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ESTIMATIONS OF THE BEST M -TERM APPROXIMATIONS OF FUNCTIONS IN THE LORENTZ SPACE WITH CONSTRUCTIVE METHODS

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ABSTRACT. This paper considers the Lorentz space of periodic functions of many variables with the anisotropic norm. Estimations of the best M -term approximations of Nikol'ski-Besov's classes in the Lorentz space with the anisotropic norm are given.

INTRODUCTION

Let $\bar{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$, $I^m = [0, 2\pi)^m$ and numbers $\theta_j, q_j \in [1, +\infty)$, $j = 1, \dots, m$. Let $L_{\bar{q}, \bar{\theta}}^*(I^m)$ denotes the space of Lebesgue measurable functions $f(\bar{x})$ defined on \mathbb{R}^m with the period 2π with respect to each variable such that the quantity

$$\|f\|_{\bar{q}, \bar{\theta}}^* = \left[\int_0^{2\pi} t_m^{\frac{\theta_m}{q_m} - 1} \left[\dots \left[\int_0^{2\pi} \left(f^{*1, \dots, *m}(t_1, \dots, t_m) \right)^{\theta_1} t_1^{\frac{\theta_1}{q_1} - 1} dt_1 \right]^{\frac{\theta_2}{q_2}} \dots \right]^{\frac{\theta_{m-1}}{q_{m-1}}} dt_m \right]^{\frac{1}{\theta_m}}$$

is finite, where $f^{*1, \dots, *m}(t_1, \dots, t_m)$ is a non-increasing rearrangement of the function $|f(\bar{x})|$ in each variable x_j , whereas the other variables are fixed (see [1]).

In case when the $q_1 = \dots = q_m = \theta_1 = \dots = \theta_m = q$, the space of Lorentz $L_{\bar{q}, \bar{\theta}}^*(I^m)$ coincides with the space of Lebesgue $L_q(I^m)$ with the norm (see [2], Ch. I, item 1.1)

$$\|f\|_q = \left[\int_0^{2\pi} \dots \int_0^{2\pi} |f(x_1, \dots, x_m)|^q dx_1 \dots dx_m \right]^{\frac{1}{q}}.$$

Let $\overset{\circ}{L}_{\bar{q}, \bar{\theta}}^*(I^m)$ be the set of all functions $f \in L_{\bar{q}, \bar{\theta}}^*(I^m)$ such that

$$\int_0^{2\pi} f(\bar{x}) dx_j = 0, \quad \forall j = 1, \dots, m.$$

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For any function $f \in L_1(I^m) = L(I^m)$, let

$$\sum_{\bar{n} \in \mathbb{Z}^m} a_{\bar{n}}(f) e^{i\langle \bar{n}, \bar{x} \rangle}$$

be function's Fourier series with respect to the multiple trigonometric system $\{e^{i\langle \bar{n}, \bar{x} \rangle}\}_{\mathbb{Z}^m}$, where \mathbb{Z}^m is the set of points in \mathbb{R}^m with integer coordinates.

Suppose

$$\delta_{\bar{s}}(f, \bar{x}) = \sum_{\bar{n} \in \rho(\bar{s})} a_{\bar{n}}(f) e^{i\langle \bar{n}, \bar{x} \rangle},$$

where $\langle \bar{y}, \bar{x} \rangle = \sum_{j=1}^m y_j x_j$, $s_j = 1, 2, \dots$, and

$$\rho(\bar{s}) = \{\bar{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m : 2^{s_j-1} \leq |k_j| < 2^{s_j}, j = 1, \dots, m\}.$$

For a number sequence, we will write $\{a_{\bar{n}}\}_{\bar{n} \in \mathbb{Z}^m} \in l_{\bar{p}}$ if

$$\|\{a_{\bar{n}}\}_{\bar{n} \in \mathbb{Z}^m}\|_{l_{\bar{p}}} = \left\{ \sum_{n_m=-\infty}^{\infty} \left[\dots \left[\sum_{n_1=-\infty}^{\infty} |a_{\bar{n}}|^{p_1} \right]^{\frac{p_2}{p_1}} \dots \right]^{\frac{p_m}{p_{m-1}}} \right]^{\frac{1}{p_m}} < +\infty,$$

where $\bar{p} = (p_1, \dots, p_m)$, $1 \leq p_j < +\infty$, $j = 1, 2, \dots, m$.

The spaces $S_p^{\bar{r}}H$ and $S_{p,\theta}^{\bar{r}}B$ of functions with the dominating mixed derivative were introduced by S.M. Nikol'skii [3] and T.I. Amanov ([4] Ch.I, item 17). The spaces $S_p^{\bar{r}}H$, $S_{p,\theta}^{\bar{r}}B$ are called Nikolski-Besov's space, or, sometimes, Nikolski-Besov-Amanov's space.

P.I. Lizorkin and S.M. Nikol'skii [5] investigated a decomposition of elements of the space $S_{p,\theta}^{\bar{r}}B$. We will use their definition.

Let $\bar{r} = (r_1, \dots, r_m)$, $r_j > 0$, $j = 1, \dots, m$, $1 \leq p, \theta \leq +\infty$. Suppose $S_{p,\theta}^{\bar{r}}B$ is the space all of functions $f \in L_{\bar{q},\bar{\theta}}^*(I^m)$ such that

$$\|f\|_{S_{p,\theta}^{\bar{r}}B} = \left[\int_0^{2\pi} \dots \int_0^{2\pi} \|\Delta_{\bar{t}}^{\bar{k}} f(\bullet)\|_p^\theta \prod_{j=1}^m \frac{dt_j}{t_j^{1+\theta r_j}} \right]^{\frac{1}{\theta}} < +\infty,$$

where $\Delta_{\bar{t}}^{\bar{k}} f(\bar{x}) = \Delta_{t_m}^{k_m}(\dots \Delta_{t_1}^{k_1} f(\bar{x}))$ is the mixed difference of order \bar{k} with step $\bar{t} = (t_1, \dots, t_m)$ and $k_j > r_j$, $j = 1, \dots, m$.

In [5], it is given that the function $f \in S_{p,\theta}^{\bar{r}}B$ can be decomposed into Fourier series in the following form

$$\sum_{\bar{n} \in \mathbb{Z}^m, \prod_{j=1}^m n_j \neq 0} a_{\bar{n}}(f) e^{i\langle \bar{n}, \bar{x} \rangle}.$$

Moreover, it is known (see [5]) that $\|f\|_{S_{p,\theta}^{\bar{r}}B}$ is a norm and

$$\|f\|_{S_{p,\theta}^{\bar{r}}B} \asymp \left\{ \sum_{\bar{s} \in \mathbb{Z}_+^m} 2^{\langle \bar{s}, \bar{r} \rangle \theta} \|\delta_{\bar{s}}(f)\|_p^\theta \right\}^{\frac{1}{\theta}}$$

provided $1 < p < +\infty$, $1 \leq \theta \leq +\infty$.

