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Highest weight categories and recollements

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HIGHEST WEIGHT CATEGORIES AND RECOLLEMENTS

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Abstract. We provide several equivalent descriptions of a highest weight category using recollements of abelian categories. The settings for this are abelian length categories and k-linear exact categories, where k denotes any commutative base ring. Also, we discuss Ringel duality and establish a precise connection with Serre duality.

Contents

1. Introduction

Highest weight categories and quasi-hereditary algebras were introduced in a series of papers by Cline, Parshall, and Scott [5, 26, 33]; see also the work of Dlab and Ringel [7, 8, 29]. The intimate connection between highest weight categories and recollements of derived categories was noticed right from the beginning.

In this note we provide several equivalent descriptions of a highest weight category, using the following concepts:

– recollements of abelian categories,
– chains of heredity ideals,
– filtrations via standard objects,
– exceptional sequences.

Thus *we characterise highest weight categories in terms of recollements of abelian categories*; see Theorems 5.2 and 6.6 for precise formulations. This makes explicit the underlying intuition that a highest weight category is built in a specific way from semisimple categories.

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For another interesting approach towards highest weight categories via A_∞-categories and boxes we refer to recent work in [22].

We consider two settings:

– abelian length categories, and
– k-linear exact categories, where k denotes a commutative ring.

In both cases we assume that the set of weights is finite and totally ordered, leaving the generalisation to locally finite posets to the interested reader.

A highest weight category is determined by its standard objects (usually denoted by Δ_i, where the index i refers to the weight). An efficient way to formulate this for a k-linear highest weight category is given by the following result; it is an immediate consequence of Theorem 6.5 and the analogue of a result of Dlab and Ringel [8] for finite dimensional algebras over a field.

Theorem 1.1. *Let $\mathcal{A}$ be a k-linear exact category. Suppose there is a k-algebra Λ that is finitely generated projective over k such that $\mathcal{A}$ is equivalent to the category of Λ-modules that are finitely generated projective over k. Then $\mathcal{A}$ is a k-linear highest weight category if and only if there are objects $\Delta_1, \dots, \Delta_n$ in $\mathcal{A}$ having the following properties:*

(1) $\operatorname{Hom}_{\mathcal{A}}(\Delta_i, \Delta_j) = 0$ *for all* $i > j$.
(2) $\operatorname{End}_{\mathcal{A}}(\Delta_i) \cong k$ *for all* i.
(3) $\operatorname{Ext}^1_{\mathcal{A}}(\Delta_i, \Delta_j) = 0$ *for all* $i \geq j$.
(4) *A projective generator of* $\operatorname{Filt}(\Delta_1, \dots, \Delta_n)$ *is also one for* $\mathcal{A}$. □

A k-algebra is by definition k-split quasi-hereditary if it arises as the endomorphism ring of a projective generator of a k-linear highest weight category.

The description of a highest weight category via its sequence of standard objects in Theorem 1.1 suggests a close connection with the concept of an exceptional sequence, as introduced in the study of vector bundles [3, 17, 18, 31]. It seems important to point out that both concepts are basically equivalent, even though their origins are quite different. To this end we say that an exceptional sequence $(E_1, \dots, E_n)$ of an exact category $\mathcal{A}$ is *strictly full* if the inclusion $\operatorname{Filt}(E_1, \dots, E_n) \to \mathcal{A}$ induces, up to direct summands, a triangle equivalence $\mathbf{D}^b(\operatorname{Filt}(E_1, \dots, E_n)) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A})$. Then we have the following result; see Theorem 6.11.

Theorem 1.2. *Let $\mathcal{A}$ be a k-linear highest weight category. Then the sequence of standard objects $(\Delta_1, \dots, \Delta_n)$ forms a strictly full exceptional sequence in $\mathcal{A}$.*

Conversely, let $\mathcal{A}$ be an idempotent complete k-linear exact category and suppose that Ext-groups are finitely generated over k. If $(E_1, \dots, E_n)$ is a strictly full exceptional sequence in $\mathcal{A}$ such that the endomorphism ring of a projective generator of $\operatorname{Filt}(E_1, \dots, E_n)$ *is finitely generated projective over k, then this identifies with the sequence of standard objects of a highest weight category $\mathcal{A}'$ via a triangle equivalence* $\mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A}')$.

Note that a full exceptional sequence need not be strictly full; see Example 6.12.

An important feature of highest weight categories is Ringel duality which is based on the notion of a characteristic tilting object [29]. This is discussed at the end of this note. The following result formulates the close connection between Ringel and Serre duality; see Theorem 7.15.

Theorem 1.3. *Let Λ be a k-split quasi-hereditary algebra with characteristic tilting module T and set $\Gamma = \operatorname{End}_\Lambda(T)$. Then $\operatorname{Hom}_k(T,k)$ is a characteristic tilting module over Γ and the composite of tilting functors*

$$\mathbf{D}^{\mathrm{perf}}(\Lambda) \xrightarrow{-\otimes^{\mathbf{L}}_{\Lambda}\operatorname{Hom}_k(T,k)} \mathbf{D}^{\mathrm{perf}}(\Gamma) \xrightarrow{\quad -\otimes^{\mathbf{L}}_{\Gamma} T \quad} \mathbf{D}^{\mathrm{perf}}(\Lambda)$$

is a Serre functor. □

Categories of strict polynomial functors [12] form an interesting class of examples of highest weight categories. This is explained in some detail in [24] and provided the motivation for this work; it complements the fact that Schur algebras are split quasi-hereditary [6]. We refer to [4, 11] for recent applications in the study of derived categories of Grassmannians in arbitrary characteristic.

These notes are organised as follows. In §2 we begin with an informal discussion of the concept of a highest weight category, and §3 provides basic material about recollements. The sections §4, §5, and §6 contain the main results and are independent from each other. Section §7 discusses Ringel duality and continues §6.

Given a ring Λ (associative with identity), we consider the category $\operatorname{Mod}\Lambda$ of right Λ-modules. We write $\operatorname{mod}\Lambda$ for the full subcategory of finitely presented Λ-modules and $\operatorname{proj}\Lambda$ for the full subcategory of finitely generated projective Λ-modules.

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2. The concept of a highest weight category

We explain informally the concept of a highest weight category, restricting ourselves to the case that the set of weights is finite and totally ordered.

A highest weight category is an exact category $\mathcal{A}$ together with a distinguished set of standard objects $\Delta_1, \dots, \Delta_n$ satisfying

$$\operatorname{Hom}_{\mathcal{A}}(\Delta_i, \Delta_j) = 0 \quad (i > j) \qquad \text{and} \qquad \operatorname{Ext}^1_{\mathcal{A}}(\Delta_i, \Delta_j) = 0 \quad (i \geq j).$$

There are further finiteness assumptions which are needed to make the theory work; they are added along the way.

Let $\operatorname{Filt}(\Delta_1, \dots, \Delta_n)$ denote the full subcategory of objects X in $\mathcal{A}$ that admit a finite filtration $0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_t = X$ such that each factor X_i/X_{i-1} is isomorphic to an object of the form Δ_j.

For each $1 \leq i \leq n$ the object Δ_i is projective in $\operatorname{Filt}(\Delta_1, \dots, \Delta_i)$, and therefore each X in $\operatorname{Filt}(\Delta_1, \dots, \Delta_i)$ fits into an exact sequence $0 \to X'' \to X \to X' \to 0$ such that $X'' \cong \Delta_i^r$ for some $r \geq 0$ and X' belongs to $\operatorname{Filt}(\Delta_1, \dots, \Delta_{i-1})$. In fact, the assignment $X \mapsto X'$ yields a left adjoint of the inclusion

$$\operatorname{Filt}(\Delta_1, \dots, \Delta_{i-1}) \longrightarrow \operatorname{Filt}(\Delta_1, \dots, \Delta_i).$$

Define inductively a sequence of full subcategories

$$0 = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_n = \mathcal{A}$$

by setting

$$\mathcal{A}_{i-1} = \{X \in \mathcal{A}_i \mid \operatorname{Hom}_{\mathcal{A}}(\Delta_i, X) = 0\}$$

and note that

$$\operatorname{Filt}(\Delta_1, \dots, \Delta_i) \subseteq \mathcal{A}_i.$$

The exact category $\mathcal{A}_i$ admits a projective generator P_i which one constructs inductively as a projective generator of $\operatorname{Filt}(\Delta_1, \dots, \Delta_i)$. An appropriate Ext-finiteness condition guarantees that $\operatorname{Ext}^1_{\mathcal{A}_i}(P_{i-1}, \Delta_i)$ is finitely generated as an $\operatorname{End}_{\mathcal{A}}(\Delta_i)^{\mathrm{op}}$-module, and therefore P_i fits into an exact sequence $0 \to \Delta_i^{r_i} \to P_i \to P_{i-1} \to 0$.

Set $\Lambda_i = \operatorname{End}_{\mathcal{A}}(P_i)$. Then the functor $\operatorname{Hom}_{\mathcal{A}_i}(P_i, -)$ induces an exact embedding

$$\mathcal{A}_i \longrightarrow \operatorname{Mod} \Lambda_i, \quad X \longmapsto \bar{X}.$$

Note that P_i depends on choices but $\operatorname{Mod} \Lambda_i$ does not because we can identify it with the category of left exact functors $\mathcal{A}_i^{\mathrm{op}} \to \mathrm{Ab}$.

The left adjoint of the inclusion $\operatorname{Filt}(\Delta_1, \dots, \Delta_{i-1}) \to \operatorname{Filt}(\Delta_1, \dots, \Delta_i)$ induces a surjective ring homomorphism $\Lambda_i \to \Lambda_{i-1}$. Restriction along it makes the following square commutative

$$\begin{tikzcd} \mathcal{A}_{i-1} \arrow[r] \arrow[d] & \mathcal{A}_i \arrow[d] \\ \operatorname{Mod} \Lambda_{i-1} \arrow[r] & \operatorname{Mod} \Lambda_i \end{tikzcd}$$

and induces a recollement of abelian categories

$$\begin{tikzcd} \operatorname{Mod} \Lambda_{i-1} \arrow[r, tail] & \operatorname{Mod} \Lambda_i \arrow[l, two heads, shift right=2] \arrow[l, two heads, shift left=2] \arrow[r, two heads] & \operatorname{Mod} \Gamma_i \arrow[l, tail, shift right=2] \arrow[l, tail, shift left=2] \end{tikzcd}$$

where $\Gamma_i = \operatorname{End}_{\mathcal{A}}(\Delta_i)$ and the functor $\operatorname{Mod} \Lambda_i \to \operatorname{Mod} \Gamma_i$ is given by $\operatorname{Hom}_{\Lambda_i}(\bar{\Delta}_i, -)$. The recollement is homological by construction, which means by definition that $\operatorname{Mod} \Lambda_{i-1} \to \operatorname{Mod} \Lambda_i$ induces isomorphisms

$$\operatorname{Ext}^p_{\Lambda_{i-1}}(-, -) \xrightarrow{\sim} \operatorname{Ext}^p_{\Lambda_i}(-, -) \qquad (p \geq 0).$$

The above recollement extends the sequence of functors

$$\mathcal{A}_{i-1} \rightarrowtail \mathcal{A}_i \twoheadrightarrow \operatorname{proj} \Gamma_i$$

given by $\mathrm{Hom}_{\mathcal{A}_i}(\Delta_i, -)$. The left adjoint of this functor sends Γ_i to the standard object Δ_i, while some further assumption on $\mathcal{A}$ ensures that a right adjoint exists, sending then Γ_i to the costandard object ∇_i.

The definition of a highest weight category imposes assumptions on the endomorphisms rings of the standard objects. If $\mathcal{A}$ is an abelian length category, then each Γ_i is a division ring. If $\mathcal{A}$ is a k-linear exact category over some commutative ring k, then each Γ_i is isomorphic to k.

The endomorphism ring of a projective generator of a highest weight category is by defintion a quasi-hereditary algebra.

3. Recollements

Recollements of abelian and triangulated categories. We recall the definition of a recollement using the standard notation [2, 1.4]. In fact, any recollement is built from two diagrams involving 'localisation' and 'colocalisation' [14, 32].

Definition 3.1. A *localisation sequence* of abelian (triangulated) categories is a diagram of functors

$$\begin{tikzcd} \mathcal{A}' \arrow[r, shift left, tail, "i_!"] & \mathcal{A} \arrow[l, shift left, two heads, "i^!"] \arrow[r, shift left, two heads, "j^*"] & \mathcal{A}'' \arrow[l, shift left, tail, "j_*"] \end{tikzcd} \tag{3.1}$$

satisfying the following conditions:

(1) $i_!$ and j^* are exact functors of abelian (triangulated) categories.
(2) $(i_!, i^!)$ and (j^*, j_*) are adjoint pairs.
(3) $i_!$ and j_* are fully faithful functors.
(4) An object in $\mathcal{A}$ is annihilated by j^* iff it is in the essential image of $i_!$.

