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# GENERALIZED ADJOINT ACTIONS

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ABSTRACT. The aim of this paper is to generalize the classical formula  $e^x y e^{-x} = \sum_{k \geq 0} \frac{1}{k!} (ad\ x)^k(y)$ .

## 1. NOTATION AND MAIN RESULTS

One of the most fundamental tools in Lie theory, the adjoint action of Lie groups on their Lie algebras, is based on the following formula:

$$(1.1) \quad e^x y e^{-x} = e^{ad\ x}(y) = \sum_{k \geq 0} \frac{1}{k!} (ad\ x)^k(y) ,$$

where  $(ad\ x)^k(y) = [x, [x, \dots, [x, y], \dots]]$  and  $[a, b] = ab - ba$ .

The aim of this paper is to generalize (1.1) by replacing  $e^t$  with any formal power series

$$(1.2) \quad f = f(t) = 1 + \sum_{k \geq 1} a_k t^k$$

with central coefficients.

For any formal power series (1.2) define a polynomials

$$P_k(t) = P_{f,k}(t) = (-1)^k \det \begin{pmatrix} 1 & a_1 t & a_2 t^2 & \dots & a_k t^k \\ 1 & a_1 & a_2 & \dots & a_k \\ 0 & 1 & a_1 & \dots & a_{k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & a_1 \end{pmatrix}$$

for  $k = 0, 1, 2, \dots$  (with the convention that  $P_0(t) = 1$ ). Clearly,  $P_k(1) = 0$  for  $k \geq 1$ .

The following result is, probably, well-known (for readers' convenience, we prove it in Section 2).

**Theorem 1.1.**  $\frac{f(tx)}{f(x)} = \sum_{k \geq 0} P_{f,k}(t) \cdot x^k$ . In particular,  $P_{f,k}(st) = \sum_{i=0}^k P_{f,i}(s) P_{f,k-i}(t) t^i$ .

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Furthermore, for any algebra  $\mathcal{A}$ , a subset  $\mathbf{q} = \{q_1, \dots, q_k\} \subset Z(\mathcal{A})$ ,  $x, y \in \mathcal{A}$ , and  $k \geq 1$  define

$$(\text{ad } x)^{\mathbf{q}}(y) = [x, [x, \dots, [x, y]_{q_1}, \dots]_{q_{k-1}}]_{q_k}$$

where  $[a, b]_q = ab - qba$ . It is easy to see that

$$(\text{ad } x)^{\mathbf{q}}(y) = \sum_{j=0}^k (-1)^j e_j(q_1, \dots, q_k) \cdot x^{k-j} y x^j,$$

where  $e_j(q_1, \dots, q_k)$  is the  $j$ -th elementary symmetric function.

**Theorem 1.2.** *Let  $\mathcal{A}$  be any topological algebra and suppose that  $f$  is any power series (1.2) with  $a_k \neq 0$  for  $k \geq 1$ . Then*

$$(1.3) \quad f(x) y f(x)^{-1} = y + \sum_{k \geq 1} a_k (\text{ad } x)^{\mathbf{q}_k}(y),$$

for any  $x, y \in \mathcal{A}$ , where  $\mathbf{q}_k = \{q_{1k}, \dots, q_{kk}\}$  is the set of roots of  $P_{f,k}(t)$ .

**Remark 1.3.** A formula for  $f(x) y f(x)^{-1}$  without assumption that  $a_k \neq 0$  is given in Proposition 2.3.

It is easy to see that if  $a_k = \frac{1}{k!}$  for all  $k$ , then  $P_k(t) = \frac{(t-1)^k}{k!}$  which immediately recovers (1.1). Suppose now that  $a_k = \frac{1}{[k]_q!}$  for all  $k$ , where  $k_q! = [1]_q \cdots [k]_q$  is the  $q$ -factorial and  $[\ell]_q = 1 + q + \cdots + q^{\ell-1}$ . We will show (Proposition 2.5) that  $P_{f,k}(t) = \frac{(t-1)(t-q) \cdots (t-q^{k-1})}{[k]_q!}$  for  $f(t) = e_q^t = \sum_{k \geq 0} \frac{t^k}{[k]_q!}$ , therefore, recover the following famous result (see e.g., [3]).

**Theorem 1.4.** *Let  $e_q^x = \sum_{k \geq 0} \frac{x^k}{[k]_q!}$  be the  $q$ -exponential. Then  $e_q^x \cdot y \cdot (e_q^x)^{-1} = \sum_{k \geq 0} \frac{1}{[k]_q!} (\text{ad } x)^{\{1, q, \dots, q^{k-1}\}}(y)$ .*

On the other hand, combining Theorem 1.1 and Proposition 2.5, we recover the following well-known properties of  $q$ -exponentials and  $q$ -binomials:

$$e_q^{q^n x} = e_q^x \left( 1 + \sum_{k=1}^n \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{[k]_q!} x^k \right)$$

for  $n \geq 0$ , in particular,

$$e_q^{q^n x} = e_q^x \cdot (1 + (q-1)x)$$

and

$$1 + \sum_{k=1}^n \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{[k]_q!} x^k = \prod_{i=1}^n (1 + (q-1)q^{i-1}x).$$

We conclude with a curious observation that the polynomials  $P_{f,k}(t)$  are related to the Hall-Littlewood symmetric polynomials.