Therefore, in the anisotropic Lorentz space $L_{\bar{p},\bar{\theta}}^*(I^m)$, we will consider an analogous space. Suppose $S_{\bar{p},\bar{\theta},\bar{\tau}}^{\bar{r}}B$ denotes the space of all functions $f \in L_{\bar{p},\bar{\theta}}^*(I^m)$ such that

$$\|f\|_{S_{\bar{p},\bar{\theta},\bar{\tau}}^{\bar{r}}B} = \left\| \left\{ 2^{\langle \bar{s}, \bar{r} \rangle} \|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{\theta}}^* \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} < \infty,$$

where $\bar{p} = (p_1, \dots, p_m)$, $\bar{\theta} = (\theta_1, \dots, \theta_m)$, $\bar{\tau} = (\tau_1, \dots, \tau_m)$, $1 < p_j < \infty$, $1 < \theta_j < \infty$, $1 \leq \tau_j \leq +\infty$, $r_j > 0$, $j = 1, \dots, m$.

In this space, let's consider the unit ball (with keeping the notation)

$$S_{\bar{p},\bar{\theta},\bar{\tau}}^{\bar{r}}B = \left\{ f \in L_{\bar{p},\bar{\theta}}^*(I^m) : \|f\|_{S_{\bar{p},\bar{\theta},\bar{\tau}}^{\bar{r}}B} = \left\| \left\{ 2^{\langle \bar{s}, \bar{r} \rangle} \|\delta_{\bar{s}}(f)\|_{\bar{p},\bar{\theta}}^* \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} \leq 1 \right\}.$$

For a fixed vector $\bar{\gamma} = (\gamma_1, \dots, \gamma_m)$, $\gamma_j > 0$, $j = 1, \dots, m$, set

$$Q_n^{\bar{\gamma}} = \bigcup_{\langle \bar{s}, \bar{\gamma} \rangle < n} \rho(\bar{s}), \quad T(Q_n^{\bar{\gamma}}) = \left\{ t(\bar{x}) = \sum_{\bar{k} \in Q_n^{\bar{\gamma}}} b_{\bar{k}} e^{i\langle \bar{k}, \bar{x} \rangle} \right\},$$

$$Y^m(\bar{\gamma}, n) = \left\{ \bar{s} = (s_1, \dots, s_m) \in \mathbb{Z}_+^m : \sum_{j=1}^m s_j \gamma_j \geq n \right\}.$$

Let X, Y be spaces with the norm of 2π -periodic functions of several variables. For a function $f \in X$ the following quantity is called the best M -term approximation of f (see [6, 7, 8])

$$e_M(f)_X = \inf_{\bar{k}^{(j)}, b_j} \left\| f - \sum_{j=1}^M b_j e^{i\langle \bar{k}^{(j)}, \bar{x} \rangle} \right\|_X,$$

where $\{\bar{k}^{(j)}\}_{j=1}^M$ is a system of vectors $\bar{k}^{(j)} = (k_1^{(j)}, \dots, k_m^{(j)})$ with integer coordinates and b_j are arbitrary numbers.

For a given class F , let

$$e_M(F)_X = \sup_{f \in F} e_M(f)_X.$$

In the case $X = L_2$, the quantity $e_M(f)_{L_2}$ for a function of one variable was introduced by S.B. Stechkin [6] and it was used in a criteria for an absolute convergence of Fourier series by complete orthonormal systems. Order estimations of the quantity $e_M(F)_X$ were investigated by R.S. Ismagilov [7], B.I. Mayorov [8] (for $X = L_p$, one-dimensional case), E.S. Belinskii [9, 10, 11] (multi-dimensional

case in the case $Y = L_q(I^m)$, $X = L_p(I^m)$, $F = W_p^r$, V.N. Temlyakov [12] (in the case $Y = L_q(I^m)$, $F = H_p^r$), A.S. Romanyuk [13, 14], R. De Vore and V.N. Temlyakov [15], V.N. Temlyakov [16] (in the case $Y = L_q(I^m)$, $F = B_{p,\theta}^r$), Dinh Dung [17]. We should note that B.S. Kashin [18] established an estimation of the quantity $e_M(f)_X$ in the case $X = L_2$ by orthonormal systems. The latest results in this direction can be found in [19, 20, 21].

In particular, the following theorem is well known.

Theorem 1 (A.S. Romanyuk [13]). *Let $\bar{r} = (r_1, \dots, r_m)$, $0 < r_1 = \dots = r_\nu < r_{\nu+1} \leq \dots \leq r_m$, $1 \leq p \leq 2 < q < +\infty$, and $1 \leq \theta \leq +\infty$.*

1) *If $r_1 > \frac{1}{p}$ then*

$$e_M(S_{p,\theta}^{\bar{r}}B)_q \asymp M^{-(r_1 + \frac{1}{2} - \frac{1}{p})} (\log M)^{(\nu-1)(r_1 - \frac{1}{p} + \frac{1}{2}) + \sum_{j=2}^{\nu} (\frac{1}{2} - \frac{1}{\theta})_+}$$

2) *If $\frac{1}{p} - \frac{1}{q} < r_1 < \frac{1}{p}$, then*

$$e_M(S_{p,\theta}^{\bar{r}}B)_q \asymp M^{-\frac{q}{2}(r_1 + \frac{1}{q} - \frac{1}{p})} (\log M)^{(q-1)(\nu-1)(r_1 - \frac{1}{p} + \frac{q}{q} \frac{1}{\theta'})_+}$$

3) *If $r_1 = \frac{1}{p}$, then*

$$e_M(S_{p,\theta}^{\bar{r}}B)_q \asymp M^{-\frac{1}{2}} (\log^\nu M)^{\frac{1}{\theta'}},$$

where $a_+ = \max\{a, 0\}$, $\frac{1}{b} + \frac{1}{b'} = 1$.

Here $\log M$ is the logarithm with the base 2 of $M > 0$.

V.N. Temlyakov [22, 23, 24] developed a constructive method of estimation of the M -term best approximations of functions of the Nikol'skii-Besov's class $S_{p,\tau}^{\bar{r}}B$ in the space $L_q(I^m)$ in the case $1 < p < q < \infty$. This method is based on greedy algorithms.