Note that condition (3) admits an equivalent formulation; see [13, I.1.3]. In the presence of (2), the functor $i_!$ is fully faithful iff the unit $\mathrm{id}_{\mathcal{A}'} \to i^! i_!$ is an isomorphism. Also, the functor j_* is fully faithful iff the counit $j^* j_* \to \mathrm{id}_{\mathcal{A}''}$ is an isomorphism.

Definition 3.2. A *colocalisation sequence* of abelian (triangulated) categories is a diagram of functors

$$\begin{tikzcd} \mathcal{A}' \arrow[r, shift right, tail, "i_*"'] & \mathcal{A} \arrow[l, shift right, two heads, "i^*"'] \arrow[r, shift right, two heads, "j^!"'] & \mathcal{A}'' \arrow[l, shift right, tail, "j_!"'] \end{tikzcd} \tag{3.2}$$

satisfying the following conditions:

(1) i_* and $j^!$ are exact functors of abelian (triangulated) categories.
(2) (i^*, i_*) and $(j_!, j^!)$ are adjoint pairs.
(3) i_* and $j_!$ are fully faithful functors.
(4) An object in $\mathcal{A}$ is annihilated by $j^!$ iff it is in the essential image of i_*.

Definition 3.3. A *recollement* of abelian (triangulated) categories is a diagram of functors

$$
\begin{tikzcd}
\mathcal{A}' \arrow[r, "i_*=i_!" description] & \mathcal{A} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*" description] & \mathcal{A}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}
\tag{3.3}
$$

such that the subdiagram (3.1) is a localisation sequence and the subdiagram (3.2) is a colocalisation sequence.

The recollement or one of its subdiagrams (3.1) or (3.2) is called *homological* if the functor i_* induces for all $X, Y \in \mathcal{A}'$ and $p \geq 0$ isomorphisms

$$\operatorname{Ext}^p_{\mathcal{A}'}(X,Y) \xrightarrow{\sim} \operatorname{Ext}^p_{\mathcal{A}}(i_*(X), i_*(Y)).$$

The terminology follows that used in [27], where i_* is called *homological embedding.*

The above definitions also make sense for an exact category [28].

Given a colocalisation sequence (3.2) and an object X in $\mathcal{A}$, we have the counit $j_!j^!(X) \to X$ and the unit $X \to i_*i^*(X)$. These fit into an exact sequence

$$j_!j^!(X) \longrightarrow X \longrightarrow i_*i^*(X) \longrightarrow 0 \qquad (\mathcal{A} \text{ abelian})$$

and an exact triangle

$$j_!j^!(X) \longrightarrow X \longrightarrow i_*i^*(X) \longrightarrow \qquad (\mathcal{A} \text{ triangulated}).$$

Recollements with semisimple quotient. In this work we are mainly interested in recollements of abelian categories that are homological and have a semisimple quotient. A basic tool are derived categories.

For an abelian category $\mathcal{A}$ let $\mathbf{D}^b(\mathcal{A})$ denote its bounded derived category. An exact functor $f\colon \mathcal{A} \to \mathcal{B}$ between abelian categories induces a canonical functor $\mathbf{D}^b(\mathcal{A}) \to \mathbf{D}^b(\mathcal{B})$ by sending a complex X to $f(X)$.

Lemma 3.4. *Fix a colocalisation sequence* (3.2) *of abelian categories. Suppose that $\mathcal{A}$ has enough projective objects and that $\mathcal{A}''$ is semisimple. Then the following are equivalent:*

(1) *The sequence* (3.2) *is homological.*
(2) *There is a colocalisation sequence of triangulated categories*

$$
\begin{tikzcd}
\mathbf{D}^b(\mathcal{A}') \arrow[r, shift right, "i_*"'] & \mathbf{D}^b(\mathcal{A}) \arrow[l, shift right] \arrow[r, shift right, "j^!"'] & \mathbf{D}^b(\mathcal{A}'') \arrow[l, shift right]
\end{tikzcd}.
$$

(3) *For every projective object X in $\mathcal{A}$ the counit of the adjoint pair $(j_!, j^!)$ and the unit of the adjoint pair (i^*, i_*) give an exact sequence*

$$0 \longrightarrow j_!j^!(X) \longrightarrow X \longrightarrow i_*i^*(X) \longrightarrow 0.$$

(4) *For every projective object X in $\mathcal{A}$ there is an exact sequence*

$$0 \longrightarrow X'' \longrightarrow X \longrightarrow X' \longrightarrow 0$$

with X' in the image of i_* and X'' in the image of $j_!$.

Proof. (1) ⇒ (2): First observe that the functor $\mathbf{D}^b(\mathcal{A}') \to \mathbf{D}^b(\mathcal{A})$ induced by i_* is fully faithful iff the sequence is homological; see [20, Lemme 2.1.3]. The functor $j_!$ is exact since $\mathcal{A}''$ is semisimple, and therefore $(j_!, j^!)$ induces an adjoint pair of functors between the derived categories. It follows that i_* admits a left adjoint $\mathbf{L}i^*$. For X in $\mathbf{D}^b(\mathcal{A})$ the object $\mathbf{L}i^*(X)$ is determined by the exact triangle

$$j_!j^!(X) \longrightarrow X \longrightarrow i_*(\mathbf{L}i^*(X)) \longrightarrow$$

which is given by the counit for $(j_!, j^!)$.

(2) ⇒ (3): For X in $\mathcal{A}$ the unit $\eta_X \colon X \to i_*i^*(X)$ is an epimorphism. Viewing X as a complex concentrated in degree zero, we obtain an exact triangle

$$\operatorname{Ker}\eta_X \longrightarrow X \longrightarrow i_*i^*(X) \longrightarrow$$

in $\mathbf{D}^b(\mathcal{A})$. On the other hand, we have an exact triangle

$$j_!j^!(X) \longrightarrow X \longrightarrow i_*i^*(X) \longrightarrow$$

when X is projective, because the left adjoint of i_* sends a complex of projectives P to $i^*(P)$. Thus taking homology yields the exact sequence

$$0 \longrightarrow j_!j^!(X) \longrightarrow X \longrightarrow i_*i^*(X) \longrightarrow 0.$$

(3) ⇒ (4): Clear.

(4) ⇒ (3): The sequence in (4) identifies with the one in (3). This follows by applying $j_!j^!$ and i_*i^* to the sequence in (4), because we obtain the following commutative squares.

$$\begin{tikzcd} j_!j^!(X'') \arrow[r, "\sim"] \arrow[d, "\wr"'] & j_!j^!(X) \arrow[d] \\ X'' \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d, "\wr"] \\ i_*i^*(X) \arrow[r, "\sim"] & i_*i^*(X) \end{tikzcd}$$

(3) ⇒ (1): The functor i_* extends to an exact functor $\mathbf{D}^-(\mathcal{A}') \to \mathbf{D}^-(\mathcal{A})$ and we denote by $\mathbf{L}i^*$ its left adjoint, which takes by definition a complex P of projectives to $i^*(P)$. For an object X in $\mathcal{A}'$ the counit $\varepsilon_X \colon \mathbf{L}i^*i_*(X) \to X$ induces a morphism

$$\operatorname{Ext}^*_{\mathcal{A}'}(X, -) \longrightarrow \operatorname{Ext}^*_{\mathcal{A}'}(\mathbf{L}i^*i_*(X), -) \xrightarrow{\sim} \operatorname{Ext}^*_{\mathcal{A}}(i_*(X), i_*-),$$

and this is an isomorphism for all X if ε_X is an isomorphism for every projective object X in $\mathcal{A}'$.

Now fix a projective object P in $\mathcal{A}$ and suppose we have an exact sequence

$$0 \longrightarrow j_!j^!(P) \longrightarrow P \longrightarrow i_*i^*(P) \longrightarrow 0$$

in $\mathcal{A}$. Applying $\mathbf{L}i^*$ yields an isomorphism

$$i^*(P) = \mathbf{L}i^*(P) \xrightarrow{\sim} \mathbf{L}i^*(i_*i^*(P))$$

since $i^*j_! = 0$ and $j_!j^!(P)$ is projective. This gives an inverse for $\varepsilon_{i^*(P)}$, and it remains to observe that every projective object in $\mathcal{A}'$ is a direct summand of one of the form $i^*(P)$. □

Recall that a projective object P of an abelian (or exact) category is a *projective generator* if for every object X there is an exact sequence $0 \to N \to P^r \to X \to 0$ for some positive integer r.

Lemma 3.5. *Fix a homological colocalisation sequence* (3.2) *of abelian categories and suppose that $\mathcal{A}''$ is semisimple. Then the following are equivalent:*

(1) *There is a projective generator in $\mathcal{A}$.*
(2) *There are projective generators P' in $\mathcal{A}'$ and P'' in $\mathcal{A}''$ such that*
$$\operatorname{Ext}^1_{\mathcal{A}}(i_*(P'), j_!(P''))$$
is finitely generated over $\operatorname{End}_{\mathcal{A}''}(P'')^{\mathrm{op}}$.

Proof. (1) ⇒ (2): Let P be a projective generator of $\mathcal{A}$. Then $P' = i^*(P)$ is a projective generator of $\mathcal{A}'$ and $P'' = j^!(P)$ is a projective generator of $\mathcal{A}''$. These fit into an exact sequence
$$0 \longrightarrow j_!(P'') \longrightarrow P \longrightarrow i_*(P') \longrightarrow 0$$
by Lemma 3.4, which induces an epimorphism
$$\operatorname{Hom}_{\mathcal{A}}(j_!(P''), j_!(P'')) \to \operatorname{Ext}^1_{\mathcal{A}}(i_*(P'), j_!(P'')).$$

(2) ⇒ (1): First observe that for any object X in $\mathcal{A}$, the counit $\varepsilon_X \colon j_!j^!(X) \to X$ fits into an exact sequence
$$0 \longrightarrow X_0 \longrightarrow j_!j^!(X) \longrightarrow X \longrightarrow X_1 \longrightarrow 0 \tag{3.4}$$
with X_0, X_1 in the image of i_*, since $j^!(\varepsilon_X)$ is invertible. In fact, $X_1 = i_*i^*(X)$.

From the projective generators P' in $\mathcal{A}'$ and P'' in $\mathcal{A}''$ we can form the *universal extension*
$$0 \longrightarrow j_!(P'')^r \longrightarrow P \longrightarrow i_*(P') \longrightarrow 0$$
in $\mathcal{A}$, that is, the induced map $\operatorname{Hom}_{\mathcal{A}}(j_!(P'')^r, j_!(P'')) \to \operatorname{Ext}^1_{\mathcal{A}}(i_*(P'), j_!(P''))$ is surjective. We claim that P is projective. First one checks that $\operatorname{Ext}^1_{\mathcal{A}}(P, -)$ vanishes on the image of i_* and the image of $j_!$. Then one applies the sequence (3.4) by writing it as composite of two exact sequences
$$0 \longrightarrow X_0 \longrightarrow j_!j^!(X) \longrightarrow \bar{X} \longrightarrow 0 \quad \text{and} \quad 0 \to \bar{X} \longrightarrow X \longrightarrow X_1 \longrightarrow 0.$$
Thus $\operatorname{Ext}^1_{\mathcal{A}}(P, \bar{X}) = 0$, and finally $\operatorname{Ext}^1_{\mathcal{A}}(P, X) = 0$. It follows that $P \oplus j_!(P'')$ is a projective generator of $\mathcal{A}$. □

The following lemma provides a bound for the homological dimension in a colocalisation sequence with semisimple quotient.

