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ON ORDERS OF APPROXIMATION OF THE GENERALIZED NIKOL'SKII–BESOV CLASS IN LORENTZ SPACES

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Abstract. This paper considers the Lorentz space with mixed norm of periodic functions of many variables and of the generalized Nikol'skii–Besov classes. Estimates for the order of approximation of the generalized Nikol'skii–Besov classes by partial sums of Fourier's series for multiple trigonometric system in Lorentz spaces with mixed norm are obtained.

Introduction

Let $\overline{x} = (x_1, \ldots, x_m) \in \mathbb{I}^m = [0, 2\pi]^m$ and let $\theta_j, p_j \in [1, +\infty)$, $j = 1, \ldots, m$, $\mathbb{N}$ be the set of natural numbers.

We shall denote by $L_{\overline{p},\overline{\theta}}(\mathbb{I}^m)$ the Lorentz spaces with mixed norm of Lebesgue measurable functions $f(\overline{x})$ defined on $\mathbb{R}^m$ with of period 2π for each variable such that

$$\|f\|_{\overline{p},\overline{\theta}} = \| \cdots \|f\|_{p_1,\theta_1} \cdots \|_{p_m,\theta_m} < +\infty,$$

where

$$\|g\|_{p,\theta} = \left\{ \int\limits_0^{2\pi} (g^*(t))^{\theta} t^{\frac{\theta}{p}-1} dt \right\}^{\frac{1}{\theta}},$$

where g^* is a non-increasing rearrangement of the function $|g|$ (see [9]).

As we know, that in case when $p_j = \theta_j$, $j = 1, \ldots, m$, the space $L_{\overline{p},\overline{\theta}}(\mathbb{I}^m)$ coincides with the Lebesgue space $L_{\overline{p}}(I^m)$ with mixed norm (for the definition see [19], p. 128):

$$\|f\|_{\bar{p}} = \left[\int_0^{2\pi} \left[\cdots \left[\int_0^{2\pi} |f(\bar{x})|^{p_1} dx_1 \right]^{\frac{p_2}{p_1}} \cdots \right]^{\frac{p_m}{p_{m-1}}} dx_m \right]^{\frac{1}{p_m}}.$$

Let $\overset{\circ}{L}_{\overline{q},\overline{\theta}}(\mathbb{I}^m)$ be the set of functions $f \in L_{\overline{q},\overline{\theta}}(\mathbb{I}^m)$ such that

$$\int\limits_0^{2\pi} f(\overline{x})\, dx_j = 0, \ \ \forall j = 1, \ldots, m,$$

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and let $a_{\overline{n}}(f)$ be the Fourier coefficients of the function $f \in L_1(\mathbb{I}^m)$ with respect to the multiple trigonometric system $\{e^{i\langle \overline{n}, \overline{x}\rangle}\}_{\overline{n}\in\mathbb{Z}^m}$. Then, we set

$$\delta_{\overline{s}}(f, \overline{x}) = \sum_{\overline{n}\in\rho(\overline{s})} a_{\overline{n}}(f)\, e^{i\langle \overline{n}, \overline{x}\rangle},$$

where $\langle \bar{y}, \bar{x}\rangle = \sum_{j=1}^{m} y_j x_j$, $\rho(\bar{s}) = \big\{\overline{k} = (k_1, \ldots, k_m) \in \mathbb{Z}^m : \quad 2^{s_j - 1} \leqslant |k_j| < 2^{s_j},$ $j = 1, \ldots, m\}$.

A function $\Omega(\bar{t}) = \Omega(t_1, \ldots, t_m)$ is a function of mixed module continuity type of an order $l \in \mathbb{N}$ if it satisfies the following conditions:

1) $\Omega(\bar{t}) > 0$, $t_j > 0$, $j = 1, \ldots, m$, $\Omega(\bar{t}) = 0$, if $\prod_{j=1}^{m} t_j = 0$;

2) $\Omega(\bar{t})$ increases in each variable;

3) $\Omega(k_1 t_1, \ldots, k_m t_m) \leqslant \left(\prod_{j=1}^{m} k_j\right)^l \Omega(t_1, \ldots, t_m)$, $k_j \in \mathbb{N}$, $j = 1, \ldots, m$;

4) $\Omega(\bar{t})$ is continuous for $t_j > 0$, $j = 1, \ldots, m$.

Let us consider the following sets

$$\Gamma(N) = \Gamma(\Omega, N) = \left\{\bar{s} = (s_1, \ldots, s_m) \in \mathbb{Z}_+^m : \Omega(2^{-s_1}, \ldots, 2^{-s_m}) \geqslant \frac{1}{N}\right\},$$

$$Q(N) = \bigcup_{\bar{s}\in\Gamma(\Omega,N)} \rho(\bar{s}),$$

$$\Gamma^{\perp}(N) = \Gamma^{\perp}(\Omega, N) = \mathbb{Z}_+^m \setminus \Gamma(\Omega, N), \tag{1}$$

$$\Lambda(N) = \Gamma^{\perp}(N) \setminus \Gamma^{\perp}(2^l N). \tag{2}$$

It follows from (1) and (2) that $\Lambda(N) \subset \Gamma^{\perp}(N)$ and

$$\frac{1}{2^l N} \leqslant \Omega(2^{-\bar{s}}) < \frac{1}{N} \tag{3}$$

for $\bar{s} \in \Lambda(N)$. In [21], N.N. Pustovoitov proved that $\varkappa(N) \neq \emptyset$ and

$$|\Lambda(N)| \asymp (\log_2 N)^{m-1}, \tag{4}$$

where $|F|$ is the number of elements of the set F.

We will use the notation $S_{Q(N)}(f, \bar{x}) = \sum_{\bar{k}\in Q(N)} a_{\bar{k}}(f) \cdot e^{i\langle \bar{k}, \bar{x}\rangle}$ for a partial sum of the Fourier series of a function f.

For a sequence of numbers we write $\{a_{\overline{n}}\}_{\overline{n}\in\mathbb{Z}^m} \in l_{\overline{p}}$ if

$$\left\|\{a_{\overline{n}}\}_{\overline{n}\in\mathbb{Z}^m}\right\|_{l_{\overline{p}}} = \left\{\sum_{n_m=-\infty}^{\infty}\left[\cdots\left[\sum_{n_1=-\infty}^{\infty} |a_{\overline{n}}|^{p_1}\right]^{\frac{p_2}{p_1}}\cdots\right]^{\frac{p_m}{p_{m-1}}}\right\}^{\frac{1}{p_m}} < +\infty,$$

where $\overline{p} = (p_1, \ldots, p_m)$, $1 \leqslant p_j < +\infty$, $j = 1, 2, \ldots, m$.

For a given function of mixed module smoothness type $\Omega(\bar{t})$ consider the generalized Nikol'skii – Besov class

$$S^{\Omega}_{\overline{p},\overline{\theta},\overline{\tau}}B=\left\{f\in \overset{\circ}{L}_{\overline{p},\overline{\theta}}(I^m):\left\|\left\{\Omega^{-1}(2^{-\bar{s}})\left\|\delta_{\bar{s}}(f)\right\|_{\overline{p},\overline{\theta}}\right\}_{\bar{s}\in\mathbb{Z}^m_+}\right\|_{l_{\overline{\tau}}}\leqslant 1\right\},$$

where $\overline{p}=(p_1,\dots,p_m)$, $\overline{\theta}=(\theta_1,\dots,\theta_m)$, $\overline{\tau}=(\tau_1,\dots,\tau_m)$, $1<p_j<+\infty$, $1<\theta_j<\infty$, $1\leqslant\tau_j\leqslant+\infty$, $j=1,\dots,m$, and $\Omega(2^{-\bar{s}})=\Omega(2^{-s_1},\dots,2^{-s_m})$.

If $\Omega(\bar{t})=\prod_{j=1}^m t_j^{r_j}$, $r_j>0$, $j=1,\dots,m$, then this class is denoted by $S^{\overline{r}}_{\overline{p},\overline{\theta},\overline{\tau}}B$.

In case $p_j=\theta_j=p$ and $\Omega(\bar{t})=\prod_{j=1}^m t_j^{r_j}, r_j<l, \tau_j=+\infty, j=1,\dots,m$, $S^{\Omega}_{\overline{p},\overline{\theta},\overline{\tau}}B$ was defined by S.M. Nikol'skii [18], and for $1\leqslant\tau_j<+\infty, j=1,\dots,m$, by T.I. Amanov [4] and P.I. Lizorkin, S.M. Nikol'skii [16]. The generalized Besov class was considered by M.L. Gol'dman [15].