The main goal of the present paper is to find the exact order of the best M -term approximation of a function in the class $S_{\bar{p},\bar{\theta},\bar{\tau}}^{\bar{r}}B$ in the Lorentz spaces with anisotropic norm in the case $1 < p_j < q_j \leq 2$, $j = 1, \dots, m$, and in the case $1 < p_j < 2 \leq q_j < \infty$, $j = 1, \dots, m$, to give a constructive proof for an upper bound for the quantity $e_M(S_{\bar{p},\bar{\theta},\bar{\tau}}^{\bar{r}}B)_{\bar{q},\bar{\theta}}$.

Let us denote by $C(p, q, r, y)$ positive quantities which depend on the parameters in the parentheses, which are, in general, are distinct in distinct formulas. $A(y) \asymp B(y)$ means that there are positive C_1, C_2 such that $C_1 \cdot A(y) \leq B(y) \leq C_2 \cdot A(y)$.

To prove the main results, we need the following auxiliary results.

Lemma 1 (see [25]). *Suppose $\alpha \in (0, +\infty)$ and $\bar{\gamma} = (\gamma_1, \dots, \gamma_m)$, $\bar{\gamma}' = (\gamma'_1, \dots, \gamma'_m)$, $\bar{\theta} = (\theta_1, \dots, \theta_m)$, $\theta_j \in [1, +\infty)$, $j = 1, \dots, m$, $1 = \gamma_1 = \dots = \gamma_\nu < \gamma_{\nu+1} \leq \dots \leq$*

$\gamma_m, 1 = \gamma'_j = \gamma_j, j = 1, \dots, \nu$, and $1 = \gamma'_j < \gamma_j, j = \nu + 1, \dots, m$. Then the following relation holds

$$\left\| \left\{ 2^{-\alpha \langle \bar{s}, \bar{\gamma} \rangle} \right\}_{\bar{s} \in Y^m(\bar{\gamma}', n)} \right\|_{l_{\bar{\theta}}} \asymp 2^{-n\alpha} n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}}.$$

Remark 1. For the case $\theta_1 = \dots = \theta_m$, Lemma 1 was proved by V.N. Temlyakov [12].

In what follows we denote by $\chi_{\mathcal{X}(n)}(\bar{s})$ the characteristic function of the set $\mathcal{X}(n) = \{\bar{s} = (s_1, \dots, s_m) \in \mathbb{Z}_+^m : \langle \bar{s}, \bar{\gamma} \rangle = n\}$.

Lemma 2 (see [25], Lemma 2.3 and [26], Lemma 4). Let $\bar{\tau} = (\tau_1, \dots, \tau_m)$, $1 \leq \tau_j < +\infty$, and $j = 1, \dots, m$. Then the following relation holds

$$\left\| \left\{ \chi_{\mathcal{X}(n)}(\bar{s}) \right\}_{\bar{s} \in \mathcal{X}(n)} \right\|_{l_{\bar{\tau}}} \asymp n^{\sum_{j=2}^m \frac{1}{\tau_j}}, \quad n \in \mathbb{N}.$$

Theorem 2 (see [27]). Let $\bar{p} = (p_1, \dots, p_m)$, $\bar{q} = (q_1, \dots, q_m)$, $\bar{r} = (r_1, \dots, r_m)$, $\bar{\theta} = (\theta_1, \dots, \theta_m)$, $\bar{\tau} = (\tau_1, \dots, \tau_m)$, $1 \leq p_j < 2 < q_j, j = 1, \dots, m$, $1 \leq \theta_j, \tau_j < +\infty$, $0 < r_1 + \frac{1}{q_1} - \frac{1}{p_1} = \dots = r_{\nu} + \frac{1}{q_{\nu}} - \frac{1}{p_{\nu}} < r_{\nu+1} + \frac{1}{q_{\nu+1}} - \frac{1}{p_{\nu+1}} \leq \dots \leq r_m + \frac{1}{q_m} - \frac{1}{p_m}$.

1) If $r_j > \frac{1}{p_j}, j = 1, \dots, m$, $(r_1 - \frac{1}{p_1})\frac{1}{q_j} < (r_j - \frac{1}{p_j})\frac{1}{q_1}, j = \nu + 1, \dots, m$, then

$$e_M \left(S_{\bar{p}, \bar{\theta}, \bar{\tau}}^{\bar{r}} B \right)_{\bar{q}, \bar{\theta}} \asymp M^{-\left(r_1 + \frac{1}{2} - \frac{1}{p_1}\right)} (\log M)^{(\nu-1)\left(r_1 - \frac{1}{p_1} + \frac{1}{2}\right) + \sum_{j=2}^{\nu} \left(\frac{1}{2} - \frac{1}{\tau_j}\right)}.$$

2) If $\frac{1}{p_j} - \frac{1}{q_j} < r_j < \frac{1}{p_j}, \theta_j < \tau_j, j = 1, \dots, m$, $(r_1 - \frac{1}{p_1})\frac{1}{q_j} < (r_j - \frac{1}{p_j})\frac{1}{q_1}, j = \nu + 1, \dots, m$, then

$$e_M \left(S_{\bar{p}, \bar{\theta}, \bar{\tau}}^{\bar{r}} B \right)_{\bar{q}, \bar{\theta}} \asymp M^{-\frac{q_1}{2} \left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} (\log M)^{q_1 \left(r_1 - \frac{1}{p_1}\right) \sum_{j=2}^{\nu} \frac{1}{\theta_j} + \sum_{j=2}^{\nu} \frac{1}{\tau_j}}$$

3) If $\nu \leq \mu$, $r_j = \frac{1}{p_j}, j = 1, \dots, \mu$ and $r_j > \frac{1}{p_j}, j = \mu + 1, \dots, m$, then

$$e_M \left(S_{\bar{p}, \bar{\theta}, \bar{\tau}}^{\bar{r}} B \right)_{\bar{q}, \bar{\theta}} \asymp M^{-\frac{1}{2}} (\log M)^{\sum_{j=1}^{\mu} \frac{1}{\tau_j}}$$

Theorem 3 (see [25]). Let $\bar{p} = (p_1, \dots, p_m)$, $\bar{q} = (q_1, \dots, q_m)$, $\bar{\theta}^{(1)} = (\theta_1^{(1)}, \dots, \theta_m^{(1)})$, $\bar{\theta}^{(2)} = (\theta_1^{(2)}, \dots, \theta_m^{(2)})$ and $1 \leq p_j < q_j < +\infty, 1 \leq \theta_j^{(1)}, \theta_j^{(2)} < +\infty, j = 1, \dots, m$.