Lemma 3.6. *Fix a homological colocalisation sequence* (3.2) *of abelian categories. Suppose that $\mathcal{A}$ has enough projective objects and that $\mathcal{A}''$ is semisimple. Then* $\operatorname{Ext}^n_{\mathcal{A}'}(-,-) = 0$ *implies* $\operatorname{Ext}^{n+2}_{\mathcal{A}}(-,-) = 0$ *for every integer $n \geq 0$.*

Proof. Fix an object X in $\mathcal{A}$ and write $\operatorname{pd}_{\mathcal{A}}(X) < n$ if $\operatorname{Ext}^n_{\mathcal{A}}(X,-) = 0$. We view $\mathcal{A}'$ as a full subcategory of $\mathcal{A}$ and note that $\operatorname{pd}_{\mathcal{A}'}(X) < n$ implies $\operatorname{pd}_{\mathcal{A}}(X) < n+1$ for $X \in \mathcal{A}'$. This is shown by induction on n. For the case $n = 1$ observe that every projective object in $\mathcal{A}'$ is a direct summand of $i^*(P)$ for some projective object P in $\mathcal{A}$. Then one uses the exact sequence

$$0 \longrightarrow j_!j^!(P) \longrightarrow P \longrightarrow i_*i^*(P) \longrightarrow 0$$

from Lemma 3.4. If $n > 1$, choose an exact sequence $0 \to N \to P \to X \to 0$ in $\mathcal{A}'$. Then $\operatorname{pd}_{\mathcal{A}}(X) < n+1$ since $\operatorname{pd}_{\mathcal{A}}(N) < n$. Now let X be arbitrary. Then we use the exact sequence (3.4). If $\operatorname{pd}_{\mathcal{A}'}(X_i) < n$ for $i = 0,1$, then $\operatorname{pd}_{\mathcal{A}}(X_i) < n+1$, and $\operatorname{pd}_{\mathcal{A}}(X) < n+2$ follows since $j_!j^!(X)$ is projective. $\square$

Remark 3.7. The assertions of Lemmas 3.4, 3.5, and 3.6 remain true (with same proofs) when the abelian category $\mathcal{A}$ is replaced by an exact category.

Recollements of module categories. Let Λ be a ring. We recall some basic facts about subcategories of $\operatorname{Mod}\Lambda$ consisting of modules that are annihilated by a fixed ideal. Note that all ideals in this work are two-sided.

Proposition 3.8 ([1, Proposition 7.1])**.** *A full subcategory $\mathcal{C}$ of $\operatorname{Mod}\Lambda$ is of the form $\operatorname{Mod}\Lambda/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of Λ if and only if the following holds:*

(1) *If $0 \to X' \to X \to X'' \to 0$ is an exact sequence in $\operatorname{Mod}\Lambda$ and X is in $\mathcal{C}$, then X' and X'' are in $\mathcal{C}$.*
(2) *If $(X_i)_{i\in I}$ is a family of modules in $\mathcal{C}$, then their product $\prod_{i\in I} X_i$ is in $\mathcal{C}$.*

In this case we have $\mathfrak{a} = \bigcap_{X\in\mathcal{C}} \operatorname{ann} X$. Moreover, $\mathfrak{a}^2 = \mathfrak{a}$ if and only if $\mathcal{C}$ is closed under extensions. $\square$

Given an idempotent $e \in \Lambda$, the inclusion $i_*\colon \operatorname{Mod}\Lambda/\Lambda e\Lambda \to \operatorname{Mod}\Lambda$ and

$$j^* := \operatorname{Hom}_\Lambda(e\Lambda,-) \cong - \otimes_\Lambda \Lambda e$$

induce a recollement

$$\begin{tikzcd} \operatorname{Mod}\Lambda/\Lambda e\Lambda \arrow[r, tail, "i_*"] & \operatorname{Mod}\Lambda \arrow[l, two heads, shift right=2ex] \arrow[l, two heads, shift left=2ex] \arrow[r, two heads, "j^*"] & \operatorname{Mod} e\Lambda e \,. \arrow[l, tail, shift right=2ex] \arrow[l, tail, shift left=2ex] \end{tikzcd} \tag{3.5}$$

In fact, any recollement of module categories

$$\begin{tikzcd} \operatorname{Mod}\Lambda' \arrow[r, tail] & \operatorname{Mod}\Lambda \arrow[l, two heads, shift right=2ex] \arrow[l, two heads, shift left=2ex] \arrow[r, two heads] & \operatorname{Mod}\Lambda'' \arrow[l, tail, shift right=2ex] \arrow[l, tail, shift left=2ex] \end{tikzcd}$$

is up to Morita equivalence of this form.

If Λ is right artinian, then (3.5) restricts to a colocalisation sequence

$$\begin{tikzcd} \operatorname{mod}\Lambda/\Lambda e\Lambda \arrow[r, tail, shift right=1ex] & \operatorname{mod}\Lambda \arrow[l, two heads, shift right=1ex] \arrow[r, two heads, shift right=1ex] & \operatorname{mod} e\Lambda e \,. \arrow[l, tail, shift right=1ex] \end{tikzcd}$$

Recall that a full subcategory $\mathcal{C} \subseteq \mathcal{A}$ of an abelian category is a *Serre subcategory* if for every exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$ we have $X \in \mathcal{C}$ iff $X', X'' \in \mathcal{C}$.

Lemma 3.9. *Let Λ be a right artinian ring and $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ a Serre subcategory. Then there is an idempotent $e \in \Lambda$ such that $\mathcal{C} = \operatorname{mod}\Lambda/\Lambda e\Lambda$. Moreover, the following holds:*

1. *The inclusion $\operatorname{mod}\Lambda/\Lambda e\Lambda \to \operatorname{mod}\Lambda$ admits a left and a right adjoint.*
2. *The functor $\operatorname{Hom}_\Lambda(e\Lambda,-)\colon \operatorname{mod}\Lambda \to \operatorname{mod} e\Lambda e$ admits a left adjoint.*
3. *The functor $\operatorname{Hom}_\Lambda(e\Lambda,-)\colon \operatorname{mod}\Lambda \to \operatorname{mod} e\Lambda e$ admits a right adjoint provided that $\operatorname{mod}\Lambda$ has enough injective objects.*

Proof. The annihilator $\mathfrak{a} \subseteq \Lambda$ of the modules in $\mathcal{C}$ is idempotent since $\mathcal{C}$ is closed under forming extension. Thus $\mathfrak{a} = \Lambda e\Lambda$ for some idempotent $e \in \Lambda$.

(1) The right adjoint sends a Λ-module X to the maximal submodule belonging to $\mathcal{C}$. The left adjoint sends X to the maximal factor module belonging to $\mathcal{C}$.

(2) Take $-\otimes_{e\Lambda e} e\Lambda$.

(3) Let E be an injective cogenerator and set $\Gamma = \operatorname{End}_\Lambda(E)^{\mathrm{op}}$. Then we have $(\operatorname{mod}\Lambda)^{\mathrm{op}} \xrightarrow{\sim} \operatorname{mod}\Gamma$ and can apply (2). $\square$

Example 3.10. The ring $\Lambda = \left[\begin{smallmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{Q} \end{smallmatrix}\right]$ is right artinian. Let $e = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and set $S = e\Lambda/\operatorname{rad} e\Lambda$. Note that the injective envelope of S is not finitely generated. Thus $\operatorname{Hom}_\Lambda(e\Lambda,-)\colon \operatorname{mod}\Lambda \to \operatorname{mod} e\Lambda e$ admits no right adjoint because it would send $e\Lambda e$ to an injective envelope of S.

We recall a well known criterion for a recollement of module categories to be homological.

Lemma 3.11. *Let Λ be a ring and $\mathfrak{a} \subseteq \Lambda$ an ideal. Then the following are equivalent:*

1. *$\Lambda/\mathfrak{a} \otimes_\Lambda \Lambda/\mathfrak{a} \cong \Lambda/\mathfrak{a}$ and $\operatorname{Tor}_n^\Lambda(\Lambda/\mathfrak{a},\Lambda/\mathfrak{a}) = 0$ for all $n > 0$.*
2. *$\operatorname{Ext}^*_{\Lambda/\mathfrak{a}}(X,Y) \xrightarrow{\sim} \operatorname{Ext}^*_\Lambda(X,Y)$ for all $\Lambda/\mathfrak{a}$-modules X,Y.*

Proof. See [16, Theorem 4.4] $\square$

Heredity ideals. Recall that a ring Λ is *semi-primary* if its Jacobson radical $J(\Lambda)$ is nilpotent and $\Lambda/J(\Lambda)$ is semisimple.

Definition 3.12. An ideal $\mathfrak{a} \subseteq \Lambda$ of a semi-primary ring Λ is an *heredity ideal* if $\mathfrak{a}$ is idempotent, $\mathfrak{a}$ is a projective Λ-module, and $\mathfrak{a}J(\Lambda)\mathfrak{a} = 0$.

Note that an ideal $\mathfrak{a}$ of a semi-primary ring Λ is idempotent iff there exists an idempotent $e \in \Lambda$ such that $\mathfrak{a} = \Lambda e\Lambda$; see [7, Appendix].

Lemma 3.13. *For an ideal $\mathfrak{a}$ of a semi-primary ring Λ the following are equivalent:*

1. *The ideal $\mathfrak{a}$ is an heredity ideal.*
2. *There is a semisimple ring Γ such that the inclusion $\operatorname{Mod}\Lambda/\mathfrak{a} \to \operatorname{Mod}\Lambda$ induces a homological recollement*

$$\begin{tikzcd} \operatorname{Mod}\Lambda/\mathfrak{a} \arrow[r, tail] & \operatorname{Mod}\Lambda \arrow[l, two heads, shift right=2ex] \arrow[l, two heads, shift left=2ex] \arrow[r, two heads] & \operatorname{Mod}\Gamma\,. \arrow[l, tail, shift right=2ex] \arrow[l, tail, shift left=2ex] \end{tikzcd}$$

When Λ is right artinian, in addition the following is equivalent:

(3) *There is a semisimple ring Γ such that the inclusion $\operatorname{mod}\Lambda/\mathfrak{a} \to \operatorname{mod}\Lambda$ induces a homological colocalisation sequence*

$$\begin{tikzcd} \operatorname{mod}\Lambda/\mathfrak{a} \arrow[r, shift right, tail] & \operatorname{mod}\Lambda \arrow[l, shift right, two heads] \arrow[r, shift right, two heads] & \operatorname{mod}\Gamma \arrow[l, shift right, tail] \end{tikzcd},$$

Proof. (1) $\Rightarrow$ (2): We have $\mathfrak{a} = \Lambda e \Lambda$ for some idempotent $e \in \Lambda$ since $\mathfrak{a}$ is idempotent. Then $\Gamma = e\Lambda e$ is semisimple since $\mathfrak{a}J(\Lambda)\mathfrak{a} = 0$. The fact that $\mathfrak{a}$ is a projective Λ-module implies $\operatorname{Tor}_*^\Lambda(\mathfrak{a}, \Lambda/\mathfrak{a}) = 0$. Thus the exact sequence $0 \to \mathfrak{a} \to \Lambda \to \Lambda/\mathfrak{a} \to 0$ induces an isomorphism $\operatorname{Tor}_*^\Lambda(\Lambda/\mathfrak{a}, \Lambda/\mathfrak{a}) \cong \Lambda/\mathfrak{a}$. It follows from Lemma 3.11 that the inclusion $\operatorname{Mod}\Lambda/\mathfrak{a} \to \operatorname{Mod}\Lambda$ induces an isomorphism $\operatorname{Ext}^*_{\Lambda/\mathfrak{a}}(-,-) \xrightarrow{\sim} \operatorname{Ext}^*_\Lambda(-,-)$.

(2) $\Rightarrow$ (1): The fact that $\operatorname{Mod}\Lambda/\mathfrak{a}$ viewed as subcategory of $\operatorname{Mod}\Lambda$ is closed under extensions implies that $\mathfrak{a}$ is idempotent, say $\mathfrak{a} = \Lambda e \Lambda$ for some idempotent $e \in \Lambda$; see Proposition 3.8. Then $\operatorname{Mod} e\Lambda e \xrightarrow{\sim} \operatorname{Mod}\Gamma$, and therefore $\mathfrak{a}J(\Lambda)\mathfrak{a} = 0$ since Γ is semisimple. The property of the recollement to be homological implies that $\mathfrak{a}$ belongs to the image of the left adjoint $\operatorname{Mod}\Gamma \to \operatorname{Mod}\Lambda$, by Lemma 3.4. It remains to observe that every module in the image of this functor is projective since Γ is semisimple.

(1) $\Leftrightarrow$ (3): This is analogous to (1) $\Leftrightarrow$ (2). □

Recollements of abelian length categories. Let $\mathcal{A}$ be an abelian *length category*. Thus $\mathcal{A}$ is an abelian category and every object in $\mathcal{A}$ has a finite composition series.

Recall that $\mathcal{A}$ is *Ext-finite* if for every pair of simple objects S and T

$$\dim_{\operatorname{End}_{\mathcal{A}}(T)^{\mathrm{op}}} \operatorname{Ext}^1_{\mathcal{A}}(S,T) < \infty.$$

Proposition 3.14 ([15, 8.2])**.** *An abelian length category $\mathcal{A}$ is equivalent to the category $\operatorname{mod}\Lambda$ of finitely generated Λ-modules for some right artinian ring Λ if and only if the following holds:*

(1) *$\mathcal{A}$ has only finitely many simple objects.*
(2) *$\mathcal{A}$ is Ext-finite.*
(3) *The supremum of the Loewy lengths of the objects in $\mathcal{A}$ is finite.* □

We provide another criterion for a length category to be a module category.