As pointed out in [32, 33] one of the difficulties in the theory of approximation of functions of several variables is the choice of harmonics of the approximating polynomials. The first author, who suggested to approximate functions of several variables by polynomials with harmonics in hyperbolic crosses, was K.I. Babenko [5]. After that approximations of various classes of smooth functions by this method were considered by S.A. Telyakovskii [29], B.S. Mityagin [17], Ya.S. Bugrov [10], N.S. Nikol'skaya [20], E.M. Galeev [13, 14], V.N. Temlyakov [30, 31], Dinh Dung [12], A.R. DeVore, S.V. Konyagin and V.N. Temlyakov [11], H.-J. Schmeisser and W. Sickel [26], W. Sickel and T. Ullrich [24], A.S. Romanyuk [22, 23], and N.N. Pustovoitov [21].

For the generalized Besov class this problem was considered by Sun Yongsheng and Wang Heping [28], D.B. Bazakhanov [7], M. Sikhov [25], and S.A. Stasyuk [27].

Exact orders of the approximation of the Nikol'skii–Besov classes in the metric of the Lorentz space were found by the author [1, 2] and K.A. Bekmaganbetov [8].

An order of approximation of the class $S^{\overline{r}}_{\overline{p},\overline{\theta},\overline{\tau}}B$ by partial Fourier sums $S^{\bar{\gamma}}_n(f,\bar{x}) = \sum_{\langle\bar{s},\bar{\gamma}\rangle<n}\delta_{\bar{s}}(f,\bar{x})$ was found in [1]. It is stated in the following theorem.

Theorem (see [1]). *Let* $\bar{\theta}^{(1)}=(\theta_1^{(1)},\dots,\theta_m^{(1)})$, $\bar{\theta}^{(2)}=(\theta_1^{(2)},\dots,\theta_m^{(2)})$, $\bar{\tau}=(\tau_1,\dots,\tau_m)$, $\bar{p}=(p_1,\dots,p_m)$, $\bar{q}=(q_1,\dots,q_m)$, $\bar{r}=(r_1,\dots,r_m)$, $\bar{\gamma}=(\gamma_1,\dots,\gamma_m)$, $\gamma_j=\frac{r_j}{r_1}$, *and assume that* $1\leqslant\theta_j^{(1)},\theta_j^{(2)},\tau_j<+\infty$, $1\leqslant p_j<q_j<+\infty$, $\max_{j=1,\dots,m-1}\{\theta_j^{(2)}\}<\min_{j=2,\dots,m}\{q_j\}$, $\frac{1}{p_j}-\frac{1}{q_j}<r_j$, $j=1,\dots,m$, $0<r_1=\dots=r_\nu<r_{\nu+1}\leqslant\dots\leqslant r_m$, $\frac{1}{p_1}-\frac{1}{q_1}=\dots=\frac{1}{p_\nu}-\frac{1}{q_\nu}$, $r_1(\frac{1}{p_j}-\frac{1}{q_j})<r_j(\frac{1}{p_1}-\frac{1}{q_1}), j=\nu+1,\dots,m$. *Then the following relation holds:*

$$\sup_{f\in S^{\bar{r}}_{\bar{p},\bar{\theta}^{(1)},\bar{\tau}}B}\left\|f-S^{\bar{\gamma}}_n(f)\right\|_{\bar{q},\bar{\theta}^{(2)}}\asymp\begin{cases}2^{-n\left(r_1+\frac{1}{q_1}-\frac{1}{p_1}\right)}\cdot n^{\sum\limits_{j=2}^{\nu}\left(\frac{1}{\theta_j^{(2)}}-\frac{1}{\tau_j}\right)}, & \theta_j^{(2)}<\tau_j,\ j=1,\dots,m\\ 2^{-n\left(r_1+\frac{1}{q_1}-\frac{1}{p_1}\right)}, & \tau_j\leqslant\theta_j^{(2)},\ j=1,\dots,m\end{cases}$$

The notation $A(y) \asymp B(y)$ means that there exist positive constants C_1, C_2 such that $C_1 A(y) \leqslant B(y) \leqslant C_2 A(y)$. If $B \leqslant C_2 A$ or $A \leqslant C_1 B$, then we write $B << A$ or $A << B$.

The main aim of the present paper is to estimate the order of the quantity

$$\sup_{f \in S^{\Omega}_{\bar{p},\bar{\theta},\bar{\tau}} B} \|f - S_{Q(N)}(f)\|_{\bar{q},\bar{\theta}}.$$

This paper is organized as follows. In the second section some auxiliary lemmas are given. The third section establishes the estimate of the order approximation of the Nikol'skii–Besov classes in the Lorentz space with mixed norm.

Auxiliary lemmas

In what follows, we denote by $\chi_{\varkappa(n)}(\bar{s})$ the characteristic function of the set $\varkappa(n) = \{\bar{s} = (s_1, \ldots, s_m) \in \mathbb{Z}^m_+ : \quad \langle \bar{s}, \bar{\gamma} \rangle = n\}$.

Lemma 1. *Let $\bar{\tau} = (\tau_1, \ldots, \tau_m)$, $1 \leqslant \tau_j < +\infty$, $j = 1, \ldots, m$. Then the following relation holds:*

$$\left\| \left\{ \chi_{\varkappa(n)}(\bar{s}) \right\}_{\bar{s} \in \varkappa(n)} \right\|_{l_{\bar{\tau}}} \asymp n^{\sum\limits_{j=2}^{m} \frac{1}{\tau_j}}.$$

Lemma 2. *Let $\bar{\gamma} = (\gamma_1, \ldots, \gamma_m)$, $\bar{\gamma}' = (\gamma_1', \ldots, \gamma_m')$, $\gamma_j' = \gamma_j, j = 1, \ldots, \nu$, $1 < \gamma_j < \gamma_j', j = \nu + 1, \ldots, m$, and let $\bar{\tau} = (\tau_{l+1}, \ldots, \tau_m)$, where $1 \leqslant \tau_j < +\infty$, $j = 1, \ldots, m$, and $\alpha > 0$. Then the following relation holds:*

$$I_n^{(l)} = \left\| \left\{ 2^{-\alpha \langle \bar{s}, \bar{\gamma}' \rangle} \right\}_{\bar{s} \in \varkappa(n)} \right\|_{l_{\bar{\tau}}} \asymp 2^{-n\alpha} \cdot n^{\sum\limits_{j=2}^{\nu} \frac{1}{\tau_j}}.$$

Lemma 1, 2 are proved in [2].

Let us recall definitions of the conditions (S), (S_l) given by S.B. Stechkin and N.K. Bary [6].

Definition. A function $g(t)$ satisfies the condition (S), if for some $\alpha \in (0,1)$ the function $t^{-\alpha} g(t)$ almost increases on $(0,1]$.

We say that a function $\Omega(\bar{t})$ satisfies the condition (S) on $(0,1]^m$, if it satisfies this condition on each variable.

Definition. A function $g(t)$ satisfies the condition (S_l), if for some $\alpha \in (0,l)$ the function $t^{-\alpha} g(t)$ almost decreases on $(0,1]$.

We say that a function $\Omega(\bar{t})$ satisfies the condition (S_l) on $(0,1]^m$, if it satisfies this condition on each variable.

