in the Theorem 2.1 is such that the reduced flow has only two singular points,* $(0,0)$ *and* $(\pi, 0)$*, and the slow manifold is the curve* $\frac{x}{2}(1 - 3\psi(\theta)) = Q(\psi(\theta))$ *that joins the two folds and tends to* $\pm\infty$ *if* θ *tens to* θ_0*, where* $\psi(\theta_0) = 1/3$*. Moreover, the reduced flow is singular.*

Example 8. Consider the PSVF $X_N = (X_+, X_-)$ with $X_+ = (1,x)$, $X_- = (1,-2x)$ and the continuous combination

$$\widetilde{X} = \big(\lambda^2, -1 + \lambda^2 + (-1/2 + 3\lambda/2)x\big).$$

The corresponding slow–fast system is

$$r\dot{\theta} = -\sin\theta\big(-1 + \lambda^2 + (-1/2 + 3\lambda/2)x\big), \quad \dot{x} = \lambda^2,$$

with $\lambda = \varphi(\cot\theta)$. The equation $-1 + \lambda^2 + (-1/2 + 3\lambda/2)x = 0$ defines implicitly the slow manifold as a pair of smooth curves with an asymptote on θ_0 with

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$\varphi(\cot\theta_0) = 1/3$. The slow flow is determined by $\dot{x} = \lambda^2$, thus it is locally equivalent to $\dot{x} = 1$. See figure (5).

5.4. **Parabolic fold.** We say that $X_N = (X_+, X_-)$ is the normal form of the parabolic fold PSVF defined on $\mathbb{R}^2$ if $X_+(x, y) = (-1, -x)$, $X_-(x, y) = (1, 2x)$, $h(x, y) = y$ and $\Sigma = h^{-1}(0)$.

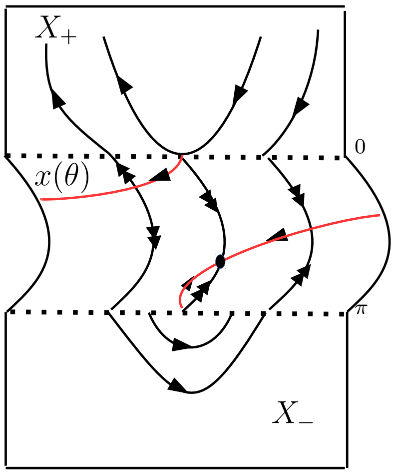

FIGURE 6. Slow-fast system obtained blowing up the nonlinear regularization of the normal form of parabolic fold PSVF.

Proposition 5.4. *Let $X_N = (X_+, X_-)$ be the normal form of the parabolic fold PSVF and the continuous combination*

$$\widetilde{X} = \big(-\lambda + P(\lambda), Q(\lambda) + (1/2 - 3\lambda/2)x\big)$$

The singular perturbation problem (2) *is*

$$r\dot{\theta} = -\sin\theta\big(Q(\psi(\theta)) + (1/2 - 3\psi(\theta)/2)x\big), \quad \dot{x} = -\psi(\theta) + P\big(\psi(\theta)\big).$$

The slow manifold a graphic $x = x(\theta)$ given implicitly by $x(3\psi(\theta)-1) = 2Q(\psi(\theta))$, satisfying that $x(0) = x(\pi) = 0$ and tends to $\pm\infty$ if θ tends to θ_0 with $\psi(\theta_0) = 1/3$. The reduced flow is given by $\dot{x} = \mathrm{sgn}(-\psi(\theta_0) + P(\psi(\theta_0)))$.

Example 9. Consider the PSVF $X_N = (X_+, X_-)$ with $X_+ = (-1, -x)$, $X_- = (1, 2x)$ and the continuous combination

$$\widetilde{X} = \big(-1 - \lambda + \lambda^2, -1 + \lambda^2 + (1/2 - 3\lambda/2)x\big).$$

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The corresponding slow–fast system is

$$r\dot{\theta} = -\sin\theta\big(-1+\lambda^2+(1/2-3\lambda/2)x\big), \quad \dot{x} = -1-\lambda+\lambda^2,$$

with $\lambda = \varphi(\cot\theta)$. The equation $-1+\lambda^2+(1/2-3\lambda/2)x = 0$ defines implicitly the slow manifold as a smooth curve with an asymptote on θ_0 with $\varphi(\cot\theta_0) = 1/3$. The slow flow is determined by $\dot{x} = -1-\lambda+\lambda^2$. Thus $\dot{x} < 0$ if $\lambda \in (\lambda_0, 1)$ and $\dot{x} > 0$ if $\lambda \in (-1, \lambda_0)$, where $\lambda_0 = \varphi(\cot\theta_0), \theta_0 \in (\pi/2, \pi)$. See figure (6).