Lemma 3.15. *Let $\mathcal{A}$ be an abelian length category and suppose that $\mathcal{A}$ is Ext-finite. For a semisimple ring Γ and a homological colocalisation sequence*

$$\begin{tikzcd} \mathcal{A}' \arrow[r, shift right, tail] & \mathcal{A} \arrow[l, shift right, two heads] \arrow[r, shift right, two heads] & \operatorname{mod}\Gamma \arrow[l, shift right, tail] \end{tikzcd}$$

the following are equivalent:

(1) *There is a ring Λ such that $\mathcal{A}$ is equivalent to $\operatorname{mod}\Lambda$.*
(2) *There is a ring Λ' such that $\mathcal{A}'$ is equivalent to $\operatorname{mod}\Lambda'$.*

Proof. (1) ⇒ (2): View $\mathcal{A}'$ as a full subcategory of $\mathcal{A} = \operatorname{mod}\Lambda$ and let $\mathfrak{a} \subseteq \Lambda$ be the annihilator of the modules in $\mathcal{A}'$. Then $\mathcal{A}' = \operatorname{mod}\Lambda/\mathfrak{a}$.

(2) ⇒ (1): We apply Lemma 3.5. Clearly, $\mathcal{A}'$ and $\operatorname{mod}\Gamma$ have projective generators, say P' and P'', which we view as objects in $\mathcal{A}$. Then $\mathcal{A} \to \operatorname{mod}\Gamma$ sends the canonical morphism $P'' \to P''/\operatorname{rad}P''$ to an isomorphism, and therefore $\operatorname{rad}P''$ belongs $\mathcal{A}'$. Thus $\operatorname{Ext}^1_{\mathcal{A}}(P', P'') \cong \operatorname{Ext}^1_{\mathcal{A}}(P', P''/\operatorname{rad}P'')$ and invoking the Ext-finiteness of $\mathcal{A}$, it follows that $\mathcal{A}$ admits a projective generator, say P. Then $\operatorname{Hom}_{\mathcal{A}}(P, -)\colon \mathcal{A} \to \operatorname{mod}\operatorname{End}_{\mathcal{A}}(P)$ is an equivalence. □

4. Quasi-hereditary rings

We recall the definition of a quasi-hereditary ring and characterise the module categories of such rings in terms of recollements.

Quasi-hereditary algebras over a field were introduced by Scott [33]; the definition given here for semi-primary rings is due Dlab and Ringel [7].

Definition 4.1. A semi-primary ring Λ is *quasi-hereditary* if there is a finite sequence of surjective ring homomorphisms

$$\Lambda = \Lambda_n \longrightarrow \Lambda_{n-1} \longrightarrow \cdots \longrightarrow \Lambda_1 \longrightarrow \Lambda_0 = 0$$

such that for each $1 \leq i \leq n$ the kernel of $\Lambda_i \to \Lambda_{i-1}$ is a heredity ideal. Clearly, such a sequence is equivalent to a finite chain of ideals

$$0 = \mathfrak{a}_n \subseteq \mathfrak{a}_{n-1} \subseteq \cdots \subseteq \mathfrak{a}_0 = \Lambda$$

such that $\mathfrak{a}_{i-1}/\mathfrak{a}_i$ is an heredity ideal in $\Lambda/\mathfrak{a}_i$ for all i.

Theorem 4.2. *Let $\mathcal{A}$ be an abelian length category. Suppose that $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ are Ext-finite. Then the following are equivalent:*

1. *There is a quasi-hereditary ring Λ such that $\mathcal{A}$ is equivalent to the category of finitely generated Λ-modules.*
2. *There is a finite chain of full subcategories*

$$0 = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_n = \mathcal{A}$$

and a sequence of semisimple rings $\Gamma_1, \dots, \Gamma_n$ such that each inclusion $\mathcal{A}_{i-1} \to \mathcal{A}_i$ induces a homological recollement of abelian categories

$$\begin{tikzcd} \mathcal{A}_{i-1} \arrow[r, tail] & \mathcal{A}_i \arrow[l, two heads, shift right=2] \arrow[l, two heads, shift left=2] \arrow[r, two heads] & \operatorname{mod}\Gamma_i \arrow[l, tail, shift right=2] \arrow[l, tail, shift left=2] \end{tikzcd}.$$

Remark 4.3. Consider an abelian length category $\mathcal{A}$ satisfying the equivalent conditions of Theorem 4.2.

(1) The chain $0 = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_n = \mathcal{A}$ can be refined such that each semisimple ring Γ_i is a division ring. Then the length of this chain equals the number of pairwise non-isomorphic simple objects in $\mathcal{A}$ and the Γ_i are their endomorphism rings.

(2) Each inclusion $\mathcal{A}_{i-1} \to \mathcal{A}_i$ induces a recollement of triangulated categories

$$\begin{tikzcd}[column sep=large] \mathbf{D}^b(\mathcal{A}_{i-1}) \arrow[r, tail] & \mathbf{D}^b(\mathcal{A}_i) \arrow[l, two heads, bend right] \arrow[l, two heads, bend left] \arrow[r, two heads] & \mathbf{D}^b(\operatorname{mod}\Gamma_i) \arrow[l, tail, bend right] \arrow[l, tail, bend left] \end{tikzcd}.$$

Proof of Theorem 4.2. (1) $\Rightarrow$ (2): Suppose that Λ is a quasi-hereditary ring with an equivalence $\mathcal{A} \xrightarrow{\sim} \operatorname{mod}\Lambda$. The defining sequence of homomorphisms $\Lambda_i \to \Lambda_{i-1}$ yields a sequence of homological recollements

$$\begin{tikzcd}[column sep=large] \operatorname{mod}\Lambda_{i-1} \arrow[r, tail] & \operatorname{mod}\Lambda_i \arrow[l, two heads, bend right] \arrow[l, two heads, bend left] \arrow[r, two heads] & \operatorname{mod}\Gamma_i \arrow[l, tail, bend right] \arrow[l, tail, bend left] \end{tikzcd}$$

by Lemma 3.13. Note that the right adjoints exist by Lemma 3.9. In fact, $\mathcal{A}$ has enough injective objects by Proposition 3.14 because $\mathcal{A}^{\mathrm{op}}$ is Ext-finite.

(2) $\Rightarrow$ (1): Suppose there is a chain of full subcategories $\mathcal{A}_i \subseteq \mathcal{A}$ satisfying the conditions in (2). Using induction on n we find a quasi-hereditary ring Λ with an equivalence $\mathcal{A} \xrightarrow{\sim} \operatorname{mod}\Lambda$. In fact, an application of Lemmas 3.13 and 3.15 provides rings Λ_i and surjective homomorphisms $f_i \colon \Lambda_i \to \Lambda_{i-1}$ such that $\mathcal{A}_i \xrightarrow{\sim} \operatorname{mod}\Lambda_i$ and $\operatorname{Ker} f_i$ is a heredity ideal. $\square$

Remark 4.4. There is a variation of Theorem 4.2 without the assumption on $\mathcal{A}^{\mathrm{op}}$ to be Ext-finite. Then one drops in (2) the right adjoints of $\mathcal{A}_{i-1} \to \mathcal{A}_i$ and $\mathcal{A}_i \to \operatorname{mod}\Gamma_i$.

5. Filtrations via standard objects

We recall the definition of a highest weight category in terms of filtrations via standard objects. Then we characterise the highest weight categories using recollements of abelian categories.

Highest weight categories. Let $\Delta_1, \dots, \Delta_n$ be objects in an abelian category $\mathcal{A}$. We write $\operatorname{Filt}(\Delta_1, \dots, \Delta_n)$ for the full subcategory of objects X in $\mathcal{A}$ that admit a finite filtration $0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_t = X$ such that each factor X_i/X_{i-1} is isomorphic to an object of the form Δ_j.

Highest weight categories were introduced by Cline, Parshall, and Scott [5] in the context of k-linear categories over a field k. The definition given here is more general since the endomorphism ring of a standard object can be any division ring.

Definition 5.1. An abelian length category $\mathcal{A}$ with finitely many simple objects is called *highest weight category* if there are finitely many exact sequences

$$0 \longrightarrow U_i \longrightarrow P_i \longrightarrow \Delta_i \longrightarrow 0 \qquad (1 \le i \le n) \tag{5.1}$$

in $\mathcal{A}$ satisfying the following:

(1) $\operatorname{Hom}_{\mathcal{A}}(\Delta_i, \Delta_j) = 0$ for all $i > j$.
(2) $\operatorname{End}_{\mathcal{A}}(\Delta_i)$ is a division ring for all i.
(3) U_i belongs to $\operatorname{Filt}(\Delta_{i+1}, \dots, \Delta_n)$ for all i.
(4) $\bigoplus_{i=1}^n P_i$ is a projective generator of $\mathcal{A}$.

The objects $\Delta_1, \dots, \Delta_n$ are called *standard objects*.

The following result establishes the precise connection between highest weight categories and recollements of abelian categories with semisimple factors. For a similar result involving recollements of derived categories, see [26, Theorem 5.13].

Theorem 5.2. *Let $\mathcal{A}$ be an abelian length category with finitely many simple objects. Suppose that $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ are Ext-finite. Then the following are equivalent:*

(1) *The category $\mathcal{A}$ is a highest weight category.*
(2) *There is a finite chain of full subcategories*

$$0 = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_n = \mathcal{A}$$

and a sequence of division rings $\Gamma_1, \dots, \Gamma_n$ such that each inclusion $\mathcal{A}_{i-1} \to \mathcal{A}_i$ induces a homological recollement of abelian categories

(5.2)
$$\begin{tikzcd} \mathcal{A}_{i-1} \arrow[r, tail] & \mathcal{A}_i \arrow[l, shift right=2, two heads] \arrow[l, shift left=2, two heads] \arrow[r, two heads] & \operatorname{mod}\Gamma_i \arrow[l, shift right=2, tail] \arrow[l, shift left=2, tail] \end{tikzcd} .$$

Proof. (1) $\Rightarrow$ (2): Suppose $\mathcal{A}$ is a highest weight category satisfying the conditions in Definition 5.1. We identify $\mathcal{A} \xrightarrow{\sim} \operatorname{mod}\Lambda$ for $\Lambda = \operatorname{End}_{\mathcal{A}}(\bigoplus_i P_i)$ and give a recursive construction of a chain

$$0 = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_n = \mathcal{A}$$

of full subcategories satisfying the conditions in (2). Let $\mathcal{A}_{n-1}$ denote the full subcategory of objects X in $\mathcal{A}$ such that $\operatorname{Hom}_{\mathcal{A}}(\Delta_n, X) = 0$ and set $\Gamma_n = \operatorname{End}_{\mathcal{A}}(\Delta_n)$. The object Δ_n is projective and $\operatorname{Hom}_{\mathcal{A}}(\Delta_n, -)$ induces a recollement

$$\begin{tikzcd} \mathcal{A}_{n-1} \arrow[r, tail] & \mathcal{A} \arrow[l, shift right=2, two heads] \arrow[l, shift left=2, two heads] \arrow[r, two heads] & \operatorname{mod}\Gamma_n \arrow[l, shift right=2, tail] \arrow[l, shift left=2, tail] \end{tikzcd}$$

by Lemma 3.9. We claim that $\mathcal{A}_{n-1}$ is a highest weight category. Applying the left adjoint of the inclusion $\mathcal{A}_{n-1} \to \mathcal{A}$ to (5.1) yields exact sequences

(5.3)
$$0 \longrightarrow \bar{U}_i \longrightarrow \bar{P}_i \longrightarrow \Delta_i \longrightarrow 0 \qquad (1 \le i \le n-1)$$

in $\mathcal{A}_{n-1}$. More precisely, because U_i belongs to $\operatorname{Filt}(\Delta_{i+1}, \dots, \Delta_n)$ and Δ_n is projective, we have an exact sequence

$$0 \longrightarrow U_i' \longrightarrow U_i \longrightarrow \bar{U}_i \longrightarrow 0$$

with U_i' a direct sum of copies of Δ_n and $\bar{U}_i$ in $\mathrm{Filt}(\Delta_{i+1}, \dots, \Delta_{n-1})$. Thus (5.3) equals the composite of (5.1) with $U_i \to \bar{U}_i$. This yields the following commutative diagram with exact rows and columns.