Lemma 3. *Let $1 \leqslant \theta_j < +\infty$, $j = 1, \ldots, m$, and $\Omega(\bar{t})$ be a function of mixed module continuity type of an order l which satisfies the (S)-condition for $\bar{\alpha} =$*

$(\alpha_1,\dots,\alpha_m)$, $\alpha_j>\beta_j\geqslant 0$, $j=1,\dots,m$. Then for $1\leqslant\theta_j<+\infty$, $j=1,\dots,m$, the following relation holds

$$\left\|\left\{\Omega(2^{-s_1},\dots,2^{-s_m})\prod_{j=1}^{m}2^{s_j\beta_j}\right\}_{\bar s\in\Gamma^{\perp}(\Omega,N)}\right\|_{l_{\bar\theta}}\asymp$$

$$\asymp\left\|\left\{\Omega(2^{-s_1},\dots,2^{-s_m})\prod_{j=1}^{m}2^{s_j\beta_j}\right\}_{\bar s\in\Lambda(N)}\right\|_{l_{\bar\theta}}.$$

Proof. Let $m=2$. Suppose

$$s_2^N=\min\left\{s_1\in N:\Omega(1,2^{-s_2})<\frac{1}{N}\right\}.$$

For each fixed $s_2\in\mathbb{N}$, suppose

$$s_1^0(s_2)=\min\left\{s_1\in N:\Omega(2^{-s_1},2^{-s_2})<\frac{1}{N}\right\},$$

$$s_1^{'}(s_2)=\min\left\{s_1\in N:\Omega(2^{-s_1},2^{-s_2})\geqslant\frac{1}{2^lN}\right\}.$$

Let us use the following notation: $B_{\bar s}=\Omega(2^{-s_1},2^{-s_2})\prod_{j=1}^{2}2^{s_j\beta_j}$, if $\bar s\in\Gamma^{\perp}(\Omega,N)$, and $B_{\bar s}=0$, if $\bar s\notin\Gamma^{\perp}(\Omega,N)$. Then

$$J_N=\left\|\left\{\Omega(2^{-s_1},2^{-s_2})\prod_{j=1}^{2}2^{s_j\beta_j}\right\}_{\bar s\in\Gamma^{\perp}(\Omega,N)}\right\|_{l_{\bar\theta}}<<$$

(5)

$$<<C(\theta)\left\{\left[\sum_{s_2=1}^{s_2^N}\left[\sum_{s_1:\bar s\in\Gamma^{\perp}(\Omega,N)}B_{\bar s}^{\theta_1}\right]^{\frac{\theta_2}{\theta_1}}\right]^{\frac{1}{\theta_2}}+\left[\sum_{s_2=s_2^N+1}^{\infty}\left[\sum_{s_1:\bar s\in\Gamma^{\perp}(\Omega,N)}B_{\bar s}^{\theta_1}\right]^{\frac{\theta_2}{\theta_1}}\right]^{\frac{1}{\theta_2}}\right\}.$$

Next, by definitions of $s_1^{'}(s_2)$ and $s_1^0(s_2)$,

$$\sum_{s_1:\bar s\in\Gamma^{\perp}(\Omega,N)}B_{\bar s}^{\theta_1}=\sum_{s_1=s_1^0(s_2)}^{s_1^{'}(s_2)}B_{\bar s}^{\theta_1}+\sum_{s_1=1+s_1^{'}(s_2)}^{\infty}B_{\bar s}^{\theta_1}. \tag{6}$$

By the (S)-condition of a function of mixed module continuity type, we have

$$\sum_{s_1=1+s_1^{'}(s_2)}^{\infty}B_{\bar s}^{\theta_1}<<\sum_{s_1=1+s_1^{'}(s_2)}^{\infty}2^{s_1(\beta_1-\alpha_1)\theta_1}(2^{s_1^{'}(s_2)\alpha_1}\Omega(2^{-s_1^{'}(s_2)},2^{-s_2}))^{\theta_1}\cdot 2^{s_2\beta_2\theta_1}<<$$

$$<<\Omega^{\theta_1}(2^{-s_1^{'}(s_2)},2^{-s_2})2^{s_2\beta_2\theta_1}\cdot 2^{s_1^{'}(s_2)\beta_1\theta_1}.$$

Therefore from (6) we obtain

$$\sum_{s_1:\bar{s}\in\Gamma^{\perp}(\Omega,N)} B_{\bar{s}}^{\theta_1} << \sum_{s_1=s_1^0(s_2)}^{s_1^{'}(s_2)} \Omega(2^{-s_1},2^{-s_2})^{\theta_1}\cdot\prod_{j=1}^{2}2^{s_j\beta_j\theta_1}$$

for each fixed $s_2=1,\ldots,s_2^N$. Hence

$$\left[\sum_{s_2=1}^{s_2^N}\left[\sum_{s_1:\bar{s}\in\Gamma^{\perp}(\Omega,N)} B_{\bar{s}}^{\theta_1}\right]^{\frac{\theta_2}{\theta_1}}\right]^{\frac{1}{\theta_2}} << \left\|\left\{\Omega(2^{-s_1},2^{-s_2})\prod_{j=1}^{2}2^{s_j\beta_j}\right\}_{\bar{s}\in\Lambda(N)}\right\|_{l_{\bar{\theta}}}. \tag{7}$$

Next using the (S)-condition twice and taking into account that $\alpha_j>\beta_j$, $j=1,2$, we have

$$\sum_{s_2=1}^{\infty}\left[\sum_{s_1:\bar{s}\in\Gamma^{\perp}(\Omega,N)} B_{\bar{s}}^{\theta_1}\right]^{\frac{\theta_2}{\theta_1}} <<$$

$$<< \sum_{s_2=s_2^N+1}^{\infty}\Omega^{\theta_2}(1,2^{-s_2})2^{s_2\beta_2\theta_2}\times\left[\sum_{s_1:\bar{s}\in\Gamma^{\perp}(\Omega,N)}2^{-s_1\theta_1(\alpha_1-\beta_1)}.\right]^{\frac{\theta_2}{\theta_1}}$$

$$<< \sum_{s_2=s_2^N+1}^{\infty}\Omega^{\theta_2}(1,2^{-s_2})2^{s_2\beta_2\theta_2} <<$$

$$<< (2^{s_2^N\alpha}\Omega^{\theta_2}(1,2^{-s_2^N}))\sum_{s_2=s_2^N+1}^{\infty}2^{-s_2\theta_2(\alpha_2-\beta_2)} <<$$

$$<< (\Omega(1,2^{-s_2^N}))^{\theta_2}2^{s_2^N\beta_2\theta_2}.$$

So

$$\left[\sum_{s_2=1}^{\infty}\left[\sum_{s_1:\bar{s}\in\Gamma^{\perp}(\Omega,N)} B_{\bar{s}}^{\theta_1}\right]^{\frac{\theta_2}{\theta_1}}\right]^{\frac{1}{\theta_2}} \leqslant C(q,\theta,\alpha)\Omega(1,2^{-s_2^N})(2^{s_2^N})^{\beta_2} <<$$

$$<< \left\|\left\{\Omega(2^{-s_1},2^{-s_2})\prod_{j=1}^{2}s_j^{\beta_j}\right\}_{\bar{s}\in\Lambda(N)}\right\|_{l_{\bar{\theta}}}. \tag{8}$$

From (5) by the inequalities (7) and (8), we get

$$J_N << \left\|\left\{\Omega(2^{-s_1},2^{-s_2})\prod_{j=1}^{2}s_j^{\beta_j}\right\}_{\bar{s}\in\Lambda(N)}\right\|_{l_{\bar{\theta}}}.$$

Thus, the lemma has been proved for $m=2$.

Now let us consider $m > 2$. Suppose that Lemma 3 is true for $m-1$, i.e.

$$\left\|\left\{\Omega(2^{-s_1},\dots,2^{-s_m})\prod_{j=1}^{m-1}2^{s_j\beta_j}\right\}_{\overline{s}^m\in\Gamma^{\perp}_m(\Omega,N)}\right\|_{l_{\bar{\theta}^m}} <<$$

$$<< \left\|\left\{\Omega(2^{-s_1},\dots,2^{-s_m})\prod_{j=1}^{m-1}2^{s_j\beta_j}\right\}_{\overline{s}^m\in\Lambda_m(N)}\right\|_{l_{\bar{\theta}^m}}, \tag{9}$$

where $\overline{s}^m=(s_1,\dots,s_{m-1}), \bar{\theta}^m=(\theta_1,\dots,\theta_{m-1})$, $\Gamma^{\perp}_m(\Omega,N)=\{\overline{s}^m : \overline{s}\in\Gamma^{\perp}_m(\Omega,N)\}$, $\Lambda_m(N)=\{\overline{s}^m : \overline{s}\in\Lambda(N)\}$. Then, by the (S)-condition and (9), we have

$$\left\|\left\{\Omega(2^{-s_1},\dots,2^{-s_m})\prod_{j=1}^{m}2^{s_j\beta_j}\right\}_{\overline{s}\in\Gamma^{\perp}(\Omega,N)}\right\|_{l_{\bar{\theta}}} <<$$