6. Some Examples of PSVF's on $\mathbb{R}^3$

In this section we present some examples of nonlinear regularization of PSVF's on $\mathbb{R}^3$. In the previous section, the continuous combinations of X_+ and X_- had nonlinear terms depending only on λ. Indeed, in all the examples we considered $\widetilde{X}$ of the kind (5).

Now, in order to obtain some generic situations, the examples of continuous combinations of discontinuous systems in $\mathbb{R}^3$ will have the nonlinear terms also depending on $p \in \Sigma$, with $\Sigma = \{(x, y, 0) \in \mathbb{R}^3\}$.

Example 10. Consider the PSVF $X = (X_+, X_-)$ with $X_+ = (0, 0, -1)$, $X_- = (0, 1, 1)$ and the continuous combination

$$\widetilde{X} = \big(P(\lambda, x, y, z), 1/2 - \lambda/2 + Q(\lambda, x, y, z), -\lambda + R(\lambda, x, y, z)\big),$$

where $P(\lambda, x, y, z) = x(\lambda^2 - 1)$, $Q(\lambda, x, y, z) = -y/2(\lambda^2 - 1)$ and $R(\lambda, x, y, z) = -\lambda(\lambda^2 - 1)$ are continuous polynomials satisfying the condition $P(\pm1, x, y, z) = Q(\pm1, x, y, z) = R(\pm1, x, y, z) = 0$. Thus we have

$$\widetilde{X} = \big(x(\lambda^2-1), 1/2 - \lambda/2 - y/2(\lambda^2-1), -\lambda - \lambda(\lambda^2-1)\big).$$

The nonlinear sliding region, Σ_n^{sl}, is Σ and for any $p \in \Sigma_n^{sl}$ the nonlinear sliding vector field is given by

$$\dot{x} = -x, \; \dot{y} = 1/2 + y/2.$$

Now, let us consider the nonlinear regularization given by $X^\varepsilon = \widetilde{X}(\varphi(z/\varepsilon), x, y, z)$ and apply Theorem (2.1). We get the singular perturbation problem

$$\begin{aligned} r\dot{\theta} &= -\sin\theta(-\psi(\theta)^3) \\ \dot{x} &= x(\psi(\theta)^2 - 1) \\ \dot{y} &= (1+y)/2 - \psi(\theta)/2 - y\psi(\theta)^2/2. \end{aligned}$$

The slow manifold is $\theta = \pi/2$ and the corresponding slow system is determined by $\dot{x} = -x, \quad \dot{y} = 1/2 + y/2$, and it has a saddle point in $(0, -1)$. See figure (7).

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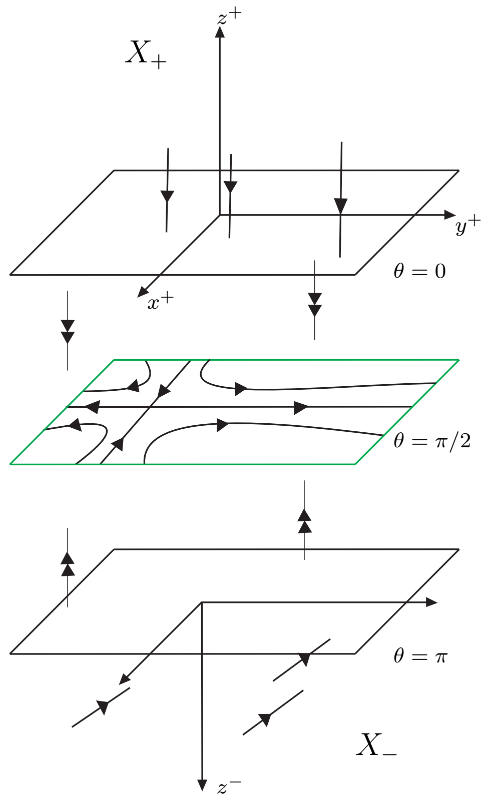

FIGURE 7. Slow-fast system obtained blowing up the nonlinear regularization of a sewing PSVF on $\mathbb{R}^3$.