If $f \in L_{\bar{p}, \bar{\theta}^{(1)}}^*(I^m)$ and the quantity

$$\sigma(f) \equiv$$

$$\left\{ \sum_{s_m=1}^{\infty} 2^{s_m \theta_m^{(2)} \left(\frac{1}{p_m} - \frac{1}{q_m}\right)} \left[\dots \left[\sum_{s_1=1}^{\infty} 2^{s_1 \theta_1^{(2)} \left(\frac{1}{p_1} - \frac{1}{q_1}\right)} \left(\|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^* \right)^{\theta_1^{(2)}} \right]_{\theta_1^{(2)}}^{\frac{\theta_2^{(2)}}{\theta_1^{(2)}}} \dots \right]_{\theta_{m-1}^{(2)}}^{\frac{\theta_m^{(2)}}{\theta_{m-1}^{(2)}}} \right\}^{\frac{1}{\theta_m^{(2)}}}$$

is finite, then $f \in L_{\bar{q}, \bar{\theta}^{(2)}}^{\circ}(I^m)$ and

$$\|f\|_{\bar{q}, \bar{\theta}^{(2)}}^* \leq C(p, q, \theta) \cdot \sigma(f).$$

Theorem 4. Let $1 < q_j < 2 \leq p_j < +\infty$, $1 < \theta_j, \lambda_j < +\infty$, $j = 1, \dots, m$. If $f \in L_{\bar{q}, \bar{\theta}}^*(I^m)$, then

$$\|f\|_{\bar{q}, \bar{\theta}}^* \geq$$

$$C(q, \theta, p, m) \left\{ \sum_{s_m=1}^{\infty} 2^{s_m \theta_m (\frac{1}{p_m} - \frac{1}{q_m})} \left[\dots \left[\sum_{s_1=1}^{\infty} 2^{s_1 \theta_1 (\frac{1}{p_1} - \frac{1}{q_1})} \left(\|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\lambda}}^* \right)^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \dots \right]^{\frac{\theta_{m-1}}{\theta_{m-1}}} \right\}^{\frac{1}{\theta_m}}.$$

Proof. Firstly, we will prove the theorem for the case $p_j = \lambda_j^{(1)} = 2$, $j = 1, \dots, m$. Then

$$\|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\lambda}}^* = \|\delta_{\bar{s}}(f)\|_2, \quad \bar{s} \in \mathbb{Z}_+^m.$$

By the conditions of the theorem $f \in L_{\bar{q}, \bar{\theta}}^{\circ}(I^m)$, $1 < q_j, \theta_j < +\infty$, $j = 1, \dots, m$. Therefore (see [1])

$$\|f\|_{\bar{q}, \bar{\theta}}^* \geq \sup_{\|g\|_{\bar{q}', \bar{\theta}'}^* \leq 1} \int_{I^m} f(\bar{x})g(\bar{x})d\bar{x}, \quad (1)$$

where $\bar{q}' = (q'_1, \dots, q'_m)$, $\bar{\theta}' = (\theta'_1, \dots, \theta'_m)$, $\frac{1}{q_j} + \frac{1}{q'_j} = 1$, $\frac{1}{\theta_j} + \frac{1}{\theta'_j} = 1$, $j = 1, \dots, m$.

From the inequality (1), we have

$$\|f\|_{\bar{q}, \bar{\theta}}^* \geq \sup_{\|g\|_{\bar{q}', \bar{\theta}'}^* \leq 1} \sum_{\bar{s} \in \mathbb{Z}_+^m} \int_{I^m} \delta_{\bar{s}}(f, \bar{x}) \delta_{\bar{s}}(g, \bar{x}) d\bar{x}. \quad (2)$$

Since by the assumption of the theorem $1 < q_j < 2$, $j = 1, \dots, m$, we have $2 < q'_j < \infty$, $j = 1, \dots, m$. Therefore, by Theorem 3 for $p_j = \theta_j^{(1)} = 2$, $j = 1, \dots, m$, we obtain

$$\|g\|_{\bar{q}', \bar{\theta}'}^* \leq C_0 \left\| \left\{ \prod_{j=1}^m 2^{s_j (\frac{1}{2} - \frac{1}{q'_j})} \|\delta_{\bar{s}}(g)\|_2 \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}'}}. \quad (3)$$

Let's introduce the following notation

$$U_{\bar{q}', \bar{\theta}'} = \left\{ g \in L_{\bar{q}', \bar{\theta}'}^* : C_0 \left\| \left\{ \prod_{j=1}^m 2^{s_j (\frac{1}{2} - \frac{1}{q'_j})} \|\delta_{\bar{s}}(g)\|_2 \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}'}} \leq 1 \right\}$$

Then it follows from the inequality (3) that the set $U_{\bar{q}', \bar{\theta}'}$ is a subset of the unit ball of the the space $L_{\bar{q}', \bar{\theta}'}^*(I^m)$. Therefore, the formula (2) implies that

$$\|f\|_{\bar{q}, \bar{\theta}}^* \geq C \sup_{g \in U_{\bar{q}', \bar{\theta}'}} \sum_{\bar{s} \in \mathbb{Z}_+^m} \int_{I^m} \delta_{\bar{s}}(f, \bar{x}) \delta_{\bar{s}}(g, \bar{x}) d\bar{x}. \quad (4)$$

If $g \in L_{\bar{q}, \bar{\theta}}^*(I^m)$, such that $\|\delta_{\bar{s}}(g)\|_2 \leq b_{\bar{s}}$, $\bar{s} \in \mathbb{Z}_+^m$ and

$$\left\| \left\{ \prod_{j=1}^m 2^{s_j \left(\frac{1}{2} - \frac{1}{q_j}\right)} b_{\bar{s}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}'}} \leq 1,$$

then from inequality (4) we obtain

$$\|f\|_{\bar{q}, \bar{\theta}}^* \geq C \sup_{\{b_{\bar{s}}\}} \sum_{\bar{s} \in \mathbb{Z}_+^m} \sup_{\|\delta_{\bar{s}}(g)\|_2 \leq b_{\bar{s}}} \int_{I^m} \delta_{\bar{s}}(f, \bar{x}) \delta_{\bar{s}}(g, \bar{x}) d\bar{x}. \quad (5)$$