$$<< \Big[\sum_{s_m=1}^{s_m^N}2^{s_m\beta_m\theta_m}\Big\|\Big\{\Omega(2^{-s_1},\dots,2^{-s_m})\prod_{j=1}^{m-1}2^{s_j\beta_j}\Big\}_{\overline{s}^m\in\Gamma^{\perp}_m(\Omega,N)}\Big\|^{\theta_m}_{l_{\bar{\theta}^m}}\Big]^{\frac{1}{\theta_m}}+$$

$$+\Big[\sum_{s_2=s_2^N+1}^{\infty}\Big\|\Big\{\Omega(2^{-s_1},\dots,2^{-s_m})\prod_{j=1}^{m-1}2^{s_j\beta_j}\Big\}_{\overline{s}^m\in\Gamma^{\perp}_m(\Omega,N)}\Big\|^{\theta_m}_{l_{\bar{\theta}^m}}\Big]^{\frac{1}{\theta_m}} <<$$

$$<< C(\theta,\alpha)\Big\{\Big[\sum_{s_m=1}^{s_m^N}2^{s_m\beta_m\theta_m}\Big\|\Big\{\Omega(2^{-s_1},\dots,2^{-s_m})\prod_{j=1}^{m-1}2^{s_j\beta_j}\Big\}_{\overline{s}^m\in\Lambda_m(N)}\Big\|^{\theta_m}_{l_{\bar{\theta}^m}}\Big]^{\frac{1}{\theta_m}}+$$

$$+\Big[\sum_{s_2=s_2^N+1}^{\infty}\Omega^{\theta_m}(1,\dots,1,2^{-s_m})2^{s_m\beta_m\theta_m}\Big]^{\frac{1}{\theta_m}}\Big\} <<$$

$$<< \Big\|\Big\{\Omega(2^{-s_1},\dots,2^{-s_m})\prod_{j=1}^{m}2^{s_j\beta_j}\Big\}_{\overline{s}\in\Lambda(N)}\Big\|_{l_{\bar{\theta}}}.$$

So Lemma 3 has been proved.

Lemma 4. *Let $\Omega(\bar{t})$ be a function of mixed module continuity type of an order l, which satisfies the conditions (S) and (S_l), $1\leqslant\tau_j<+\infty, j=1,\dots,m$, and $\Lambda(N)=\Gamma^{\perp}(N)\setminus\Gamma^{\perp}(2^lN)$. Then*

$$\Big\|\big\{\chi_{\Lambda(N)}(\bar{s})\big\}_{\bar{s}\in\Lambda(N)}\Big\|_{l_{\bar{\tau}}} \asymp (\log_2 N)^{\sum_{j=2}^{m}\frac{1}{\tau_j}}.$$

Proof. It is known (see [21], p. 114) that there exists a one-to-one mapping between the set $\Lambda(N)$ and some subset of the set $A = \cup_{n=m_1}^{m_1+m_2} \varkappa_n$, where

$$\varkappa_n = \Big\{ \bar{s} = (s_1, \ldots, s_m) \in \mathbb{Z}_+^m : \sum_{j=1}^{m} s_j = n \Big\},$$

and $m_1 = \left[\frac{1}{\alpha} \log_2(C_2 2^l N)\right] + 1$ and $m_2 = \frac{l+\log_2 C_1}{\alpha}$ (here $[y]$ is the integer part of y). By the property of quasinorm, we have

$$\left\| \left\{ \chi_{\Lambda(N)}(\bar{s}) \right\}_{\bar{s}\in\Lambda(N)} \right\|_{l_{\bar{\tau}}} \leqslant \left\| \left\{ \chi_A(\bar{s}) \right\}_{\bar{s}\in A} \right\|_{l_{\bar{\tau}}} =$$

$$= \left\| \sum_{n=m_1}^{m_1+m_2} \left\{ \chi_{\varkappa(n)}(\bar{s}) \right\}_{\bar{s}\in\varkappa(n)} \right\|_{l_{\bar{\tau}}} = \sum_{n=m_1}^{m_1+m_2} \left\| \left\{ \chi_{\varkappa(n)}(\bar{s}) \right\}_{\bar{s}\in\varkappa(n)} \right\|_{l_{\bar{\tau}}}. \tag{10}$$

By Lemma 1,

$$\left\| \left\{ \chi_{\varkappa(n)}(\bar{s}) \right\}_{\bar{s}\in\theta(n)} \right\|_{l_{\bar{\tau}}} << n^{\sum\limits_{j=2}^{m} \frac{1}{\tau_j}}.$$

Therefore, from the inequality (10) taking into account the definitions of m_1 and m_2, we get

$$\left\| \left\{ \chi_{\Lambda(N)}(\bar{s}) \right\}_{\bar{s}\in\Lambda(N)} \right\|_{l_{\bar{\tau}}} << \sum_{n=m_1}^{m_1+m_2} n^{\sum\limits_{j=2}^{m} \frac{1}{\tau_j}} <<$$

$$<< m_2 \left(\frac{1}{\alpha} \log_2(C_2 2^l N) + 1 + m_2 \right)^{\sum\limits_{j=2}^{m} \frac{1}{\tau_j}}. \tag{11}$$

Since m_2 does not depend on N, by the property of logarifmic functions from the estimation (11) for some sufficiently big N, we obtain

$$\left\| \left\{ \chi_{\Lambda(N)}(\bar{s}) \right\}_{\bar{s}\in\Lambda(N)} \right\|_{l_{\bar{\tau}}} << (\log_2 N)^{\sum\limits_{j=2}^{m} \frac{1}{\tau_j}},$$

which gives us the upper bound in the lemma. Let us prove the lower bound. By Holder's inequality, we have

$$\sum_{\bar{s}\in\Lambda(N)} 1 \leqslant \left\| \{1\}_{\bar{s}\in\Lambda(N)} \right\|_{l_{\bar{\tau}}} \left\| \{1\}_{\bar{s}\in\Lambda(N)} \right\|_{l_{\bar{\tau}'}}, \tag{12}$$

where $\frac{1}{\tau_j} + \frac{1}{\tau_j'} = 1$, $j = 1, \ldots, m$. By the proved fact, we have

$$\left\| \{1\}_{\bar{s}\in\Lambda(N)} \right\|_{l_{\bar{\tau}'}} << (\log_2 N)^{\sum\limits_{j=2}^{m} \frac{1}{\tau_j'}}$$

and N.N. Pustovoitov [21] proved that

$$\sum_{\bar{s}\in\Lambda(N)} 1 \asymp \log_2 N.$$

Therefore from (12) we obtain

$$(\log_2 N)^{\sum\limits_{j=2}^{m}\frac{1}{\tau_j}} << \left\| \left\{ \chi_{\Lambda(N)}(\bar{s}) \right\}_{\bar{s}\in\Lambda(N)} \right\|_{l_{\bar{\tau}}}.$$

The lemma is proved.

Remark. Note that for the case $\tau_1 = \cdots = \tau_m = 1$ Lemma 4 was proved by N.N. Pustovoitov [21].

Theorem 1. *Let* $\bar{q}=(q_1,\ldots,q_m)$, $1<q_j<\infty$, $j=1,\ldots,m$, $\beta=min\{q_1,\ldots,q_m,2\}$. *Then, for any function* $f \in L_{\bar{q}}(I^m)$, *the following inequality holds*

$$\|f\|_{\bar{q}} << \left\{ \sum_{\bar{s}\in\mathbb{Z}_+^m} \|\delta_{\bar{s}}(f)\|_{\bar{q}}^{\beta} \right\}^{\frac{1}{\beta}}.$$

The proof of theorem is given in [3].