(5.4)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & U_i' \arrow[r, equal] \arrow[d] & U_i' \arrow[d] & & \\
0 \arrow[r] & U_i \arrow[r] \arrow[d] & P_i \arrow[r] \arrow[d] & \Delta_i \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \bar{U}_i \arrow[r] \arrow[d] & \bar{P}_i \arrow[r] \arrow[d] & \Delta_i \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

In particular there is an exact sequence

$$0 \longrightarrow U_i' \longrightarrow P_i \longrightarrow \bar{P}_i \longrightarrow 0,$$

and therefore the above recollement is homological by Lemma 3.4. It remains to observe that $\bigoplus_{i=1}^{n-1} \bar{P}_i$ is a projective generator of $\mathcal{A}_{n-1}$.

(2) $\Rightarrow$ (1): Fix a chain of full subcategories $\mathcal{A}_i \subseteq \mathcal{A}$ satisfying the conditions in (2). We show by induction on n that $\mathcal{A}$ is a highest weight category. Let Δ_n denote the image of Γ_n under the left adjoint $\mathrm{mod}\,\Gamma_n \to \mathcal{A}$. Clearly, $\mathrm{End}_{\mathcal{A}}(\Delta_n) \cong \Gamma_n$ and Δ_n is a projective object. The induction hypothesis for $\mathcal{A}_{n-1}$ yields a collection of exact sequences (5.3). We modify them as follows to obtain exact sequences (5.1). First observe that $\Delta_n / \operatorname{rad} \Delta_n$ is a simple object and that

$$\mathrm{Ext}^1_{\mathcal{A}}(\bar{P}_i, \Delta_n) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{A}}(\bar{P}_i, \Delta_n / \operatorname{rad} \Delta_n)$$

since $\operatorname{rad} \Delta_n$ belongs to $\mathcal{A}_{n-1}$. Using Ext-finiteness we can form the universal extension

$$0 \longrightarrow \Delta_n^r \longrightarrow P_i \longrightarrow \bar{P}_i \longrightarrow 0$$

in $\mathcal{A}$, that is, the induced map $\mathrm{Hom}_{\mathcal{A}}(\Delta_n^r, \Delta_n) \to \mathrm{Ext}^1_{\mathcal{A}}(\bar{P}_i, \Delta_n)$ is surjective. The argument given in the proof of Lemma 3.5 shows that P_i is a projective object, and we obtain exact sequences (5.1) with U_i in $\mathrm{Filt}(\Delta_{i+1}, \dots, \Delta_n)$, where $P_n := \Delta_n$ and $U_n := 0$. Finally observe that $\bigoplus_{i=1}^{n} P_i$ is a projective generator of $\mathcal{A}$. $\square$

Remark 5.3. Consider an abelian length category $\mathcal{A}$ satisfying the equivalent conditions of Theorem 5.2.

(1) For each $1 \le i \le n$ the standard object Δ_i is obtained from the recollement (5.2) by taking the image of Γ_i under the left adjoint $\mathrm{mod}\,\Gamma_i \to \mathcal{A}_i$. Dually, there is a *costandard object* ∇_i which equals the image of Γ_i under the right adjoint $\mathrm{mod}\,\Gamma_i \to \mathcal{A}_i$.

(2) The category $\mathcal{A}^{\mathrm{op}}$ is a highest weight category because the second condition in Theorem 5.2 is self-dual. The costandard objects of $\mathcal{A}$ are precisely the standard objects of $\mathcal{A}^{\mathrm{op}}$.

Finite global dimension. The chain of recollements (5.2) for a highest weight category provides a bound for the homological dimension. In fact, it is well known that a quasi-hereditary algebra has finite global dimension with bound $2n-2$ if there is a chain of heredity ideals of length n; see [7]. We include a proof which has the advantage that it carries over to split quasi-hereditary algebras over any commutative base ring.

For a ring Λ let $\mathbf{D}^{\mathrm{perf}}(\Lambda)$ denote the category of perfect complexes over Λ which equals $\mathbf{D}^b(\operatorname{proj}\Lambda)$ by definition. We recall the following well known fact [34, III.2.4].

Lemma 5.4. *Let $\mathcal{A}$ be an abelian (or exact) category with projective generator P and set $\Lambda = \operatorname{End}_{\mathcal{A}}(P)$. Then the inclusion* $\operatorname{proj}\Lambda \to \mathcal{A}$ *induces a triangle equivalence $\mathbf{D}^{\mathrm{perf}}(\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A})$ iff all objects of $\mathcal{A}$ have finite projective dimension.* □

Corollary 5.5. *Let $\mathcal{A}$ be a highest weight category with n simple objects and let Λ denote the endomorphism ring of a projective generator. Then we have $\operatorname{Ext}^{2n-1}_{\mathcal{A}}(-,-) = 0$ and a triangle equivalence $\mathbf{D}^{\mathrm{perf}}(\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A})$.*

Proof. The first assertion follows from the chain of recollements (5.2) in Theorem 5.2 in combination with Lemma 3.6. For the second assertion apply Lemma 5.4. □

6. k-linear highest weight categories

Let k be a commutative ring. Following [10, 11, 30] we provide another setting for the theory of heighest weight categories and consider k-linear exact categories.

k-linear exact categories. Fix a k-linear exact category $\mathcal{A}$. Thus $\mathcal{A}$ is an exact category [28], the morphisms sets in $\mathcal{A}$ are k-modules, and the composition maps are k-linear. The exact structure of $\mathcal{A}$ is given by a distinguished class of *exact sequences* $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ which are kernel-cokernel pairs and sometimes called *admissible*.

We recall from [21, Appendix A] a useful construction. Suppose that $\mathcal{A}$ is essentially small and let $\widehat{\mathcal{A}}$ denote the category of left exact functors $\mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$ into the category of abelian groups. Then $\widehat{\mathcal{A}}$ is a Grothendieck abelian category and the Yoneda functor

$$\mathcal{A} \longrightarrow \widehat{\mathcal{A}}, \quad X \longmapsto h_X = \operatorname{Hom}_{\mathcal{A}}(-,X)$$

is fully faithful and exact, inducing a bijection

$$\operatorname{Ext}^1_{\mathcal{A}}(X,Y) \xrightarrow{\sim} \operatorname{Ext}^1_{\widehat{\mathcal{A}}}(h_X,h_Y).$$

Note that any exact functor $f\colon \mathcal{A} \to \mathcal{B}$ extends uniquely to an exact and colimit preserving functor $\widehat{\mathcal{A}} \to \widehat{\mathcal{B}}$ (the left adjoint of the functor $\widehat{\mathcal{B}} \to \widehat{\mathcal{A}}$ given by precomposition with f).

Lemma 6.1. *Suppose that $\mathcal{A}$ admits a projective generator P and set $\Lambda = \operatorname{End}_{\mathcal{A}}(P)$. Then evaluation at P induces an equivalence $\widehat{\mathcal{A}} \xrightarrow{\sim} \operatorname{Mod}\Lambda$.*

Conversely, if $\widehat{\mathcal{A}}$ is equivalent to $\operatorname{Mod}\Gamma$ for some ring Γ, then the equivalence identifies Γ with a projective generator of $\mathcal{A}$.

Proof. Sending a Λ-module M to $\operatorname{Hom}_\Lambda(\operatorname{Hom}_{\mathcal{A}}(P,-),M)$ gives a quasi-inverse $\operatorname{Mod}\Lambda \to \widehat{\mathcal{A}}$. For the converse observe that any functor in $\widehat{\mathcal{A}}$ is the epimorphic image of a direct sum of representable functors. Thus Γ identifies with a direct summand of a finite direct sum of representables. $\square$

For a k-algebra Λ we denote by $\operatorname{mod}(\Lambda,k)$ the category of Λ-modules that are finitely generated projective over k. This is a k-linear exact category where a sequence is by definition exact if it is split exact when restricted to the category of k-modules. Note that $\operatorname{Hom}_k(-,k)$ induces a k-linear equivalence

$$\operatorname{mod}(\Lambda,k)^{\mathrm{op}} \xrightarrow{\sim} \operatorname{mod}(\Lambda^{\mathrm{op}},k).$$

Suppose that $\mathcal{A}$ admits a projective generator P and set $\Lambda = \operatorname{End}_{\mathcal{A}}(P)$. If $\operatorname{Hom}_{\mathcal{A}}(P,X)$ is finitely generated projective over k for all X in $\mathcal{A}$, then $\operatorname{Hom}_{\mathcal{A}}(P,-)$ and evaluation at P make the following diagram commutative.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\operatorname{Hom}_{\mathcal{A}}(P,-)"] \arrow[d, tail] & \operatorname{mod}(\Lambda,k) \arrow[d, tail] \\
\widehat{\mathcal{A}} \arrow[r, "\sim"] & \operatorname{Mod}\Lambda
\end{tikzcd}$$

All functors are fully faithful and exact, but the top one need not be an equivalence.

k-linear highest weight categories. We give the definition of a highest weight category relative to a base ring k, following closely Rouquier [30].

Definition 6.2. Let $\mathcal{A}$ be a k-linear exact category. Suppose that $\mathcal{A} \xrightarrow{\sim} \operatorname{mod}(\Lambda,k)$ for some k-algebra Λ that is finitely generated projective over k. Then $\mathcal{A}$ is called *k-linear highest weight category* if there are finitely many exact sequences

$$0 \longrightarrow U_i \longrightarrow P_i \longrightarrow \Delta_i \longrightarrow 0 \qquad (1 \le i \le n) \tag{6.1}$$

in $\mathcal{A}$ satisfying the following:

(1) $\operatorname{Hom}_{\mathcal{A}}(\Delta_i,\Delta_j) = 0$ for all $i > j$.
(2) $\operatorname{End}_{\mathcal{A}}(\Delta_i) \cong k$ for all i.
(3) U_i belongs to $\operatorname{Filt}(\Delta_{i+1},\dots,\Delta_n)$ for all i.
(4) $\bigoplus_{i=1}^n P_i$ is a projective generator of $\mathcal{A}$.

The objects $\Delta_1,\dots,\Delta_n$ are called *standard objects*.

Note that the sequence (6.1) implies

$$\operatorname{Ext}^1_{\mathcal{A}}(\Delta_i, \Delta_j) = 0 \qquad \text{for all} \qquad i \geq j.$$

The structure of a highest weight category is determined by the ordered set of standard objects; see Theorem 6.5. Thus an *equivalence* of highest weight categories is an equivalence of categories which preserves standard objects and their ordering.

Following [6] we call a k-algebra *split quasi-hereditary* if it is the endomorphism ring of a projective generator of a k-linear highest weight category. Later we will see that the standard objects $\Delta_1, \dots, \Delta_n$ in $\operatorname{mod}(\Lambda, k)$ give rise to a sequence of surjective algebra homomorphism

$$\Lambda = \Lambda_n \longrightarrow \Lambda_{n-1} \longrightarrow \cdots \longrightarrow \Lambda_1 \longrightarrow \Lambda_0 = 0$$

which makes it possible to define split quasi-hereditary algebras in terms of ideal chains.

Standardisation. We give an axiomatic description of the standard objects of a highest weight category.

Let $\mathcal{A}$ be an exact category and fix a sequence of objects $\Delta_1, \dots, \Delta_n$. We consider the following conditions:

(Δ1) $\operatorname{Hom}_{\mathcal{A}}(\Delta_i, \Delta_j) = 0$ for all $i > j$.
(Δ2) $\operatorname{Ext}^1_{\mathcal{A}}(\Delta_i, \Delta_j) = 0$ for all $i \geq j$.
(Δ3) $\operatorname{Ext}^1_{\mathcal{A}}(X, \Delta_j)$ is finitely generated over $\operatorname{End}_{\mathcal{A}}(\Delta_j)^{\mathrm{op}}$ for all $X \in \mathcal{A}$.

For a ring Λ let $\operatorname{free}\Lambda$ denote the category of free Λ-modules of finite rank.