Now let's prove that the following inequality holds

$$\sup_{\|\delta_{\bar{s}}(g)\|_2 \leq b_{\bar{s}}} \int_{I^m} \delta_{\bar{s}}(f, \bar{x}) \delta_{\bar{s}}(g, \bar{x}) d\bar{x} \quad (6)$$

for each $\bar{s} \in \mathbb{Z}_+^m$. So consider the function

$$g_0(\bar{x}) = \sum_{\bar{k} \in \rho(\bar{s})} a_{\bar{k}}(g_0) e^{i\langle \bar{k}, \bar{x} \rangle},$$

where

$$a_{\bar{k}}(g_0) = \|\delta_{\bar{s}}(f)\|_2^{-1} a_{\bar{k}}(f) b_{\bar{s}}.$$

Then $\|\delta_{\bar{s}}(g)\|_2 \leq b_{\bar{s}}$, $\bar{s} \in \mathbb{Z}_+^m$. Hence, by the Parseval's equality, we have

$$\begin{aligned} \sup_{\|\delta_{\bar{s}}(g)\|_2 \leq b_{\bar{s}}} \int_{I^m} \delta_{\bar{s}}(f, \bar{x}) \delta_{\bar{s}}(g, \bar{x}) d\bar{x} &\geq \int_{I^m} \delta_{\bar{s}}(f, \bar{x}) \delta_{\bar{s}}(g_0, \bar{x}) d\bar{x} \\ &= \|\delta_{\bar{s}}(f)\|_2^{-1} b_{\bar{s}} \sum_{\bar{k} \in \rho(\bar{s})} a_{\bar{k}}^2(f) = \|\delta_{\bar{s}}(f)\|_2 b_{\bar{s}}, \end{aligned}$$

which proves the formula (6).

Next, it follows from formulas (5) and (6) that

$$\|f\|_{\bar{q}, \bar{\theta}}^* \geq C \sup_{\{b_{\bar{s}}\}} \sum_{\bar{s} \in \mathbb{Z}_+^m} b_{\bar{s}} \|\delta_{\bar{s}}(f)\|_2 \quad (7)$$

Suppose

$$\begin{aligned} \sigma_{\bar{s}_{m-j}}(f)_{\bar{\theta}_j} &= \\ &= \left\{ \sum_{s_j=1}^{\infty} 2^{s_j \theta_j \left(\frac{1}{2} - \frac{1}{q_j}\right)} \left[\dots \left[\sum_{s_1=1}^{\infty} 2^{s_1 \theta_1 \left(\frac{1}{2} - \frac{1}{q_1}\right)} \|\delta_{\bar{s}}(f)\|_2^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \dots \right]^{\frac{\theta_j}{\theta_{j-1}}} \right\}^{\frac{1}{\theta_j}}, \end{aligned}$$

where $\bar{s}_{m-j} = (s_{j+1}, \dots, s_m)$, $\bar{\theta}_j = (\theta_1, \dots, \theta_j)$, $j = 1, \dots, m-1$.

Consider the following sequence

$$b_{\bar{s}} =$$

$$\left\| \left\{ \prod_{j=1}^m 2^{s_j(\frac{1}{2} - \frac{1}{q_j})} \|\delta_{\bar{s}}(f)\|_2 \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}}}^{-\frac{\theta_m}{\theta_1}} \prod_{j=1}^m (\sigma_{\bar{s}_{m-j}}(f)_{\bar{\theta}_j})^{\theta_{j+1} - \theta_j} \times \|\delta_{\bar{s}}(f)\|_2^{\theta_1 - 1} \prod_{j=1}^m 2^{s_j(\frac{1}{2} - \frac{1}{q_j})\theta_j}$$

for $\bar{s} \in \mathbb{Z}_+^m$. Then

$$\left\| \left\{ \prod_{j=1}^m 2^{s_j(\frac{1}{2} - \frac{1}{q_j})} b_{\bar{s}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}}} = 1$$

and

$$\sum_{\bar{s} \in \mathbb{Z}_+^m} b_{\bar{s}} \|\delta_{\bar{s}}(f)\|_2 = \left\| \left\{ \prod_{j=1}^m 2^{s_j(\frac{1}{2} - \frac{1}{q_j})} \|\delta_{\bar{s}}(f)\|_2 \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}}}.$$

Therefore, from the inequality (7), we get

$$\|f\|_{\bar{q}, \bar{\theta}}^* \geq C \left\| \left\{ \prod_{j=1}^m 2^{s_j(\frac{1}{2} - \frac{1}{q_j})} \|\delta_{\bar{s}}(f)\|_2 \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}}}. \quad (8)$$

Next, if $2 < p_j < \infty$, $j = 1, \dots, m$, then, by applying the inequality of different metrics for trigonometric polynomials (see [28], [29]), we obtain from the estimate (8) the following

$$\|f\|_{\bar{q}, \bar{\theta}}^* \geq C \left\| \left\{ \prod_{j=1}^m 2^{s_j(\frac{1}{p_j} - \frac{1}{q_j})} \|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\lambda}}^* \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}}}$$

for $1 < q_j < 2 \leq p_j < \infty$, $1 < \theta_j, \lambda_j < \infty$, $j = 1, \dots, m$. The theorem has been proved.

S. 1. Estimates of the best M -term approximations of functions

Suppose $y_+ = \max\{0, y\}$.

Theorem 5. Let $\bar{p} = (p_1, \dots, p_m)$, $\bar{q} = (q_1, \dots, q_m)$, $\bar{r} = (r_1, \dots, r_m)$, $1 \leq p_j < q_j \leq 2$, $j = 1, \dots, m$, $1 \leq \theta_j^{(1)}, \theta_j^{(2)} < +\infty$, $1 \leq \tau_j \leq +\infty$, $0 < r_1 + \frac{1}{q_1} - \frac{1}{p_1} = \dots = r_\nu + \frac{1}{q_\nu} - \frac{1}{p_\nu} < r_{\nu+1} + \frac{1}{q_{\nu+1}} - \frac{1}{p_{\nu+1}} = \dots = r_m + \frac{1}{q_m} - \frac{1}{p_m}$.