Theorem 2 (see [1])**.** *Let* $\bar{p}=(p_1,\ldots,p_m)$, $\bar{q}=(q_1,\ldots,q_m)$, $\bar{\theta}^{(1)}=(\theta_1^{(1)},\ldots,\theta_m^{(1)})$, $\bar{\theta}^{(2)}=(\theta_1^{(2)},\ldots,\theta_m^{(2)})$. *Assume that* $1 \leqslant p_j < q_j < +\infty$, $1 \leqslant \theta_j^{(1)}, \theta_j^{(2)} < +\infty$, $j = 1,\ldots,m$. *If* $f \in \overset{\circ}{L}_{\bar{p},\bar{\theta}^{(1)}}(\mathbb{I}^m)$, $\max_{j=1,\ldots,m-1}\theta_j^{(2)} < \min_{j=2,\ldots,m} q_j$ *and the quantity*

$$\sigma(f) \equiv \left\{ \sum_{s_m=1}^{\infty} 2^{s_m\theta_m^{(2)}(\frac{1}{p_m}-\frac{1}{q_m})} \left[\cdots \left[\sum_{s_1=1}^{\infty} 2^{s_1\theta_1^{(2)}(\frac{1}{p_1}-\frac{1}{q_1})} \|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{\theta}^{(1)}}^{\theta_1^{(2)}} \right]^{\frac{\theta_2^{(2)}}{\theta_1^{(2)}}} \cdots \right]^{\frac{\theta_m^{(2)}}{\theta_{m-1}^{(2)}}} \right\}^{\frac{1}{\theta_m^{(2)}}}$$

is finite, then $f \in \overset{\circ}{L}_{\bar{q},\bar{\theta}^{(2)}}(\mathbb{I}^m)$ *and*

$$\|f\|_{\bar{q},\bar{\theta}^{(2)}} << \sigma(f).$$

Theorem 3 (see [1])**.** *Let* $\bar{q} = (q_1,\ldots,q_m)$, $\bar{\theta} = (\theta_1,\ldots,\theta_m)$, $\bar{\lambda} = (\lambda_1,\ldots,\lambda_m)$. *Assume that* $1 < q_j < \tau_j < +\infty$, $1 < \theta_j < +\infty$, $j = 1,\ldots,m$. *If* $f \in \overset{\circ}{L}_{\bar{q},\bar{\theta}}(\mathbb{I}^m)$ *and*

$$f(\bar{x}) \sim \sum_{\bar{s}\in\mathbb{Z}_+^m} b_{\bar{s}} \sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle},$$

then

$$\|f\|_{\bar{q},\bar{\theta}} >> \left\{ \sum_{s_m=1}^{\infty} 2^{s_m\theta_m\left(\frac{1}{\lambda_m}-\frac{1}{q_m}\right)} \left[\cdots \left[\sum_{s_1=1}^{\infty} 2^{s_1\theta_1\left(\frac{1}{\lambda_1}-\frac{1}{q_1}\right)} \left(\|\delta_{\bar{s}}(f)\|_{\bar{\lambda},\bar{\theta}} \right)^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \cdots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\}^{\frac{1}{\theta_m}}.$$

Main results

Let us prove the main results of the present paper.

Theorem 4. *Let* $1 \leqslant \theta_j^{(1)}, \theta_j^{(2)}, \tau_j < +\infty$, $1 < p_j < q_j < \infty$, $j = 1, \ldots, m$, *and* $\Omega(\bar{t})$ *be a function of mixed module continuity type of an order* l, *which satisfies the conditions* (S) *and* (S_l), $\alpha_j > \frac{1}{p_j} - \frac{1}{q_j}$, $j = 1, \ldots, m$.

1) *If* $1 \leqslant \theta_j^{(2)} < \tau_j < +\infty$, $j = 1, \ldots, m$, *then*

$$\frac{1}{N} (\log_2 N)^{-\sum\limits_{j=2}^{m} \frac{1}{\tau_j}} \left\| \left\{ \prod_{j=1}^{m} 2^{s_j \left(\frac{1}{p_j} - \frac{1}{q_j} \right)} \right\}_{\bar{s} \in \Lambda(N)} \right\|_{l_{\bar{\theta}^{(2)}}} << \sup_{f \in S^{\Omega}_{\bar{p}, \bar{\theta}^{(1)} \bar{\tau}} B} \|f - S_{Q(N)}(f)\|_{\bar{q}, \bar{\theta}^{(2)}} <<$$

$$<< \frac{1}{N} \left\| \left\{ \prod_{j=1}^{m} 2^{s_j \left(\frac{1}{p_j} - \frac{1}{q_j} \right)} \right\}_{\bar{s} \in \Lambda(N)} \right\|_{l_{\bar{\epsilon}}},$$

where $\bar{\epsilon} = (\epsilon_1, \ldots, \epsilon_m)$, $\epsilon_j = \frac{\tau_j \theta_j^{(2)}}{\tau_j - \theta_j^{(2)}}$, $j = 1, \ldots, m$.

2) *If* $\tau_j \leqslant \theta_j^{(2)}$, $j = 1, \ldots, m$, *then*

$$\sup_{\bar{s} \in \Lambda(N)} \Omega(2^{-\bar{s}}) \prod_{j=1}^{m} 2^{s_j \left(\frac{1}{p_j} - \frac{1}{q_j} \right)} << \sup_{f \in S^{\Omega}_{\bar{p}, \bar{\theta}^{(1)} \bar{\tau}} B} \|f - S_{Q(N)}(f)\|_{\bar{q}, \bar{\theta}^{(2)}} <<$$

$$<< \sup_{\bar{s} \in \Gamma^{\perp}(N)} \Omega(2^{-\bar{s}}) \prod_{j=1}^{m} 2^{s_j \left(\frac{1}{p_j} - \frac{1}{q_j} \right)}.$$

Proof. By Theorem 2, we have

$$\sup_{f \in S^{\Omega}_{\bar{p}, \bar{\theta}^{(1)}, \bar{\tau}} B} \|f - S_{Q(N)}(f)\|_{\bar{q}, \bar{\theta}^{(2)}} <<$$

$$<< \left\| \left\{ \prod_{j=1}^{m} 2^{s_j \left(\frac{1}{p_j} - \frac{1}{q_j} \right)} \left\| \delta_{\bar{s}}(f - S_{Q(N)}(f)) \right\|_{\bar{p}, \bar{\theta}^{(1)}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}^{(2)}}} =$$

$$= C \left\| \left\{ \prod_{j=1}^{m} 2^{s_j \left(\frac{1}{p_j} - \frac{1}{q_j} \right)} \left\| \delta_{\bar{s}}(f - S_{Q(N)}(f)) \right\|_{\bar{p}, \bar{\theta}^{(1)}} \right\}_{\bar{s} \in \Gamma^{\perp}(N)} \right\|_{l_{\bar{\theta}^{(2)}}}.$$

Since $\beta_j = \frac{\tau_j}{\theta_j^{(2)}} > 1, j = 1, \ldots, m$, and by applying Holder's inequality we obtain the following

$$\sup_{f \in S^{\Omega}_{\bar{p}, \bar{\theta}^{(1)}, \bar{\tau}} B} \|f - S_{Q(N)}(f)\|_{\bar{q}, \bar{\theta}^{(2)}} << \left\| \left\{ \Omega^{-1}(2^{-\bar{s}}) \left\| \delta_{\bar{s}}(f) \right\|_{\bar{p}, \bar{\theta}^{(1)}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} \times$$

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$$\times \left\| \left\{ \Omega(2^{-\bar{s}}) \prod_{j=1}^{m} 2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)} \right\}_{\bar{s}\in\Gamma^{\perp}(N)} \right\|_{l_{\bar{\epsilon}}}, \tag{13}$$

where $\bar{\epsilon} = (\epsilon_1, \ldots, \epsilon_m)$, $\epsilon_j = \frac{\tau_j\theta_j^{(2)}}{\tau_j-\theta_j^{(2)}}$.

By Lemma 3 and the definition of the set $\Gamma^{\perp}(N)$ in (13), we have

$$\sup_{f\in S^{\Omega}_{\bar{p},\bar{\theta}^{(1)},\bar{\tau}}B} \|f - S_{Q(N)}(f)\|_{\bar{q},\bar{\theta}^{(2)}} << \frac{1}{N} \left\| \left\{ \prod_{j=1}^{m} 2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)} \right\}_{\bar{s}\in\Lambda(N)} \right\|_{l_{\bar{\epsilon}}}.$$

In item 1) of the theorem the upper bound has been proved.