**Proposition 1.5.** *Suppose that  $f(t) = \prod_{k \geq 1} (1 - x_k t)$ . Then*

$$P_{f,k}(t) = Q_{(k)}(\mathbf{x}; t)$$

for all  $k \geq 0$ , where  $\mathbf{x} = \{x_k, k \geq 0\}$  is viewed an infinite set of variables,  $Q_\lambda(\mathbf{x}; t)$  is Hall-Littlewood polynomial ([2, Section 3.2]), and  $(k)$  is a horizontal row of length  $k$ . In particular,

$$Q_{(k)}(\mathbf{x}; t) = (-1)^k \det \begin{pmatrix} 1 & -e_1 t & e_2 t^2 & \dots & (-1)^k e_k t^k \\ 1 & e_1 & e_2 & \dots & (-1)^k e_k \\ 0 & 1 & -e_1 & \dots & (-1)^{k-1} e_{k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & -e_1 \end{pmatrix}$$

for all  $k \geq 0$ , where  $e_k = e_k(\mathbf{x})$  is the  $k$ -th elementary symmetric function.

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## 2. PROOFS

**Proof of Theorem 1.1.** We need the following well-known fact (attributed to Wronski, see e.g., [1]).

**Lemma 2.1.** *Let  $f$  be any formal power series (1.2). Then  $\frac{1}{f(t)} = 1 + \sum_{k \geq 1} D_k(f) t^k$ ,*

where

$$D_k(f) = (-1)^k \det \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_k \\ 1 & a_1 & a_2 & \dots & a_{k-1} \\ 0 & 1 & a_1 & \dots & a_{k-2} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & a_1 \end{pmatrix}$$

(with the convention  $D_0(f) = 1$ ).

The following generalization of Lemma 2.1 is, apparently, well-known (for readers' convenience we prove it here).

**Lemma 2.2.** *Let  $f(t) = 1 + \sum_{k \geq 1} a_k t^k$ ,  $g(t) = 1 + \sum_{k \geq 1} b_k t^k$  be formal power series.*

Then

$$\frac{g(t)}{f(t)} = \sum_{k \geq 0} D_k(g, f) t^k,$$

where  $D_k(g, f) = (-1)^k \det \begin{pmatrix} 1 & b_1 & \dots & b_{k-1} & b_k \\ 1 & a_1 & \dots & a_{k-1} & a_k \\ 0 & 1 & \dots & a_{k-2} & a_{k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & a_1 \end{pmatrix}$  (with the convention  $D_0(g, f) = 1$ ).

**Proof.** Indeed, using Lemma 2.1, we obtain (with the convention  $b_0 = 1$ ):

$$\frac{g(t)}{f(t)} = g(t) \cdot \frac{1}{f(t)} = \left( \sum_{i \geq 0} b_i t^i \right) \left( \sum_{j \geq 0} D_j(f) t^j \right) = \sum_{k \geq 0} d_k t^k$$

where

$$d_k = \sum_{i=0}^k b_i D_{k-i}(f) = (-1)^k \det \begin{pmatrix} b_0 & b_1 & \dots & b_{k-1} & b_k \\ 1 & a_1 & \dots & a_{k-1} & a_k \\ 0 & 1 & \dots & a_{k-2} & a_{k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & a_1 \end{pmatrix}$$

by Cramer rule because  $b_i D_{k-i}(f) = (-1)^k \det \begin{pmatrix} 0 & 0 & \dots & b_i & \dots & 0 \\ 1 & a_1 & \dots & a_i & \dots & a_k \\ 0 & 1 & \dots & a_{i+1} & \dots & a_{k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & a_1 & a_2 \\ 0 & 0 & \dots & \dots & 1 & a_1 \end{pmatrix}$ .

The lemma is proved.  $\square$

Then taking  $b_k = a_k t^k$  for  $k \geq 1$  in Lemma 2.2, we obtain the first assertion of Theorem 1.1.