1. If $\theta_j^{(2)} \leq \tau_j$, $j = 1, \dots, m$, then

$$e_M \left(S_{\bar{p}, \bar{\theta}^{(1)}, \bar{\tau}}^{\bar{r}} B \right)_{\bar{q}, \bar{\theta}^{(2)}} \asymp M^{-\left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} (\log M)^{(\nu-1)\left(r_1 - \frac{1}{p_1} + \frac{1}{q_1}\right) + \sum_{j=2}^{\nu} \left(\frac{1}{\theta_j^{(2)}} - \frac{1}{\tau_j}\right)}$$

2. If $\theta_1^{(2)} = \dots = \theta_m^{(2)} = \theta > \tau = \tau_1 = \dots = \tau_m$, then

$$e_M \left(S_{\bar{p}, \bar{\theta}^{(1)}, \bar{\tau}}^{\bar{r}} B \right)_{\bar{q}, \theta} \leq M^{-\left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} (\log^{\nu-1} M)^{\left(r_1 - \frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{\theta} - \frac{1}{\tau}\right)_+}$$

3. If $\tau_j \leq \theta_j^{(2)}$, $j = 1, \dots, m$, then

$$e_M \left(S_{\bar{p}, \bar{\theta}^{(1)}, \bar{\tau}}^{\bar{r}} B \right)_{\bar{q}, \bar{\theta}^{(2)}} \geq CM^{-\left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} (\log M)^{(\nu-1)\left(r_1 - \frac{1}{p_1} + \frac{1}{q_1}\right) + \sum_{j=2}^{\nu} \left(\frac{1}{\theta_j^{(2)}} - \frac{1}{\tau_j}\right)}$$

Proof. The upper bound estimation of the quantity $e_M \left(S_{\bar{p}, \bar{\theta}^{(1)}, \bar{\tau}}^{\bar{r}} B \right)_{\bar{q}, \bar{\theta}^{(2)}}$ was proved in [30]. Let us consider the lower bound.

For a number $M \in \mathbb{N}$ choose a natural number n such that $M \asymp 2^n n^{m-1}$ and $2^n n^{m-1} \geq 4M$.

Consider the function

$$f_0(\bar{x}) = n^{-\sum_{j=2}^m \frac{1}{\tau_j}} \sum_{\langle \bar{s}, \bar{\gamma} \rangle = n} \prod_{j=1}^m 2^{-s_j(r_j + 1 - \frac{1}{p_j})} \sum_{\bar{k} \in \rho(\bar{s})} e^{i\langle \bar{k}, \bar{x} \rangle}.$$

Then, by Lemma 2,

$$\begin{aligned} & \left\| \left\{ 2^{\langle \bar{s}, \bar{r} \rangle} \|\delta_{\bar{s}}(f_0)\|_{\bar{p}, \bar{\lambda}}^* \right\}_{\langle \bar{s}, \bar{r} \rangle = n} \right\|_{l_{\bar{\tau}}} = \\ & = n^{-\sum_{j=2}^m \frac{1}{\tau_j}} \left\| \left\{ 2^{\langle \bar{s}, \bar{r} \rangle} \prod_{j=1}^m 2^{-s_j(r_j + 1 - \frac{1}{p_j})} \left\| \sum_{\bar{k} \in \rho(\bar{s})} e^{i\langle \bar{k}, \bar{x} \rangle} \right\|_{\bar{p}, \bar{\lambda}}^* \right\}_{\langle \bar{s}, \bar{r} \rangle = n} \right\|_{l_{\bar{\tau}}} \\ & \leq C n^{-\sum_{j=2}^m \frac{1}{\tau_j}} \left\| \left\{ 2^{\langle \bar{s}, \bar{r} \rangle} \prod_{j=1}^m 2^{-s_j(r_j + 1 - \frac{1}{p_j})} \prod_{j=1}^m 2^{s_j(1 - \frac{1}{p_j})} \right\}_{\langle \bar{s}, \bar{r} \rangle = n} \right\|_{l_{\bar{\tau}}} \\ & = C n^{-\sum_{j=2}^m \frac{1}{\tau_j}} \left\| \{1\}_{\langle \bar{s}, \bar{r} \rangle = n} \right\|_{l_{\bar{\tau}}} \leq C_0. \end{aligned}$$

Thus, the function $f_0 \in S_{\bar{p}, \bar{\theta}^{(1)}, \bar{\tau}}^{\bar{r}} B$.

Let Ω_M be an arbitrary collection in M of m -dimensional vectors $\{\bar{k}^{(1)}, \dots, \bar{k}^{(M)}\}$ with integer coordinates. For every vector \bar{s} , that satisfies the condition $\langle \bar{s}, \bar{\gamma} \rangle = n$, we consider the set $\Omega_M \cap \rho(\bar{s})$. Then, according to the choice of the number n , the set S of vectors \bar{s} such that $\langle \bar{s}, \bar{\gamma} \rangle = n$ and $|\Omega_M \cap \rho(\bar{s})| \leq \frac{1}{2}|\rho(\bar{s})|$ contains at least half of all vectors \bar{s} such that $\langle \bar{s}, \bar{\gamma} \rangle = n$. Hence $|S| \asymp n^{m-1}$.

Let $T(\bar{x})$ be an arbitrary polynomial with a collection of harmonics from Ω_M . Then, by Theorem 4, for $p_j = \lambda_j = 2$, $j = 1, \dots, m$, we get

$$\begin{aligned}
\|f_0 - T\|_{\bar{q}, \bar{\theta}^{(2)}}^* &\geq C \left\| \left\{ \prod_{j=1}^m 2^{s_j(\frac{1}{2} - \frac{1}{q_j})} \|\delta_{\bar{s}}(f_0 - T)\|_2 \right\}_{\langle \bar{s}, \bar{\gamma} \rangle = n} \right\|_{l_{\bar{\theta}^{(2)}}} \\
&\geq C \left\| \left\{ \prod_{j=1}^m 2^{s_j(\frac{1}{2} - \frac{1}{q_j})} \|\delta_{\bar{s}}(f_0 - T)\|_2 \right\}_{\bar{s} \in S} \right\|_{l_{\bar{\theta}^{(2)}}} \\
&\geq C n^{-\sum_{j=2}^m \frac{1}{\tau_j}} \left\| \left\{ \prod_{j=1}^m 2^{s_j(\frac{1}{2} - \frac{1}{q_j})} \prod_{j=1}^m 2^{-s_j(r_j + 1 - \frac{1}{p_j})} \prod_{j=1}^m 2^{\frac{s_j}{2}} \right\}_{\bar{s} \in S} \right\|_{l_{\bar{\theta}^{(2)}}} \\
&\geq C n^{-\sum_{j=2}^m \frac{1}{\tau_j}} \left\| \left\{ 2^{-(r_1 + \frac{1}{q_1} - \frac{1}{p_1})\langle \bar{s}, \bar{\gamma} \rangle} \right\}_{\bar{s} \in S} \right\|_{l_{\bar{\theta}^{(2)}}} \\
&= C n^{-\sum_{j=2}^m \frac{1}{\tau_j}} 2^{-(r_1 + \frac{1}{q_1} - \frac{1}{p_1})} \|\{1\}_{\bar{s} \in S}\|_{l_{\bar{\theta}^{(2)}}}
\end{aligned}$$