Let us prove the lower bound. Consider the function

$$f_0(\bar{x}) = (\log_2 N)^{-\sum_{j=2}^{m}\frac{1}{\tau_j}} \sum_{\bar{s}\in\Lambda(N)} \prod_{j=1}^{m} \Omega(2^{-\bar{s}}) 2^{-s_j\left(1-\frac{1}{p_j}\right)} \sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle}.$$

In one-dimensional case for Dirichlet's kernel $D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} e^{ikx}$ the following statement holds

$$\|D_n\|_{p,\theta} \asymp n^{1-\frac{1}{p}}, \quad 1 < p < +\infty, \quad 1 < \theta < +\infty.$$

Then, by the property of the norm, we have

$$\left\| \sum_{k_j=2^{s_j-1}}^{2^{s_j}-1} e^{ik_jx_j} \right\|_{p_j,\theta_j^{(1)}} \leqslant \|D_{2^{s_j}-1}\|_{p_j,\theta_j^{(1)}} + \left\|D_{2^{s_j-1}-1}\right\|_{p_j,\theta_j^{(1)}} << 2^{s_j(1-\frac{1}{p_j})},$$

provided $1 < p_j < +\infty$, $1 < \theta_j^{(1)} < +\infty$, $j = 1, \ldots, m$. Hence

$$\left\| \sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle} \right\|_{\bar{p},\bar{\theta}^{(1)}} = \prod_{j=1}^{m} \left\| \sum_{k_j=2^{s_j-1}}^{2^{s_j}-1} e^{ik_jx_j} \right\|_{p_j,\theta_j^{(1)}} << \prod_{j=1}^{m} 2^{s_j\left(1-\frac{1}{p_j}\right)}.$$

Let us prove the rest of the equality. By Lemma B in [1], the following inequality holds

$$\max_{\bar{x}\in I^m} |\sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle}| << \prod_{j=1}^{m} 2^{s_j\left(1-\frac{1}{p_j}\right)} \left\| \sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle} \right\|_{\bar{p},\bar{\theta}^{(1)}}. \tag{14}$$

It is known that

$$\max_{\bar{x}\in I^m} |\sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle}| \geqslant |\sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{0}\rangle}| \geqslant 2^{-m} \prod_{j=1}^{m} 2^{s_j}.$$

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Therefore, it follows from (14) that

$$\prod_{j=1}^{m} 2^{s_j\left(1-\frac{1}{p_j}\right)} << \left\| \sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle} \right\|_{\overline{p},\bar{\theta}^{(1)}}.$$

Thus, we have proved the relation

$$\left\| \sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle} \right\|_{\overline{p},\bar{\theta}^{(1)}} \asymp \prod_{j=1}^{m} 2^{s_j\left(1-\frac{1}{p_j}\right)}. \tag{15}$$

Therefore, by Lemma 4 and by this estimation, we have

$$\left\| \left\{ \Omega^{-1}(2^{-\bar{s}}) \left\|\delta_{\bar{s}}(f_0)\right\|_{\overline{p},\bar{\theta}^{(1)}} \right\}_{\bar{s}\in\mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} =$$

$$= \left\| \left\{ \Omega^{-1}(2^{-\bar{s}}) (\log_2 N)^{-\sum\limits_{j=2}^{m}\frac{1}{\tau_j}} \Omega(2^{-\bar{s}}) \prod_{j=1}^{m} 2^{-s_j\left(1-\frac{1}{p_j}\right)} \left\| \sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle} \right\|_{\overline{p},\bar{\theta}^{(1)}} \right\}_{\bar{s}\in\Lambda(N)} \right\|_{l_{\bar{\tau}}}$$

$$= (\log_2 N)^{-\sum\limits_{j=2}^{m}\frac{1}{\tau_j}} \left\| \left\{ \chi_{\Lambda(N)}(\bar{s}) \right\}_{\bar{s}\in\Lambda(N)} \right\|_{l_{\bar{\tau}}} \leqslant C_0.$$

Hence $C_0^{-1} f_0 \in S^{\Omega}_{\overline{p},\overline{\tau}}B$. Now taking into account that $S^{\bar{\gamma}}_{Q(N)}(f_0,\bar{x}) = 0$ and using Theorem 4 and (15), we obtain

$$\left\|f_0 - S_{Q(N)}(f_0)\right\|_{\overline{q},\bar{\theta}^{(2)}} = \left\|f_0\right\|_{\overline{q},\bar{\theta}^{(2)}} >>$$

$$>> \left\| \left\{ \prod_{j=1}^{m} 2^{s_j\left(\frac{1}{\lambda_j}-\frac{1}{q_j}\right)} \left\|\delta_{\bar{s}}(f_0)\right\|_{\overline{\lambda},\bar{\theta}^{(1)}} \right\}_{\bar{s}\in\mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}^{(2)}}} =$$

$$= C \left\| \left\{ \prod_{j=1}^{m} 2^{s_j\left(\frac{1}{\lambda_j}-\frac{1}{q_j}\right)} (\log_2 N)^{-\sum\limits_{j=2}^{m}\frac{1}{\tau_j}} \Omega(2^{-\bar{s}}) \prod_{j=1}^{m} 2^{-s_j\left(1-\frac{1}{p_j}\right)} \right.\right.$$

$$\left.\left. \left\| \sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle} \right\|_{\overline{\lambda},\bar{\theta}^{(1)}} \right\}_{\bar{s}\in\Lambda(N)} \right\|_{l_{\bar{\theta}^{(2)}}} >>$$

$$>> (\log_2 N)^{-\sum\limits_{j=2}^{m}\frac{1}{\tau_j}} \left\| \left\{ \prod_{j=1}^{m} 2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)} \Omega(2^{-\bar{s}}) \right\}_{\bar{s}\in\Lambda(N)} \right\|_{l_{\bar{\theta}^{(2)}}} >>$$

$$>> \frac{1}{N} (\log_2 N)^{-\sum\limits_{j=2}^{m}\frac{1}{\tau_j}} \left\| \left\{ \prod_{j=1}^{m} 2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)} \right\}_{\bar{s}\in\Lambda(N)} \right\|_{l_{\bar{\theta}^{(2)}}}.$$

Thus,

$$\sup_{f\in S^{\Omega}_{\overline{p},\bar{\theta}^{(1)},\overline{\tau}}B}\|f-S_{Q(N)}(f)\|_{\overline{q},\bar{\theta}^{(2)}}>>\frac{1}{N}\left(\log_2 N\right)^{-\sum\limits_{j=2}^{m}\frac{1}{\tau_j}}\left\|\left\{\prod_{j=1}^{m}2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)}\right\}_{\bar{s}\in\Lambda(N)}\right\|_{l_{\bar{\theta}^{(2)}}}.$$

Item 1) of the theorem has been proved.

Let us prove item 2) of the theorem. Since $\tau_j\leqslant\theta_j^{(2)}$, $j=1,\dots,m$, and by applying Jensen's inequality (see [19], p. 125), we obtain

$$\|f-S_{Q(N)}(f)\|_{\overline{q},\bar{\theta}^{(2)}}\leqslant$$
$$\leqslant C\left\|\left\{\prod_{j=1}^{m}2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)}\|\delta_{\bar{s}}(f)\|_{\overline{p},\bar{\theta}^{(1)}}\right\}_{\bar{s}\in\Gamma^{\perp}(N)}\right\|_{l_{\bar{\tau}}}<<$$
$$<<\left\|\left\{\Omega^{-1}(2^{-\bar{s}})\|\delta_{\bar{s}}(f)\|_{\overline{p},\bar{\theta}^{(1)}}\right\}_{\bar{s}\in\mathbb{Z}^m_+}\right\|_{l_{\bar{\tau}}}\sup_{\bar{s}\in\Gamma^{\perp}(N)}\Omega(2^{-\bar{s}})\prod_{j=1}^{m}2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)}$$

for any function $f\in S^{\Omega}_{\overline{p},\bar{\theta}^{(1)},\overline{\tau}}B$, which proves the upper bound in item (2). For the lower bound, consider the function

$$f_1(\bar{x})=\Omega(2^{-\bar{\tilde{s}}})2^{-\sum\limits_{j=1}^{m}\tilde{s}_j\left(1-\frac{1}{p_j}\right)}\sum_{\bar{k}\in\rho(\bar{\tilde{s}})}e^{i\langle\bar{k},\bar{x}\rangle},$$

where $\bar{\tilde{s}}=(\tilde{s}_1,\dots,\tilde{s}_m)\in\Lambda(N)$. Then $f_1\in S^{\Omega}_{\overline{p},\bar{\theta}^{(1)},\overline{\tau}}B$. Next, by (15), we have

$$\|f_1-S_{Q(N)}(f_1)\|_{\overline{q},\bar{\theta}^{(2)}}=\|f_1\|_{\overline{q},\bar{\theta}^{(2)}}>>$$
$$>>\Omega(2^{-\bar{\tilde{s}}})2^{-\sum\limits_{j=1}^{m}\tilde{s}_j\left(1-\frac{1}{p_j}\right)}\prod_{j=1}^{m}2^{\tilde{s}_j\left(1-\frac{1}{q_j}\right)}$$
$$=C\Omega(2^{-\bar{\tilde{s}}})\prod_{j=1}^{m}2^{\tilde{s}_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)},\quad\forall\bar{\tilde{s}}\in\Lambda(N).$$

This proves the lower bound in item 2).