Lemma 6.3. *Suppose that* (Δ1)–(Δ2) *hold and set* $\Gamma_i = \operatorname{End}_{\mathcal{A}}(\Delta_i)$ *for* $1 \leq i \leq n$. *Then the functor* $\operatorname{Hom}_{\mathcal{A}}(\Delta_i, -)$ *induces a colocalisation sequence*

$$\operatorname{Filt}(\Delta_1, \dots, \Delta_{i-1}) \underset{i_*}{\overset{i^*}{\leftleftarrows}} \operatorname{Filt}(\Delta_1, \dots, \Delta_i) \underset{j^!}{\overset{j_!}{\leftrightarrows}} \operatorname{free}\Gamma_i \tag{6.2}$$

and each X in $\operatorname{Filt}(\Delta_1, \dots, \Delta_i)$ *fits into an exact sequence*

$$0 \longrightarrow j_! j^! (X) \longrightarrow X \longrightarrow i_* i^* (X) \longrightarrow 0.$$

Proof. The object Δ_i is projective in $\operatorname{Filt}(\Delta_1, \dots, \Delta_i)$ and therefore any X in $\operatorname{Filt}(\Delta_1, \dots, \Delta_i)$ fits into an exact sequence $0 \to X'' \to X \to X' \to 0$ with X' in $\operatorname{Filt}(\Delta_1, \dots, \Delta_{i-1})$ and $X'' \cong \Delta_i^r$ for some $r \geq 0$. We have $i_* i^*(X) \cong X'$ and $j_! j^!(X) \cong X''$. ☐

Lemma 6.4. *Suppose that* (Δ2)–(Δ3) *hold. Then there are exact sequences*

$$0 \longrightarrow U_i \longrightarrow P_i \longrightarrow \Delta_i \longrightarrow 0 \qquad (1 \leq i \leq n) \tag{6.3}$$

in $\mathcal{A}$ such that U_i belongs to $\operatorname{Filt}(\Delta_{i+1}, \dots, \Delta_n)$ *for all i and* $\bigoplus_{i=1}^n P_i$ *is a projective generator of* $\operatorname{Filt}(\Delta_1, \dots, \Delta_n)$.

Proof. We use induction on n. The induction hypothesis yields a collection of exact sequences

$$0 \longrightarrow \bar{U}_i \longrightarrow \bar{P}_i \longrightarrow \Delta_i \longrightarrow 0 \qquad (1 \leq i < n)$$

in $\mathrm{Filt}(\Delta_1, \dots, \Delta_{n-1})$. We modify them as follows to obtain exact sequences (6.3). Using Ext-finiteness we can form the universal extension

$$0 \longrightarrow \Delta_n^r \longrightarrow P_i \longrightarrow \bar{P}_i \longrightarrow 0$$

in $\mathcal{A}$, that is, the induced map $\mathrm{Hom}_{\mathcal{A}}(\Delta_n^r, \Delta_n) \to \mathrm{Ext}^1_{\mathcal{A}}(\bar{P}_i, \Delta_n)$ is surjective. Standard arguments then show that P_i is a projective object in $\mathrm{Filt}(\Delta_1, \dots, \Delta_n)$. More precisely, Δ_n is projective and therefore each object X in $\mathrm{Filt}(\Delta_1, \dots, \Delta_n)$ fits into an exact sequence $0 \to \Delta_n^s \to X \to X' \to 0$ for some $s \geq 0$ with X' in $\mathrm{Filt}(\Delta_1, \dots, \Delta_{n-1})$. Using this one shows that $\mathrm{Ext}^1_{\mathcal{A}}(P_i, X) = 0$. Now we get exact sequences (6.3) with U_i in $\mathrm{Filt}(\Delta_{i+1}, \dots, \Delta_n)$, where $P_n := \Delta_n$ and $U_n := 0$. It remains to observe that $\bigoplus_i P_i$ is a projective generator of $\mathrm{Filt}(\Delta_1, \dots, \Delta_n)$. $\square$

The following result characterises the standard objects of a k-linear highest weight category and is an analogue of a result of Dlab and Ringel [8, Theorem 2]. This gives rise to an alternative definition of highest weight categories which does not involve the choice of exact sequences; see Theorem 1.1.

Theorem 6.5. *Let $\mathcal{A}$ be a k-linear exact category and assume that $\mathrm{Ext}^1_{\mathcal{A}}(X, Y)$ is finitely generated over k for all $X, Y \in \mathcal{A}$. Then a sequence of objects $\Delta_1, \dots, \Delta_n$ in $\mathcal{A}$ identifies with the standard objects $\Delta'_1, \dots, \Delta'_n$ of a k-linear highest weight category $\mathcal{A}'$ via an exact equivalence*

$$\mathcal{A} \supseteq \mathrm{Filt}(\Delta_1, \dots, \Delta_n) \xrightarrow{\sim} \mathrm{Filt}(\Delta'_1, \dots, \Delta'_n) \subseteq \mathcal{A}'$$

if and only if the following holds:

(1) $\mathrm{Hom}_{\mathcal{A}}(\Delta_i, \Delta_j) = 0$ *for all* $i > j$.
(2) $\mathrm{End}_{\mathcal{A}}(\Delta_i) \cong k$ *for all* i.
(3) $\mathrm{Ext}^1_{\mathcal{A}}(\Delta_i, \Delta_j) = 0$ *for all* $i \geq j$.
(4) *The endomorphism ring of a projective generator of* $\mathrm{Filt}(\Delta_1, \dots, \Delta_n)$ *is finitely generated projective over* k.

Proof. Clearly, the standard objects of a k-linear highest weight category satisfy the above properties. In order to show the converse choose any projective generator P of $\mathrm{Filt}(\Delta_1, \dots, \Delta_n)$ which exists by Lemma 6.4. We claim that $\mathrm{Hom}_{\mathcal{A}}(P, X)$ is finitely generated projective over k for all X in $\mathrm{Filt}(\Delta_1, \dots, \Delta_n)$. Then the assertion of the theorem follows because we can choose $\mathcal{A}' = \mathrm{mod}(\Lambda, k)$ for $\Lambda = \mathrm{End}_{\mathcal{A}}(P)$, thanks to Lemma 6.4.

The claim is shown by induction on n. We use the colocalisation sequence (6.2) for $i = n$. Given X in $\mathrm{Filt}(\Delta_1, \dots, \Delta_n)$ set $X' = i_* i^*(X)$ and $X'' = j_! j^!(X)$. Note that P' is a projective generator of $\mathrm{Filt}(\Delta_1, \dots, \Delta_{n-1})$. The claim follows from the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P, X'') \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P, X) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P, X') \longrightarrow 0.$$

The k-module $\operatorname{Hom}_{\mathcal{A}}(P, X') \cong \operatorname{Hom}_{\mathcal{A}}(P', X')$ is finitely generated projective by induction, and (4) implies that $\operatorname{Hom}_{\mathcal{A}}(P, X'')$ is finitely generated projective. $\square$

Recollements. The following result characterises a k-linear highest weight categories in terms of recollements; it is the analogue of Theorem 5.2. We need to involve the completion $\widehat{\mathcal{A}}$ of an exact category $\mathcal{A}$, because there is in general no reason for the existence of recollements on the level of subcategories of $\mathcal{A}$.

Theorem 6.6. *Let $\mathcal{A}$ be a k-linear exact category. Suppose that $\mathcal{A} \xrightarrow{\sim} \operatorname{mod}(\Lambda, k)$ for some k-algebra Λ that is finitely generated projective over k. Then the following are equivalent:*

(1) *The category $\mathcal{A}$ is a k-linear highest weight category.*
(2) *There is a finite chain of full exact subcategories*
$$0 = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_n = \mathcal{A}$$
such that each inclusion $\mathcal{A}_{i-1} \to \mathcal{A}_i$ induces a homological recollement of exact categories
$$\begin{tikzcd} \widehat{\mathcal{A}_{i-1}} \arrow[r, tail] & \widehat{\mathcal{A}_i} \arrow[l, two heads, shift right=2] \arrow[l, two heads, shift left=2] \arrow[r, two heads] & \operatorname{Mod} k \arrow[l, tail, shift right=2] \arrow[l, tail, shift left=2] \end{tikzcd} \quad \textit{with} \quad \mathcal{A}_{i-1} = \widehat{\mathcal{A}_{i-1}} \cap \mathcal{A}_i.$$

The proof of Theorem 6.6 provides for each $1 \le i \le n$ a k-algebra Λ_i such that $\mathcal{A}_i \xrightarrow{\sim} \operatorname{mod}(\Lambda_i, k)$ and a surjective algebra homomorphism $\Lambda_i \to \Lambda_{i-1}$.

Proof. Fix a projective generator P of $\mathcal{A}$ and set $\Lambda = \operatorname{End}_{\mathcal{A}}(P)$. We identify $\mathcal{A} \xrightarrow{\sim} \operatorname{mod}(\Lambda, k)$ via $\operatorname{Hom}_{\mathcal{A}}(P, -)$ and $\widehat{\mathcal{A}} \xrightarrow{\sim} \operatorname{Mod} \Lambda$ via evaluation at P; see Lemma 6.1.

(1) $\Rightarrow$ (2): Suppose that $\mathcal{A}$ is a k-linear highest weight category satisfying the conditions in Theorem 1.1. We may assume that P belongs to $\operatorname{Filt}(\Delta_1, \ldots, \Delta_n)$. We give a recursive construction of a chain
$$0 = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_n = \mathcal{A}$$
of full subcategories satisfying the conditions in (2). To this end consider the colocalisation sequence (6.2) for $i = n$. The left adjoint
$$\operatorname{Filt}(\Delta_1, \ldots, \Delta_n) \longrightarrow \operatorname{Filt}(\Delta_1, \ldots, \Delta_{n-1})$$
takes the object P to a projective generator $\bar{P}$ of $\operatorname{Filt}(\Delta_1, \ldots, \Delta_{n-1})$. We set
$$\mathcal{A}_{n-1} = \{X \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(\Delta_n, X) = 0\} \qquad \text{and} \qquad \Lambda_{n-1} = \operatorname{End}_{\mathcal{A}}(\bar{P}).$$
Note that $\operatorname{Hom}_{\mathcal{A}}(\bar{P}, X) \cong \operatorname{Hom}_{\mathcal{A}}(P, X)$ is finitely generated projective over k for all X in $\mathcal{A}_{n-1}$. Also, it is easily checked that $\bar{P}$ is a projective generator of $\mathcal{A}_{n-1}$. Thus $\operatorname{Hom}_{\mathcal{A}}(\bar{P}, -)$ yields an equivalence $\mathcal{A}_{n-i} \xrightarrow{\sim} \operatorname{mod}(\Lambda_{n-1}, k)$. It follows from Theorem 1.1 that $\mathcal{A}_{n-1}$ is a highest weight category with standard objects $\Delta_1, \ldots, \Delta_{n-1}$. We have $\widehat{\mathcal{A}_{n-1}} \xrightarrow{\sim} \operatorname{Mod} \Lambda_{n-1}$ by Lemma 6.1, and the functor $\operatorname{Hom}_{\Lambda}(\Delta_n, -)\colon \operatorname{Mod} \Lambda \to \operatorname{Mod} k$ induces a recollement

$$\begin{tikzcd} \operatorname{Mod} \Lambda_{n-1} \arrow[r, tail] & \operatorname{Mod} \Lambda \arrow[l, two heads, shift right=2] \arrow[l, two heads, shift left=2] \arrow[r, two heads] & \operatorname{Mod} k \arrow[l, tail, shift right=2] \arrow[l, tail, shift left=2] \end{tikzcd} \tag{6.4}$$

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which is homological by Lemma 3.4 because of the sequences in Lemma 6.3.