Example 11. Consider the PSVF $X = (X_+, X_-)$ with $X_+ = (1, 0, x)$, $X_- = (0, 0, 1)$. The regular points in Σ are $\{(x, y, 0) \in \mathbb{R}^3 : x \neq 0\}$ (sewing if $x > 0$ and sliding if $x < 0$). The singular points in Σ are $\{(0, y, 0) \in \mathbb{R}^3\}$.

Let us consider a continuous combination of X_+ and X_- given by

$$\widetilde{X} = \left(1/2 + \lambda/2 + P, Q, \frac{x+1}{2} + \frac{x-1}{2}\lambda + R\right), x \neq 0,$$

where

$$P = P(\lambda, x, y, z) = (\lambda^2 - 1)(-2 + 2x + x^2 - x^3)/4x,$$
$$Q = Q(\lambda, x, y, z) = (\lambda^2 - 1)(y - 2xy + x^2 y)/4x,$$
$$R = R(\lambda, x, y, z) = 0.$$

Solving

$$\frac{x+1}{2} + \frac{x-1}{2}\lambda = 0$$

in the variable λ we have that the nonlinear sliding region is

$$\Sigma_n^{sl} = \left\{(x, y, 0) \in \mathbb{R}^3 : x < 0\right\}.$$

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The sliding vector field is given by $X^\lambda = (-x-1, y, 0)$.

Let $X^\varepsilon = \widetilde{X}(\varphi(z/\varepsilon), x, y, z)$ be the φ-nonlinear regularization. For any $p \in \Sigma_n^{sl}$, the associated singular perturbation problem is

$$\begin{aligned} r\dot{\theta} &= -\sin\theta\left(\frac{x+1}{2} + \frac{x-1}{2}\psi(\theta)\right), \\ \dot{x} &= 1/2 + \psi(\theta)/2 + \frac{-2+2x+x^2-x^3}{4x}(\psi(\theta)^2-1), \\ \dot{y} &= \frac{y-2xy+x^2y}{4x}(\psi(\theta)^2-1), \end{aligned}$$

with $\psi(\theta) = \varphi(\cot\theta)$. The slow manifold is the curve implicitly defined by

$$\psi(\theta) = -\frac{x+1}{x-1},$$

and the slow flow is determined by $\dot{x} = -x-1, \quad \dot{y} = y$. In this case, the slow flow has a saddle in $(-1, 0)$. See figure (8).

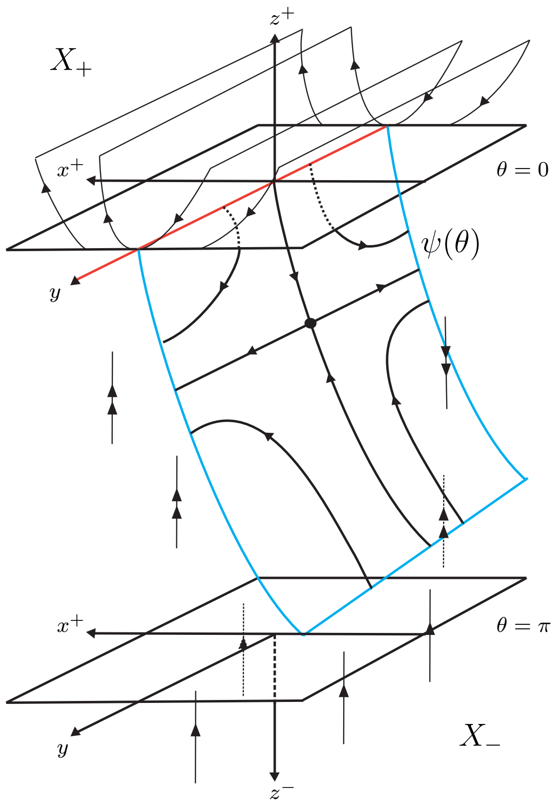

FIGURE 8. Slow-fast system obtained blowing up the nonlinear regularization of the normal form fold regular PSVF on $\mathbb{R}^3$.