Applying the Holder's inequality ($\frac{1}{\theta_j^{(2)}} + \frac{1}{\theta_j^{(2)'}} = 1$, $j = 1, \dots, m$) and Lemma 4, we have

$$\begin{aligned}
|S| = \sum_{\bar{s} \in S} 1 &\leq \|\{1\}_{\bar{s} \in S}\|_{l_{\bar{\theta}^{(2)}}} \|\{1\}_{\bar{s} \in S}\|_{l_{\bar{\theta}^{(2)'}}} \\
&\leq \|\{1\}_{\bar{s} \in S}\|_{l_{\bar{\theta}^{(2)}}} \|\{1\}_{\langle \bar{s}, \bar{\gamma} \rangle = n}\|_{l_{\bar{\theta}^{(2)'}}} \leq C n^{\sum_{j=2}^m \frac{1}{\theta_j^{(2)'}}} \|\{1\}_{\bar{s} \in S}\|_{l_{\bar{\theta}^{(2)}}}.
\end{aligned}$$

Since $|S| \asymp n^{m-1}$, we obtain

$$n^{\nu-1} \leq C n^{\sum_{j=2}^m \frac{1}{\theta_j^{(2)'}}} \|\{1\}_{\bar{s} \in S}\|_{l_{\bar{\theta}^{(2)}}}.$$

Hence

$$n^{\sum_{j=2}^m \frac{1}{\theta_j^{(2)'}}} \leq C \|\{1\}_{\bar{s} \in S}\|_{l_{\bar{\theta}^{(2)}}}.$$

Therefore,

$$\|f_0 - T\|_{\bar{q}, \bar{\theta}^{(2)}}^* \geq C 2^{-(r_1 + \frac{1}{q_1} - \frac{1}{p_1})} n^{\sum_{j=2}^m (\frac{1}{\theta_j^{(2)}} - \frac{1}{\tau_j})}$$

for any polynomial $T(\bar{x})$ with a collection of harmonics from Ω_M . Hence

$$e_M \left(S_{\bar{p}, \bar{\theta}^{(1)}, \bar{r}}^{\bar{r}} B \right)_{\bar{q}, \bar{\theta}^{(2)}} \geq C 2^{-(r_1 + \frac{1}{q_1} - \frac{1}{p_1})} n^{\sum_{j=2}^m (\frac{1}{\theta_j^{(2)}} - \frac{1}{\tau_j})}$$

$$\geq CM^{-\left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} (\log M)^{(m-1)\left(r_1 - \frac{1}{p_1} + \frac{1}{q_1}\right) + \sum_{j=2}^m \left(\frac{1}{\theta_j^{(2)}} - \frac{1}{\tau_j}\right)}. \quad (9)$$

Since $\nu \leq m$, then we obtain the lower bound estimation from (9). The theorem has been proved.

V.N. Temlyakov [22, 23] developed a constructive method of estimation of the best M -term approximations functions from Nikol'skii-Besov's classes $S_{p,\tau}^{\bar{r}}B$ in the space $L_q(I^m)$ in the case $1 < p < 2 \leq q < \infty$. Let us recall some notations.

Let $\bar{r} = (r_1, \dots, r_m)$, $r_j > 0$, $j = 1, \dots, m$, and $l \in \mathbb{N} \cap \{0\} = \mathbb{N}_0$. Let $r = r_1$ and for a function $f \in L_1(I^m)$ suppose

$$f_{l,\bar{r}}(\bar{x}) = \sum_{\bar{s}: r_l \leq \langle \bar{s}, \bar{r} \rangle < r(l+1)} \delta_{\bar{s}}(f, \bar{x}), \quad \bar{x} \in I^m,$$

$$\|f_{l,\bar{r}}\|_A = \sum_{\bar{s}: r_l \leq \langle \bar{s}, \bar{r} \rangle < r(l+1)} \sum_{\bar{k} \in \rho(\bar{s})} |a_{\bar{k}}(f)|.$$

V.N. Temlyakov [22] considered the class

$$W_A^{\bar{r},a,b} = \left\{ f \in L_1(I^m) : \|f_{l,\bar{r}}\|_A \leq 2^{-la} l^{(\nu-1)b} \right\},$$

where $a > 0, b > 0$.

Lemma 3 (see [22]). *Let $2 \leq p < +\infty$ and $a > 0, b > 0$. Then there exists a constructive method based on the greedy algorithms which leads to the following estimation*

$$e_M(W_A^{\bar{r},a,b})_p \leq CM^{-a-\frac{1}{2}} (\log M)^{(\nu-1)(a+b)}.$$

Theorem 6. *Let $1 < p_j < 2 \leq q < +\infty$, $1 < \theta_j < \infty$, $1 \leq \tau_j \leq \infty$, $j = 1, \dots, m$, $0 < r_1 = \dots = r_\nu < r_{\nu+1} \leq \dots \leq r_m$, $r_1 > \frac{1}{p_1}$ and $\frac{r_1}{p_j} = \frac{r_j}{p_1}$, $j = 1, \dots, \nu$ and $\frac{r_1}{p_j} < \frac{r_j}{p_1}$, $j = \nu + 1, \dots, m$. Then the following relation holds*

$$e_n(S_{\bar{p}, \bar{\theta}(1), \bar{\tau}}^{\bar{r}}B)_q \asymp n^{-(r+\frac{1}{2}-\frac{1}{p_1})} (\log n)^{(\nu-1)(r-\frac{1}{p_1}) + \sum_{j=2}^{\nu} \frac{1}{\tau_j}},$$

where $\tau_j' = \frac{\tau_j}{\tau_j - 1}$, $j = 1, \dots, m$.