(2) $\Rightarrow$ (1): Fix a chain of full subcategories $\mathcal{A}_i \subseteq \mathcal{A}$ satisfying the conditions in (2). We show by induction on n that $\mathcal{A}$ is a highest weight category. Let Δ_n denote the image of k under the left adjoint $\operatorname{Mod} k \to \operatorname{Mod}\Lambda$. Clearly, $\operatorname{End}_\Lambda(\Delta_n) \cong k$ and Δ_n is a finitely generated projective Λ-module that belongs therefore to $\mathcal{A}$. The inclusion $\widehat{\mathcal{A}_{n-1}} \to \operatorname{Mod}\Lambda$ identifies $\widehat{\mathcal{A}_{n-1}}$ with $\operatorname{Mod}\Lambda/\mathfrak{a}$ for some idempotent ideal $\mathfrak{a}$; see Proposition 3.8. More precisely, the left adjoint of $\widehat{\mathcal{A}_{n-1}} \to \operatorname{Mod}\Lambda$ sends Λ to $\Lambda/\mathfrak{a}$ which is a projective generator of $\mathcal{A}_{n-1}$ by Lemma 6.1. In particular, $\Lambda_{n-1} = \Lambda/\mathfrak{a}$ is finitely generated projective over k and $\mathcal{A}_{n-1} = \operatorname{mod}(\Lambda_{n-1}, k)$. The induction hypothesis for $\mathcal{A}_{n-1}$ yields a collection of exact sequences

$$0 \longrightarrow \bar{U}_i \longrightarrow \bar{P}_i \longrightarrow \Delta_i \longrightarrow 0 \qquad (1 \le i < n)$$

and we modify them as follows to obtain exact sequences (6.1). Using the fact that extension groups of objects in $\mathcal{A}$ are finitely generated over k, we can form the universal extension

$$0 \longrightarrow \Delta_n^r \longrightarrow P_i \longrightarrow \bar{P}_i \longrightarrow 0$$

in $\mathcal{A}$, that is, the induced map $\operatorname{Hom}_{\mathcal{A}}(\Delta_n^r, \Delta_n) \to \operatorname{Ext}^1_{\mathcal{A}}(\bar{P}_i, \Delta_n)$ is surjective. The argument given in the proof of Lemma 3.5 shows that P_i is a projective object, and we obtain exact sequences (6.1) with U_i in $\operatorname{Filt}(\Delta_{i+1}, \dots, \Delta_n)$, where $P_n := \Delta_n$ and $U_n := 0$. It remains to observe that $\bigoplus_i P_i$ is a projective generator of $\mathcal{A}$. $\square$

Properties of k-linear highest weight categories. We formulate some consequences of Theorem 6.6. To this end fix a k-linear highest weight category $\mathcal{A} = \operatorname{mod}(\Lambda, k)$ with chain of subcategories

$$0 = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_n = \mathcal{A} \qquad \text{and} \qquad \mathcal{A}_i = \operatorname{mod}(\Lambda_i, k).$$

For each $1 \le i \le n$ we identify $\operatorname{End}_{\mathcal{A}}(\Delta_i) = k$. Then the functor

$$\mathcal{A}_i \longrightarrow \operatorname{proj} k, \quad X \longmapsto \operatorname{Hom}_{\Lambda_i}(\Delta_i, X) \cong X \otimes_{\Lambda_i} \operatorname{Hom}_{\Lambda_i}(\Delta_i, \Lambda_i)$$

admits the left adjoint $- \otimes_k \Delta_i$ and the right adjoint $\operatorname{Hom}_k(\operatorname{Hom}_{\Lambda_i}(\Delta_i, \Lambda_i), -)$. This yields the following diagram of exact functors

$$\begin{tikzcd} \mathcal{A}_{i-1} \arrow[r, tail] & \mathcal{A}_i \arrow[r, two heads] & \operatorname{proj} k \arrow[l, tail, shift right=2ex] \arrow[l, tail, shift left=2ex] \end{tikzcd} \tag{6.5}$$

which one may think of as an incomplete recollement.

The standard object Δ_i equals the image of k under the left adjoint $\operatorname{proj} k \to \mathcal{A}_i$ while the *costandard object* ∇_i is by definition the image of k under the right adjoint $\operatorname{proj} k \to \mathcal{A}_i$. In particular we have

$$\operatorname{Hom}_{\mathcal{A}}(\nabla_j, \nabla_i) = 0 \quad (i > j) \qquad \text{and} \qquad \operatorname{Ext}^1_{\mathcal{A}}(\nabla_j, \nabla_i) = 0 \quad (i \ge j).$$

Proposition 6.7. *Let $\mathcal{A}$ be a k-linear highest weight category. Then the category $\mathcal{A}^{\mathrm{op}}$ is a k-linear highest weight category.*

Proof. We identify $\mathcal{A}^{\mathrm{op}} = \operatorname{mod}(\Lambda^{\mathrm{op}}, k)$ and use the duality $\operatorname{Hom}_k(-, k)$. Set $\Delta'_i = \operatorname{Hom}_k(\nabla_i, k)$ for $1 \leq i \leq n$. Using Theorem 1.1 it is easily checked that $\mathcal{A}^{\mathrm{op}}$ is a k-linear highest weight category with standard objects $\Delta'_1, \ldots, \Delta'_n$. $\square$

We observe that the duality $\operatorname{Hom}_k(-, k)$ induces an equivalence

$$\operatorname{mod}(\Lambda, k)^{\mathrm{op}} \supseteq \operatorname{Filt}(\Delta_1, \ldots, \Delta_n)^{\mathrm{op}} \xrightarrow{\sim} \operatorname{Filt}(\nabla_1, \ldots, \nabla_n) \subseteq \operatorname{mod}(\Lambda^{\mathrm{op}}, k) \tag{6.6}$$

which maps each Δ_i to ∇_i.

Proposition 6.8. *Let $\mathcal{A}$ be a k-linear highest weight category and let Λ denote the endomorphism ring of a projective generator. Then we have a triangle equivalence $\mathbf{D}^{\mathrm{perf}}(\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A})$ and each inclusion $\mathcal{A}_{i-1} \to \mathcal{A}_i$ induces a recollement of triangulated categories*

$$\begin{tikzcd} \mathbf{D}^b(\mathcal{A}_{i-1}) \arrow[r, tail] & \mathbf{D}^b(\mathcal{A}_i) \arrow[l, two heads, shift right=2] \arrow[l, two heads, shift left=2] \arrow[r, two heads] & \mathbf{D}^{\mathrm{perf}}(k) \arrow[l, tail, shift right=2] \arrow[l, tail, shift left=2] \end{tikzcd}.$$

Proof. The diagram (6.5) induces the recollement of derived categories; see the proof of Lemma 3.4. Using these recollements, an induction shows that each inclusion $\operatorname{proj} \Lambda_i \to \operatorname{mod}(\Lambda_i, k) = \mathcal{A}_i$ induces a triangle equivalence $\mathbf{D}^{\mathrm{perf}}(\Lambda_i) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A}_i)$. $\square$

Let $\operatorname{Filt}^{\oplus}(\Delta_1, \ldots, \Delta_n)$ denote the idempotent completion of $\operatorname{Filt}(\Delta_1, \ldots, \Delta_n)$.

Corollary 6.9. *The sequence of inclusion functors*

$$\operatorname{proj} \Lambda \longrightarrow \operatorname{Filt}^{\oplus}(\Delta_1, \ldots, \Delta_n) \longrightarrow \mathcal{A}$$

induces triangle equivalences

$$\mathbf{D}^{\mathrm{perf}}(\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{Filt}^{\oplus}(\Delta_1, \ldots, \Delta_n)) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A}).$$

Analogously, the inclusion $\operatorname{Filt}^{\oplus}(\nabla_1, \ldots, \nabla_n) \to \mathcal{A}$ *induces a triangle equivalence*

$$\mathbf{D}^b(\operatorname{Filt}^{\oplus}(\nabla_1, \ldots, \nabla_n)) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A}).$$

Proof. The argument for the inclusion $\operatorname{proj} \Lambda \to \operatorname{Filt}^{\oplus}(\Delta_1, \ldots, \Delta_n)$ is precisely that given for $\operatorname{proj} \Lambda \to \mathcal{A}$ in Proposition 6.8, using the derived version of the colocalisation sequence (6.2). The assertion for $\operatorname{Filt}^{\oplus}(\nabla_1 \ldots, \nabla_n)$ follows from the first part by duality since $\operatorname{Hom}_k(-, k)$ maps Δ_i to ∇_i. $\square$

Remark 6.10. The triangle equivalence $\mathbf{D}^{\mathrm{perf}}(\Lambda) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A})$ implies that every object in $\mathcal{A}$ has finite projective and finite injective dimension. The bound $2n-2$ from Corollary 5.5 carries over when each diagram (6.5) can be completed to a full recollement.

Exceptional sequences. There is a close connection between standard objects of a highest weight category and exceptional sequences. We wish to make this precise.

Fix a commutative ring k and a k-linear exact category $\mathcal{A}$. An object E in $\mathcal{A}$ is *exceptional* if $\operatorname{End}_{\mathcal{A}}(E) \cong k$ and $\operatorname{Ext}^p_{\mathcal{A}}(E,E) = 0$ for all $p > 0$. A sequence of objects $(E_1, \dots, E_n)$ in $\mathcal{A}$ is called *exceptional* if each E_i is exceptional and $\operatorname{Ext}^p_{\mathcal{A}}(E_i, E_j) = 0$ for all $i > j$ and $p \geq 0$. We say that the sequence is *strictly full* if the inclusion $\operatorname{Filt}(E_1, \dots, E_n) \to \mathcal{A}$ induces, up to direct summands, a triangle equivalence

$$\mathbf{D}^b(\operatorname{Filt}(E_1, \dots, E_n)) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A}).$$

Exceptional sequences were introduced in the Moscow school of vector bundles; see for instance [3, 17, 18, 31].

Theorem 6.11. *Let $\mathcal{A}$ be a k-linear highest weight category. Then the sequence of standard objects $(\Delta_1, \dots, \Delta_n)$ forms a strictly full exceptional sequence in $\mathcal{A}$.*

Conversely, let $\mathcal{A}$ be an idempotent complete k-linear exact category and suppose that Ext-groups are finitely generated over k. If $(E_1, \dots, E_n)$ is a strictly full exceptional sequence in $\mathcal{A}$ such that the endomorphism ring of a projective generator of $\operatorname{Filt}(E_1, \dots, E_n)$ *is finitely generated projective over k, then this identifies with the sequence of standard objects of a highest weight category $\mathcal{A}'$ via a triangle equivalence* $\mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A}')$.

Proof. For the first part, we need to prove that $\operatorname{Ext}^p_{\mathcal{A}}(\Delta_i, \Delta_j) = 0$ for all $i \geq j$ and $p > 1$. This follows from Theorem 6.6. In fact, Δ_i is a projective object in the full subcategory $\mathcal{A}_i$ of $\mathcal{A}$, which is defined recursively by

$$\mathcal{A}_i = \{X \in \mathcal{A}_{i+1} \mid \operatorname{Hom}_{\mathcal{A}}(\Delta_{i+1}, X) = 0\}.$$

Moreover, we use that the inclusion $\mathcal{A}_i \to \mathcal{A}$ induces bijections $\operatorname{Ext}^p_{\mathcal{A}_i}(-,-) \xrightarrow{\sim} \operatorname{Ext}^p_{\mathcal{A}}(-,-)$ for all $p \geq 0$. Thus $(\Delta_1, \dots, \Delta_n)$ is an exceptional sequence in $\mathcal{A}$, which is strictly full by Corollary 6.9.

For the second part, let P denote a projective generator of $\operatorname{Filt}(E_1, \dots, E_n)$. Set $\Lambda = \operatorname{End}_{\mathcal{A}}(P)$ and $\Delta_i = \operatorname{Hom}_{\mathcal{A}}(P, E_i)$. Then Theorem 6.5 implies that $\mathcal{A}'$ is a k-linear highest weight category with standard objects $\Delta_1, \dots, \Delta_n$. Moreover, $\operatorname{Hom}_{\mathcal{A}}(P, -)$ induces an equivalence

$$\operatorname{Filt}(E_1, \dots, E_n) \xrightarrow{\sim} \operatorname{Filt}(\Delta_1, \dots, \Delta_n).$$

Thus $\mathbf{R}\operatorname{Hom}_{\mathcal{A}}(P, -)$ yields a triangle equivalence $\mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A}')$ by Corollary 6.9, since $(E_1, \dots, E_n)$ is strictly full. □

The notion of an exceptional sequence makes equal sense in a triangulated category. For instance, an exceptional sequence $(E_1, \dots, E_n)$ in an exact category $\mathcal{A}$ yields an exceptional sequence in the derived category $\mathbf{D}^b(\mathcal{A})$. For recent applications in the study of coherent sheaves on projective varieties, see [4, 11, 19].

Let $(E_1, \dots, E_n)$ be an exceptional sequence in a k-linear exact category $\mathcal{A}$. If the objects $E_1, \dots, E_n$ generate $\mathbf{D}^b(\mathcal{A})$ as a triangulated category, then the sequence need not be strictly full. I am grateful to Martin Kalck for providing the following example.

Example 6.12 (Kalck)**.** Fix a field k and consider the finite dimensional k-algebra Λ given by the following quiver with relations.

$$\begin{tikzcd} & 1 \arrow[dl, "\alpha"'] & \\ 2 \arrow[rr, "\beta"'] & & 3 \arrow[ul, "\gamma"'] \end{tikzcd} \qquad \begin{aligned} \gamma\beta &= 0 \\ \alpha\gamma &= 0 \end{aligned}$$

For each vertex i let S_i denote the corresponding simple Λ-module and P_i its projective cover. Then (S_1, P_2, P_3) is an exceptional sequence in $\mathcal{A} = \operatorname{mod} \Lambda$ which generates $\mathbf{D}^b(\mathcal{A})$ as a triangulated category. Set $\mathcal{B} = \operatorname{Filt}(S_1, P_2, P_3)$. Then we have $\mathcal{B} = \operatorname{add}(S_1 \oplus P_2 \oplus P_3)$ but $\operatorname{Ext}^2_\Lambda(S_1, P_2) \neq 0$. Thus the canonical functor $\mathbf{D}^b(\mathcal{B}) \to \mathbf{D}^b(\mathcal{A})$ is not full.