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Now, take the continuous combination $\widetilde{X}$ of X_+ and X_- with

$$\begin{aligned}
P &= P(\lambda, x, y, z) = -(\lambda^2-1)(1+x)(-1+y+xy)/4x,\\
Q &= Q(\lambda, x, y, z) = (\lambda^2-1)(x+1)^2,\\
R &= R(\lambda, x, y, z) = \frac{x+1}{2}(\lambda^2-1).
\end{aligned}$$

By solving

$$\frac{x+1}{2} + \frac{x-1}{2}\lambda + \frac{x+1}{2}(\lambda^2-1) = 0$$

in the variable λ, we find that $\Sigma_n^{sl} = \Sigma\backslash\{x=0\}$; besides on $\{(x,y,0) \in \Sigma :\ x > 0\}$ we have two nonlinear sliding vector field and on $\{(x,y,0) \in \Sigma :\ x < 0\}$ only one.

The sliding vector field X^{λ_1} is

$$\dot{x} = 1/2 + \frac{(1+x)(-1+y+xy)}{4x}, \quad \dot{y} = -(1+x)^2, \quad \dot{z} = 0;$$

and the sliding vector field X^{λ_2} is

$$\dot{x} = y, \quad \dot{y} = -4x, \quad \dot{z} = 0.$$

The φ-nonlinear regularization $X^\varepsilon = \widetilde{X}(\varphi(z/\varepsilon), x, y, z)$. For any $p \in \Sigma_n^{sl}$, the singular perturbation problem associated is

$$\begin{aligned}
r\dot{\theta} &= -\sin\theta\left(\psi(\theta)\left(\frac{x-1}{2} + \frac{x+1}{2}\psi(\theta)\right)\right)\\
\dot{x} &= 1/2 + \psi(\theta)/2 + \frac{-(1+x)(-1+y+xy)}{4x}(\psi(\theta)^2-1)\\
\dot{y} &= (x+1)^2(\psi(\theta)^2-1)
\end{aligned}$$

where $\psi(\theta) = \varphi(\cot\theta)$. Then the slow manifold is the curve given by the plane $\theta = \pi/2$ and $\psi(\theta) = \frac{1-x}{1+x}$, $x > 0$. The slow flow on $\theta = \pi/2$ is given by

$$\dot{x} = 1/2 + \frac{(1-x)(-1+y+xy)}{4x}, \quad \dot{y} = -(x+1)^2,$$

and for $\psi(\theta) = \dfrac{1-x}{1+x}$, the slow flow is

$$\dot{x} = y, \quad \dot{y} = -4x.$$

The slow-fast system obtained blowing up the nonlinear regularization is shown in figure (9).

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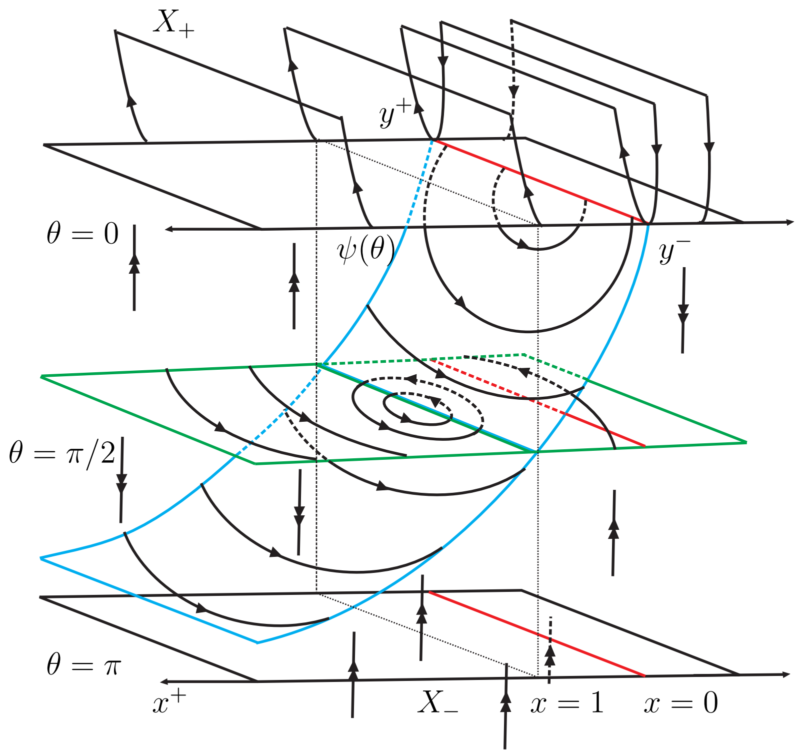

FIGURE 9. Slow-fast system obtained blowing up the nonlinear regularization of the regular fold PSVF on $\mathbb{R}^3$.

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