Proof. Let $f \in S_{\bar{p}, \bar{\theta}(1), \bar{\tau}}^{\bar{r}}B$. By the condition of the theorem $1 < p_j < 2$, $j = 1, \dots, m$, then there exists a number p_0 such that $1 < p_j < p_0 < 2$, $j = 1, \dots, m$. Therefore, by applying the Holder's inequality ($\frac{1}{p_0} + \frac{1}{p_0'} = 1$) and the Hausdorff-Young's Theorem (see [31], p. 211), we get

$$\begin{aligned} \|f_{l,\bar{r}}\|_A &\leq \sum_{\bar{s}: r_l \leq \langle \bar{s}, \bar{r} \rangle < r(l+1)} \prod_{j=1}^m 2^{(s_j-1)\frac{1}{p_0}} \left(\sum_{\bar{k} \in \rho(\bar{s})} |a_{\bar{k}}(f)|^{p_0'} \right)^{\frac{1}{p_0}} \\ &\leq C \sum_{\bar{s}: r_l \leq \langle \bar{s}, \bar{r} \rangle < r(l+1)} \prod_{j=1}^m 2^{s_j \frac{1}{p_0}} \|\delta_{\bar{s}}(f)\|_{p_0}. \end{aligned} \quad (10)$$

Now, since $p_j < p_0$, $j = 1, \dots, m$, then by applying the inequality of different metrics for trigonometric polynomials (see [28, 29]) and the Holder's inequality, we obtain from (10) the following

$$\begin{aligned} \|f_{l,\bar{r}}\|_A &\leq C \sum_{\bar{s}:rl \leq \langle \bar{s}, \bar{r} \rangle < r(l+1)} \prod_{j=1}^m 2^{s_j \frac{1}{p_j}} \|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\theta}}^* \leq \\ &\leq C \left\| \left\{ 2^{\langle \bar{s}, \bar{r} \rangle} \|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\theta}}^* \right\}_{rl \leq \langle \bar{s}, \bar{r} \rangle < r(l+1)} \right\|_{l_{\bar{r}}} \left\| \left\{ \prod_{j=1}^m 2^{s_j (\frac{1}{p_j} - r_j)} \right\}_{rl \leq \langle \bar{s}, \bar{r} \rangle < r(l+1)} \right\|_{l_{\bar{r}}}. \end{aligned} \quad (11)$$

Suppose $\gamma_j = \frac{r_j}{r_1}$, $\gamma'_j = \frac{r_j - \frac{1}{p_1}}{r_1 - \frac{1}{p_1}}$, $j = 1, \dots, m$. Then, by the assumption of the theorem, $1 = \gamma_1 = \dots = \gamma_\nu < \gamma_{\nu+1} \leq \dots \leq \gamma_m$ and $\gamma'_j = \gamma_j$, $j = 1, \dots, \nu$, $\gamma_j = \gamma'_j$, $j = \nu + 1, \dots, m$. Then using Lemma 1 we have

$$\begin{aligned} \left\| \left\{ \prod_{j=1}^m 2^{s_j (\frac{1}{p_j} - r_j)} \right\}_{rl \leq \langle \bar{s}, \bar{r} \rangle < r(l+1)} \right\|_{l_{\bar{r}}} &\leq \\ &\leq \left\| \left\{ 2^{-\langle r_1 - \frac{1}{p_1} \rangle \langle \bar{s}, \bar{\gamma}' \rangle} \right\}_{\langle \bar{s}, \bar{\gamma}' \rangle \geq l} \right\|_{l_{\bar{r}}} \leq C 2^{-l(r_1 - \frac{1}{p_1})} l^{\sum_{j=2}^{\nu} \frac{1}{\tau_j}}, \end{aligned} \quad (12)$$

where $\frac{1}{\tau_j} = 1 - \frac{1}{r_j}$, $j = 1, \dots, m$.

Next, it follows from the inequalities (11) and (12) that

$$\|f_{l,\bar{r}}\|_A \leq C 2^{-l(r_1 - \frac{1}{p_1})} l^{\sum_{j=2}^{\nu} \frac{1}{\tau_j}}$$

for any function $f \in S_{\bar{p}, \bar{\theta}(1), \bar{r}}^{\bar{r}} B$ in the case $r_1 > \frac{1}{p_1}$. Thus, the function $f \in S_{\bar{p}, \bar{\theta}(1), \bar{r}}^{\bar{r}} B$ belongs to the class $W_A^{\bar{r}, a, b}$ in the case $a = r_1 - \frac{1}{p_1}$ and $b = \frac{1}{\nu-1} \sum_{j=2}^{\nu} \frac{1}{\tau_j}$. Hence, by Lemma 3, we have

$$e_n(S_{\bar{p}, \bar{\theta}(1), \bar{r}}^{\bar{r}} B)_q \leq C e_n(W_A^{\bar{r}, a, b})_q \leq C n^{-(r + \frac{1}{2} - \frac{1}{p_1})} (\log n)^{(\nu-1)(r - \frac{1}{p_1}) + \sum_{j=2}^{\nu} \frac{1}{\tau_j}}$$

in the case $1 < p_j < 2 \leq q < +\infty$, $1 < \theta_j < \infty$, $1 \leq \tau_j \leq \infty$, $j = 1, \dots, m$, $r_1 > \frac{1}{p_1}$. It proves the upper bound estimation.

Now let's consider the lower bound estimation. Since $2 \leq q < \infty$, then $\|f\|_2 \leq C \|f\|_q$, $f \in L_q(I^m)$. Therefore

$$e_n(S_{\bar{p}, \bar{\theta}(1), \bar{r}}^{\bar{r}} B)_q \geq C e_n(S_{\bar{p}, \bar{\theta}(1), \bar{r}}^{\bar{r}} B)_2.$$

Now, by letting $\theta_j = q = 2$, $j = 1, \dots, m$, in Theorem 5, we get

$$e_n(S_{\bar{p}, \bar{\theta}(1), \bar{r}}^{\bar{r}} B)_2 \geq C n^{-(r + \frac{1}{2} - \frac{1}{p_1})} (\log n)^{(\nu-1)(r + \frac{1}{2} - \frac{1}{p_1}) + \sum_{j=2}^{\nu} (\frac{1}{2} - \frac{1}{\tau_j})}.$$

Hence

$$\begin{aligned} e_n(S_{\bar{p}, \bar{\theta}(1), \bar{\tau}} B)_q &\geq C n^{-(r+\frac{1}{2}-\frac{1}{p_1})} (\log n)^{(\nu-1)(r+\frac{1}{2}-\frac{1}{p_1})+\sum_{j=2}^{\nu}(\frac{1}{2}-\frac{1}{\tau_j})} \geq \\ &\geq C n^{-(r+\frac{1}{2}-\frac{1}{p_1})} (\log n)^{(\nu-1)(r-\frac{1}{p_1})+\sum_{j=2}^{\nu} \frac{1}{\tau_j}}. \end{aligned}$$

It proves the theorem.

Remark. Note that the upper bound estimation in the case $p_j = \theta_j = p$, $j = 1, \dots, m$, in Theorem 6 was proved by V.N. Temlyakov [22] using the constructive method.

In [32] obtained the exact estimation of the best M -term approximations of Nikol'skii's and Besov's classes in the Lebesgue space with the mixed norm.

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