To prove the second assertion of the theorem, compute  $\frac{f(stx)}{f(x)}$  in two ways, using the first assertion:

$$\frac{f(stx)}{f(x)} = \sum_{k \geq 0} P_{f,k}(st) \cdot x^k$$

and  $\frac{f(stx)}{f(x)} = \frac{f(stx)}{f(tx)} \cdot \frac{f(tx)}{f(x)} = \left( \sum_{i \geq 0} P_{f,i}(s) \cdot (tx)^i \right) \left( \sum_{j \geq 0} P_{f,j}(t) \cdot x^j \right)$ . Compar-

ing the coefficients of  $x^k$  in both series, we obtain the second assertion of Theorem 1.1.  $\square$

**Proof of Theorem 1.2.** We need the following result.

**Proposition 2.3.** *For any power series  $f$  as in (1.2) one has:*

$$(a) \ P_{f,k}(t) = \sum_{j=0}^k a_{k-j} D_j(f) \cdot t^j \text{ for all } k \geq 0.$$

(b)  $f(x)yf(x)^{-1} = y + z_1 + z_2 + \dots$ , where  $z_k = \sum_{i=0}^k a_i D_{k-i}(f) \cdot x^i y x^{k-i}$  for all  $k \geq 1$ .

**Proof.** Prove (a). Indeed, using Lemma 2.1, we obtain:

$$\frac{f(tx)}{f(x)} = \sum_{i,j \geq 0} (a_i t^i x^i) (D_j(f)x^j) = \sum_{k \geq 0} \left( \sum_{i=0}^k a_i D_{k-i}(f) \cdot t^i \right).$$

This together with Theorem 1.1 proves (a).

Prove (b) now. Indeed,

$$f(x)yf(x)^{-1} = \sum_{i,j \geq 0} (a_i x^i) y (D_j(f)x^j) = \sum_{k \geq 0} \left( \sum_{j=0}^k a_i D_{k-i}(f) \cdot x^i y x^{k-i} \right)$$

This proves (b). □

Now we can finish the proof of Theorem 1.2. Indeed, suppose that  $P_{f,k}(t)$  is factored as

$$P_{f,k}(t) = a_k(t - q_{11}) \cdots (t - q_{1k}).$$

Then, by Proposition 2.3(a),  $a_i D_{k-i} = a_k (-1)^{k-i} e_{k-i}(q_{11}, \dots, q_{1k})$  for  $i = 0, \dots, k$ . Therefore, in the notation of Proposition 2.3(b),

$$z_k = \sum_{i=0}^k a_k (-1)^{k-i} e_{k-i}(q_{11}, \dots, q_{1k}) \cdot x^i y x^{k-i} = a_k (ad x)^{\mathbf{q}_k}(y)$$

for all  $k \geq 1$ , which together with Proposition 2.3(b) verifies (1.3).

Theorem 1.2 is proved. □

**Proposition 2.4.**  $P_{e_q^t, k}(t) = \frac{(t-1)(t-q) \cdots (t-q^{k-1})}{[k]_q!}$  for all  $k \geq 1$ .

**Proof.** It suffices to show that  $P_{e_q^t, k}(q^a) = 0$  for all  $0 \leq a < k$ . We proceed by induction in such pairs  $(a, k)$ . If  $a = 0$ , then we have nothing to prove since  $P_{f,k}(1) = 0$  for all  $f$ .

Using Theorem 1.1, we obtain:

$$P_{f,k}(q^a) = \sum_{i=0}^k P_{f,i}(q^b) P_{f,k-i}(q^{a-b}) q^{(a-b)i}.$$

Taking  $f = e_q^t$ ,  $1 \leq b \leq a < k$ , and using the inductive hypothesis, this gives  $P_{f,k}(q^a) = 0$  for any  $1 \leq a < k$ .

The proposition is proved. □

**Corollary 2.5.** For all  $k \geq 1$  one has:  $\det \begin{pmatrix} 1 & \frac{t}{[1]_q} & \frac{t^2}{[2]_q} & \cdots & \frac{t^k}{[k]_q} \\ 1 & \frac{1}{[1]_q} & \frac{1}{[2]_q} & \cdots & \frac{1}{[k]_q} \\ 0 & 1 & \frac{1}{[1]_q} & \cdots & \frac{1}{[k-1]_q} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & \frac{1}{[1]_q} \end{pmatrix} =$

$$\frac{(1-t)(q-t)\cdots(q^{k-1}-t)}{[k]_q!}.$$

**Proof of Proposition 1.5.** Indeed, if  $f(t)$  is as in Proposition 1.5, then

$$\frac{f(tu)}{f(u)} = \prod_{k \geq 1} \frac{1 - x_k t u}{1 - x_k u} = \sum_{k \geq 0} Q_{(k)}(\mathbf{x}; t) u^k$$

by [2, Equations (2.10) and (2.13)]. This and Theorem 1.1 imply that  $P_{f,k} = Q_{(k)}(\mathbf{x}; t)$  for all  $k \geq 0$ , which proves the first assertion of Proposition 1.5.

To prove the second assertion, note that  $a_k = (-1)^k e_k(\mathbf{x})$  for all  $k \geq 0$  because of the well-known formula (see e.g., [2, Section 1.2]):

$$\prod_{k \geq 1} (1 - x_k t) = \sum_{k \geq 0} (-1)^k e_k(\mathbf{x}) t^k.$$

This and the first assertion of Proposition 1.5 imply the second assertion of the proposition.  $\square$

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