Theorem 5. *Let $\Omega(\bar{t})$ be a function of mixed module continuity type of an order l which satisfies the conditions (S) and (S_l), $1<q_j<p_j<\infty$, $p_j\geqslant 2$, $1<\theta_j<\infty$, $1\leqslant\tau_j\leqslant+\infty$, $j=1,\dots,m$.*

1) *If $2<\tau_j<+\infty$, $j=1,\dots,m$, then*

$$\sup_{f\in S^{\Omega}_{\overline{p},\overline{\tau}}B}\|f-S_{Q(N)}(f)\|_{\overline{q},\bar{\theta}}\asymp\frac{1}{N}\left(\log_2 N\right)^{\sum\limits_{j=2}^{m}\left(\frac{1}{2}-\frac{1}{\tau_j}\right)}.$$

2) *If* $\tau_j \leqslant 2$, $j = 1, \dots, m$, *then*

$$\sup_{f \in S^{\Omega}_{\overline{p},\overline{\tau}}B} \|f - S_{Q(N)}(f)\|_{\overline{q},\overline{\theta}} \asymp \frac{1}{N}.$$

3) *If* $\tau_j \leqslant \beta = \min\{p_1, \dots, p_m, 2\}$, $j = 1, \dots, m$, *then*

$$\sup_{f \in S^{\Omega}_{\overline{p},\overline{\tau}}B} \|f - S_{Q(N)}(f)\|_{\overline{q},\overline{\theta}} << \frac{1}{N}.$$

Proof. Since $q_j < p_j$, $j = 1, \dots, m$, we have

$$\|f\|_{\overline{q},\overline{\theta}} << \|f\|_{\overline{p}}, \quad f \in L_{\overline{p}}(I^m).$$

Therefore

$$\|f - S_{Q(N)}(f)\|_{\overline{q},\overline{\theta}} << \|f - S_{Q(N)}(f)\|_{\overline{p}} =$$

$$= C\left\| \sum_{\bar{s} \in \Gamma^{\perp}(N)} \delta_{\bar{s}}(f) \right\|_{\overline{p}}. \tag{16}$$

Now, since $2 \leqslant p_j < +\infty$, $j = 1, \dots, m$, using Theorem 1 from (16) we obtain

$$\|f - S_{Q(N)}(f)\|_{\overline{q},\overline{\theta}} << \left\{ \sum_{\bar{s} \in \Gamma^{\perp}(N)} \|\delta_{\bar{s}}(f)\|^2_{\overline{p}} \right\}^{\frac{1}{2}} =$$

$$= C\left\{ \sum_{\bar{s} \in \Gamma^{\perp}(N)} \Omega^2(2^{-\bar{s}}) \left(\Omega^{-1}(2^{-\bar{s}}) \|\delta_{\bar{s}}(f)\|_{\overline{p}} \right)^2 \right\}^{\frac{1}{2}}. \tag{17}$$

If $2 < \tau_j < +\infty$, $j = 1, \dots, m$, then by Holder's inequality from (17), we get

$$\|f - S_{Q(N)}(f)\|_{\overline{q},\overline{\theta}} << \left\| \left\{ \Omega^{-1}(2^{-\bar{s}}) \|\delta_{\bar{s}}(f)\|_{\overline{p}} \right\}_{\bar{s} \in \mathbb{Z}^m_+} \right\|_{l_{\bar{\tau}}} \times$$

$$\times \left\| \left\{ \Omega(2^{-\bar{s}}) \right\}_{\bar{s} \in \Gamma^{\perp}(N)} \right\|_{l_{\bar{\epsilon}}}, \tag{18}$$

where $\bar{\epsilon} = (\epsilon_1, \dots, \epsilon_m)$, $\epsilon_j = 2\beta'_j$, $\frac{1}{\beta_j} + \frac{1}{\beta'_j} = 1$, $\beta_j = \frac{\tau_j}{2}$, $j = 1, \dots, m$.

Now, by Lemma 3 for $\beta_j = 0$, $j = 1, \dots, m$, and Lemma 4, from (18) we obtain

$$\sup_{f \in S^{\Omega}_{\overline{p},\overline{\tau}}B} \|f - S_{Q(N)}(f)\|_{\overline{q},\overline{\theta}} <<$$

$$<< \left\| \left\{ \Omega^{-1}(2^{-\bar{s}}) \|\delta_{\bar{s}}(f)\|_{\overline{p}} \right\}_{\bar{s} \in \mathbb{Z}^m_+} \right\|_{l_{\bar{\tau}}} \times \left\| \left\{ \Omega(2^{-\bar{s}}) \right\}_{\bar{s} \in \Lambda(N)} \right\|_{l_{\bar{\epsilon}}} \asymp$$

$$\asymp \frac{1}{N} \left\| \left\{ \chi_{\Lambda(N)}(\bar{s}) \right\}_{\bar{s} \in \Lambda(N)} \right\|_{l_{\bar{\epsilon}}} << \frac{1}{N} (\log_2 N)^{\sum\limits_{j=2}^{m} \left(\frac{1}{2} - \frac{1}{\tau_j} \right)},$$

where $\chi_{\Lambda(N)}$ is the characteristic function of the set $\Lambda(N)$. This proves the upper bound in item 1).

If $\tau_j \leqslant 2$, $j = 1, \dots, m$, then using Jensen's inequality we have

$$\left\{ \sum_{\bar{s} \in \Gamma^{\perp}(N)} \|\delta_{\bar{s}}(f)\|_{\bar{p}}^2 \right\}^{\frac{1}{2}} << \left\{ \sum_{\bar{s} \in \Gamma^{\perp}(N)} \left(\Omega^{-1}(2^{-\bar{s}}) \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right)^2 \right\}^{\frac{1}{2}} \sup_{\bar{s} \in \Gamma^{\perp}(N)} \Omega(2^{-\bar{s}}).$$

Therefore, from the inequality (17) we obtain

$$\sup_{f \in S^{\Omega}_{\bar{p},\bar{\tau}} B} \|f - S_{Q(N)}(f)\|_{\bar{q},\bar{\theta}} << \sup_{\bar{s} \in \Gamma^{\perp}(N)} \Omega(2^{-\bar{s}}) << \frac{1}{N},$$

in case $2 < p_j < +\infty$, $\tau_j \leqslant 2$, $j = 1, \dots, m$.

Let us prove item 3). Let $1 < p_j < +\infty$, $j = 1, \dots, m$, $\beta = \min\{p_1, \dots, p_m, 2\}$. Then, by Theorem 1 from (16) we have

$$\|f - S_{Q(N)}(f)\|_{\bar{q},\bar{\theta}} << \left\{ \sum_{\bar{s} \in \Gamma^{\perp}(N)} \|\delta_{\bar{s}}(f)\|_{\bar{p}}^{\beta} \right\}^{\frac{1}{\beta}}. \tag{19}$$

If $\tau_j \leqslant \beta$, $j = 1, \dots, m$, then using Jensen's inequality we obtain

$$\left\{ \sum_{\bar{s} \in \Gamma^{\perp}(N)} \|\delta_{\bar{s}}(f)\|_{\bar{p}}^{\beta} \right\}^{\frac{1}{\beta}} \leqslant \left\{ \sum_{\bar{s} \in \Gamma^{\perp}(N)} \left(\Omega^{-1}(2^{-\bar{s}}) \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right)^{\beta} \right\}^{\frac{1}{\beta}} \sup_{\bar{s} \in \Gamma^{\perp}(N)} \Omega(2^{-\bar{s}}).$$

Therefore, from (19) we get

$$\sup_{f \in S^{\Omega}_{\bar{p},\bar{\tau}} B} \|f - S_{Q(N)}(f)\|_{\bar{q},\bar{\theta}} \leqslant \sup_{\bar{s} \in \Gamma^{\perp}(N)} \Omega(2^{-\bar{s}}) << \frac{1}{N},$$

in case $\tau_j \leqslant \beta$, $1 < p_j < +\infty$, $j = 1, \dots, m$. This proves the upper bound.

Let us prove the lower bound. Firstly, consider item 1). Let $2 \leqslant q_j < p_j < +\infty$, $2 < \tau_j < \infty$, $j = 1, \dots, m$. Consider the function

$$f_{N,\tau}(\bar{x}) = \left\| \left\{ \chi_{\Lambda(N)}(\bar{s}) \right\}_{\bar{s} \in \Lambda(N)} \right\|_{l_{\bar{\tau}}}^{-1} \sum_{\bar{s} \in \Lambda(N)} \Omega(2^{-\bar{s}}) e^{i\langle \bar{k}_{\bar{s}}, \bar{x} \rangle},$$

where $\bar{k}_{\bar{s}} \in \rho(\bar{s})$ is some fixed element. By the definition of the space $S^{\Omega}_{\bar{p},\bar{\tau}} B$, we have

$$\left\| \left\{ \Omega^{-1}(2^{-\bar{s}}) \|\delta_{\bar{s}}(f_{N,\tau})\|_{\bar{p}} \right\}_{\bar{s} \in \mathbb{Z}^m_+} \right\|_{l_{\bar{\tau}}} = \left\| \left\{ \chi_{\Lambda(N)}(\bar{s}) \right\}_{\bar{s} \in \Lambda(N)} \right\|_{l_{\bar{\tau}}}^{-1} \left\| \left\{ \chi_{\Lambda(N)}(\bar{s}) \right\}_{\bar{s} \in \Lambda(N)} \right\|_{l_{\bar{\tau}}} = 1,$$

i.e., $f_{N,\tau} \in S^{\Omega}_{\bar{p},\bar{\tau}} B$. By the assumption of the theorem, we have $2 \leqslant q_j < +\infty$, $j = 1, \dots, m$. Hence

$$\|f\|_2 << \|f\|_{\bar{q},\bar{\theta}}.$$

Therefore, taking into account that $S_{Q(N)}(f_{N,\tau},\bar{x})=0$ and using Parseval's identity, we get

$$\|f_{N,\tau}-S_{Q(N)}(f_{N,\tau})\|_{\bar{q},\bar{\theta}}=\|f_{N,\tau}\|_{\bar{q},\bar{\theta}}\geqslant\|f_{N,\tau}\|_2=$$
$$=\left\|\left\{\chi_{\Lambda(N)}(\bar{s})\right\}_{\bar{s}\in\Lambda(N)}\right\|_{l_{\bar{\tau}}}^{-1}\left\{\sum_{\bar{s}\in\Lambda(N)}\Omega^2(2^{-\bar{s}})\right\}^{\frac{1}{2}}>>$$
$$>>\frac{1}{N}\left\|\left\{\chi_{\Lambda(N)}(\bar{s})\right\}_{\bar{s}\in\Lambda(N)}\right\|_{l_{\bar{\tau}}}^{-1}\left\|\left\{\chi_{\Lambda(N)}(\bar{s})\right\}_{\bar{s}\in\Lambda(N)}\right\|_{l_2}.$$

Thus,
(20)
$$\sup_{f\in S^{\Omega}_{\bar{p},\bar{\tau}}B}\|f-S_{Q(N)}(f)\|_{\bar{q},\bar{\theta}}>>\frac{1}{N}\left\|\left\{\chi_{\Lambda(N)}(\bar{s})\right\}_{\bar{s}\in\Lambda(N)}\right\|_{l_{\bar{\tau}}}^{-1}\left\|\left\{\chi_{\Lambda(N)}(\bar{s})\right\}_{\bar{s}\in\Lambda(N)}\right\|_{l_2},$$
for $2\leqslant q_j<p_j<+\infty$, $2<\tau_j<\infty$, $j=1,\dots,m$.

Further, by Lemma 4,
$$\left\|\left\{\chi_{\Lambda(N)}(\bar{s})\right\}_{\bar{s}\in\Lambda(N)}\right\|_{l_{\bar{\tau}}}<<(\log_2 N)^{\sum\limits_{j=2}^{m}\frac{1}{\tau_j}}$$
and
$$(\log_2 N)^{\sum\limits_{j=2}^{m}\frac{1}{2}}<<\left\|\left\{\chi_{\Lambda(N)}(\bar{s})\right\}_{\bar{s}\in\Lambda(N)}\right\|_{l_2},$$
for $\tau_j=2, j=1,\dots,m$. Therefore, it follows from (20) that
$$\sup_{f\in S^{\Omega}_{\bar{p},\bar{\tau}}B}\|f-S_{Q(N)}(f)\|_{\bar{q},\bar{\theta}}>>\frac{1}{N}(\log_2 N)^{\sum\limits_{j=2}^{m}\left(\frac{1}{2}-\frac{1}{\tau_j}\right)}.$$

Let us prove the lower bound in item 2). Let $2\leqslant p_j<+\infty$, $j=1,\dots,m$. Suppose $\tau_j\leqslant 2$, $j=1,\dots,m$. Then consider the function
$$g_{\Omega}(\bar{x})=C_0\Omega(2^{-\bar{s}})e^{i\langle\bar{k}_{\tilde{s}},\bar{x}\rangle},\quad C_0>0,$$
where $\bar{k}_{\tilde{s}}\in\rho(\bar{x}),\tilde{\bar{s}}=(s_1,\dots,s_m)\in\Lambda(N)$. Therefore
$$\left\|\left\{\Omega^{-1}(2^{-\bar{s}})\|\delta_{\bar{s}}(g_{\Omega})\|_{\bar{p}}\right\}_{\bar{s}\in\mathbb{Z}^m_+}\right\|_{l_{\bar{\tau}}}=C_0\Omega^{-1}(2^{-\tilde{\bar{s}}})\left\|e^{i\langle\bar{k}_{\tilde{s}},\bar{x}\rangle}\right\|_{\bar{p}}\Omega(2^{-\tilde{\bar{s}}})=C_0.$$
Hence, $g_{\Omega}\in S^{\Omega}_{\bar{p},\bar{\tau}}B$. Since $S_{Q(N)}(g_{\Omega},\bar{x})=0$, we have
$$\|g_{\Omega}-S_{Q(N)}(g_{\Omega})\|_{\bar{q},\bar{\theta}}=\|g_{\Omega}\|_{\bar{q},\bar{\theta}}=C_0\Omega(2^{-\tilde{\bar{s}}})\left\|e^{i\langle\bar{k}_{\tilde{\bar{s}}},\bar{x}\rangle}\right\|_{\bar{q},\bar{\theta}}=C_1\Omega(2^{-\tilde{\bar{s}}}),$$
which means that
$$\sup_{f\in S^{\Omega}_{\bar{p},\bar{\tau}}B}\|f-S_{Q(N)}(f)\|_{\bar{q},\bar{\theta}}>>\Omega(2^{-\tilde{\bar{s}}}),\quad\tilde{\bar{s}}\in\varkappa(N)\tag{21}$$
in case $2\leqslant p_j<+\infty$, $\tau_j\leqslant 2$, $j=1,\dots,m$.

It is known that $\frac{1}{2N} \leqslant \Omega(2^{-\bar{s}})$ for $\bar{s} \in \Lambda(N)$, (see (3)). Therefore, it follows from (21) that

$$\sup_{f \in S^{\Omega}_{\bar{p},\bar{\tau}}B} \|f - S_{Q(N)}(f)\|_{\bar{q},\bar{\theta}} >> \frac{1}{N}$$

in case $2 \leqslant p_j < +\infty$, $\tau_j \leqslant 2$, $j = 1, \dots, m$.

Remark. Note that for the case $q_j = \theta_j = q$, $p_j = p$, $\tau_j = \tau$, $j = 1, \dots, m$, Theorem 5 was proved by S.A. Stasyuk [27]. For the case $p_j = \theta_j^{(1)} = p$, $q_j = \theta_j^{(2)} = q$, $\tau_j = +\infty$, $j = 1, \dots, m$, Theorem 4 was proved by N.N. Pustovoitov [21].

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