7. Ringel duality

There is a special class of tilting modules for any quasi-hereditary artin algebra which Ringel introduced in [29]. This was later extended to highest weight categories over more general base rings [9, 30].

Let k be a commutative ring. We fix a k-linear highest weight category $\mathcal{A}$ with standard objects $\Delta_1, \dots, \Delta_n$ and costandard objects $\nabla_1, \dots, \nabla_n$. To simplify notation we set

$$\operatorname{Filt}(\Delta) = \operatorname{Filt}(\Delta_1, \dots, \Delta_n) \qquad \text{and} \qquad \operatorname{Filt}(\nabla) = \operatorname{Filt}(\nabla_1, \dots, \nabla_n).$$

Given any set $X_1, \dots, X_t$ of objects in $\mathcal{A}$, we write $\operatorname{Filt}^{\oplus}(X_1, \dots, X_t)$ for the closure of $\operatorname{Filt}(X_1, \dots, X_t)$ under direct summands.

Ext-orthogonality. We compute the extensions groups between standard and costandard objects. The first lemma is an immediate consequence of the definition of a highest weight category.

Lemma 7.1. *For $1 \leq j \leq i \leq n$ and $p \geq 0$ we have*

$$\operatorname{Ext}^p_{\mathcal{A}}(\Delta_i, \nabla_j) \cong \begin{cases} k & \text{if } i = j \text{ and } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We use induction on n. The assertion is clear for $p > 0$ since Δ_n is projective. For $p = 0$ we use the recollement (6.4). In fact, $\Delta_n = j_!(k)$ and $\nabla_n = j_*(k)$. Thus $\operatorname{Hom}_{\mathcal{A}}(\Delta_n, \nabla_n) \cong k$ by adjointness, and $\operatorname{Hom}_{\mathcal{A}}(\Delta_n, \nabla_j) = 0$ for $n > j$ since ∇_j belongs to $\operatorname{Mod} \Lambda_{n-1}$. $\square$

Corollary 7.2. *For $X \in \operatorname{Filt}^{\oplus}(\Delta)$ and $Y \in \operatorname{Filt}^{\oplus}(\nabla)$ we have $\operatorname{Ext}^p_{\mathcal{A}}(X, Y) = 0$ for all $p > 0$ and the k-module $\operatorname{Hom}_{\mathcal{A}}(X, Y)$ is finitely generated projective.* $\square$

Proposition 7.3. *Let $\mathcal{A}$ be a highest weight category. For X in $\mathcal{A}$ we have:*

(1) $X \in \operatorname{Filt}^{\oplus}(\Delta)$ *if and only if* $\operatorname{Ext}^1_{\mathcal{A}}(X, \nabla_i) = 0$ *for* $1 \le i \le n$.
(2) $X \in \operatorname{Filt}^{\oplus}(\nabla)$ *if and only if* $\operatorname{Ext}^1_{\mathcal{A}}(\Delta_i, X) = 0$ *for* $1 \le i \le n$.

Proof. We prove (1) and the proof of (2) is dual. One direction is clear by Corollary 7.2. Thus assume that $\operatorname{Ext}^1_{\mathcal{A}}(X, \nabla_i) = 0$ for all i. We use induction on n and consider the recollement (6.4). First observe that the counit $j_! j^!(X) \to X$ is a monomorphism. To see this, fix an injective cogenerator Q of $\mathcal{A}$. Note that Q belongs to $\operatorname{Filt}^{\oplus}(\nabla_1, \dots, \nabla_n)$. Thus we have an exact sequence

$$0 \longrightarrow i_* i^*(Q) \longrightarrow Q \longrightarrow j_! j^!(Q) \longrightarrow 0$$

which induces the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Hom}_{\mathcal{A}}(X, i_! i^!(Q)) & \longrightarrow & \operatorname{Hom}_{\mathcal{A}}(X, Q) & \longrightarrow & \operatorname{Hom}_{\mathcal{A}}(X, j_* j^*(Q)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \operatorname{Hom}_{\mathcal{A}}(j_! j^!(X), i_! i^!(Q)) & \to & \operatorname{Hom}_{\mathcal{A}}(j_! j^!(X), Q) & \to & \operatorname{Hom}_{\mathcal{A}}(j_! j^!(X), j_* j^*(Q)) & \to & 0 \end{array}$$

We have $\operatorname{Hom}_{\mathcal{A}}(j_! j^!(X), i_! i^!(Q)) = 0$ and the map

$$\operatorname{Hom}_{\mathcal{A}}(X, j_* j^*(Q)) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(j_! j^!(X), j_* j^*(Q))$$

is a bijection by adjointness. Thus the map

$$\operatorname{Hom}_{\mathcal{A}}(X, Q) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(j_! j^!(X), Q)$$

is surjective. It follows that the sequence

$$0 \longrightarrow j_! j^!(X) \longrightarrow X \longrightarrow i_* i^*(X) \longrightarrow 0$$

given by the unit and counit for X is exact. The object $X' = i_* i^*(X)$ belongs to $\mathcal{A}_{n-1}$ and satisfies again $\operatorname{Ext}^1_{\mathcal{A}}(X', \nabla_i) = 0$ for all i. Thus X' belongs to $\operatorname{Filt}^{\oplus}(\Delta_1, \dots, \Delta_{n-1})$ by induction. It follows that X belongs to $\operatorname{Filt}^{\oplus}(\Delta_1, \dots, \Delta_n)$. $\square$

Remark 7.4. A consequence of Proposition 7.3 is the fact that the subcategory $\operatorname{Filt}^{\oplus}(\Delta)$ of $\mathcal{A}$ is closed under taking kernels of epimorphisms.

Tilting objects. We describe the special tilting objects for a k-linear highest weight category.

Proposition 7.5. *Let $\mathcal{A}$ be a k-linear highest weight category with costandard objects $\nabla_1, \dots, \nabla_n$. Then there are finitely many exact sequences*

$$0 \longrightarrow V_i \longrightarrow T_i \longrightarrow \nabla_i \longrightarrow 0 \qquad (1 \le i \le n)$$

in $\mathcal{A}$ satisfying the following:

(1) V_i *belongs to* $\operatorname{Filt}(\nabla_1, \dots, \nabla_{i-1})$ *for all* i.
(2) $T = \bigoplus_{i=1}^n T_i$ *is a projective generator of* $\operatorname{Filt}(\nabla_1, \dots, \nabla_n)$.
(3) $\operatorname{End}_{\mathcal{A}}(T)$ *is finitely generated projective over* k.

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Proof. The costandard objects satisfy $\operatorname{Ext}^1_{\mathcal{A}}(\nabla_j, \nabla_i) = 0$ for all $i \geq j$ because of the duality (6.6). Now apply Lemma 6.4. The object T belongs to $\operatorname{Filt}^{\oplus}(\Delta_1,\ldots,\Delta_n)$ by Proposition 7.3. Thus $\operatorname{End}_{\mathcal{A}}(T)$ is finitely generated projective over k by Corollary 7.2. □

We formulate some immediate consequences of Proposition 7.5.

For an object X in an additive category we denote by $\operatorname{add} X$ the full subcategory whose objects are the direct summands of finite direct sums of copies of X.

Corollary 7.6. *Let $\mathcal{A}$ be a k-linear highest weight category $\mathcal{A}$. For an object T in $\mathcal{A}$ the following are equivalent:*

(1) *T is a projective generator of* $\operatorname{Filt}^{\oplus}(\nabla)$.
(2) *T is an injective cogenerator of* $\operatorname{Filt}^{\oplus}(\Delta)$.
(3) $\operatorname{Filt}^{\oplus}(\Delta) \cap \operatorname{Filt}^{\oplus}(\nabla) = \operatorname{add} T$.

Proof. Combine Propositions 7.3 and 7.5. □

Corollary 7.7. *Let $\mathcal{A}$ be a k-linear highest weight category $\mathcal{A}$ with costandard objects $\nabla_1, \ldots, \nabla_n$ and fix a projective generator T of* $\operatorname{Filt}(\nabla_1, \ldots, \nabla_n)$. *Set $\Lambda' = \operatorname{End}_{\mathcal{A}}(T)$ and $\Delta'_i = \operatorname{Hom}_{\mathcal{A}}(T, \nabla_{n-i})$. Then* $\operatorname{mod}(\Lambda', k)$ *is a k-linear highest weight category with standard objects $\Delta'_1, \ldots, \Delta'_n$ and* $\operatorname{Hom}_{\mathcal{A}}(T, -)$ *induces an equivalence*

$$\operatorname{Filt}(\nabla_1, \ldots, \nabla_n) \xrightarrow{\sim} \operatorname{Filt}(\Delta'_1, \ldots, \Delta'_n) \tag{7.1}$$

of exact categories. □

The highest weight category $\operatorname{mod}(\Lambda', k)$ in Corollary 7.7 is called the *Ringel dual* of $\mathcal{A}$. If $\mathcal{A} \xrightarrow{\sim} \operatorname{mod}(\Lambda, k)$ for some k-split quasi-hereditary algebra Λ, then the quasi-hereditary algebra Λ' is called the *Ringel dual* of Λ; it is unique only up to Morita equivalence.

Proposition 7.8. *Let Λ be a k-split quasi-hereditary algebra. The double Ringel dual $\Lambda'' = (\Lambda')'$ is Morita equivalent to Λ. The equivalence identifies the standard modules over Λ'' and Λ.*

Proof. We have equivalences

$$\operatorname{Filt}(\Delta'') \xrightarrow{\sim} \operatorname{Filt}(\nabla') \xrightarrow{\sim} \operatorname{Filt}(\Delta')^{\mathrm{op}} \xrightarrow{\sim} \operatorname{Filt}(\nabla)^{\mathrm{op}} \xrightarrow{\sim} \operatorname{Filt}(\Delta)$$

of exact categories. Restricting this equivalence to the full subcategories of projective objects yields an equivalence $\operatorname{proj} \Lambda'' \xrightarrow{\sim} \operatorname{proj} \Lambda$. □

Recall that an object T of an exact category $\mathcal{A}$ is a *tilting object* if $\operatorname{Ext}^p_{\mathcal{A}}(T, T) = 0$ for $p > 0$ and $\mathbf{D}^b(\mathcal{A})$ admits no proper thick subcategory containing T. An equivalent statement is that $\mathbf{R}\operatorname{Hom}_{\mathcal{A}}(T, -)$ induces a triangle equivalence $\mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^{\mathrm{perf}}(\operatorname{End}_{\mathcal{A}}(T))$. In that case a quasi-inverse is denoted by $- \otimes^{\mathbf{L}}_{\operatorname{End}_{\mathcal{A}}(T)} T$.

Corollary 7.9. *Let $\mathcal{A}$ be a k-linear highest weight category $\mathcal{A}$. Then a projective generator of* $\operatorname{Filt}(\nabla)$ *is a tilting object of $\mathcal{A}$.*

Proof. Fix a projective generator T and set $\Lambda' = \operatorname{End}_{\mathcal{A}}(T)$. Then the sequence of fully faithful exact functors

$$\operatorname{proj}\Lambda' \xrightarrow{\sim} \operatorname{add} T \longrightarrow \operatorname{Filt}^{\oplus}(\nabla) \longrightarrow \mathcal{A}$$

induces a triangle equivalence

$$\mathbf{D}^{\mathrm{perf}}(\Lambda') \xrightarrow{\sim} \mathbf{D}^b(\operatorname{Filt}^{\oplus}(\nabla)) \xrightarrow{\sim} \mathbf{D}^b(\mathcal{A})$$

which is a quasi-inverse of $\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(T,-)$. The first equivalence follows from Lemma 5.4 and the second from Corollary 6.9. □

For a k-linear highest weight category $\mathcal{A}$ an object T satisfying the equivalent conditions in Corollary 7.6 is called *characteristic tilting object*.

Tor-orthogonality. Ext-orthogonality for modules over a quasi-hereditary algebra translates into Tor-orthogonality. To see this we need to recall some standard isomorphisms for derived functors.

Lemma 7.10. *Let Λ be a k-algebra and X, Y be complexes of Λ-modules. Then there are natural morphisms*

$$X \otimes^{\mathbf{L}}_{\Lambda} \mathbf{R}\mathrm{Hom}_k(Y,k) \longrightarrow \mathbf{R}\mathrm{Hom}_k(\mathbf{R}\mathrm{Hom}_{\Lambda}(X,Y),k)$$
$$Y \otimes^{\mathbf{L}}_{\Lambda} \mathbf{R}\mathrm{Hom}(X,\Lambda) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda}(X,Y)$$

which are isomorphisms when X is perfect. □

Proposition 7.11. *Let Λ be a k-split quasi-hereditary algebra. For*

$$X \in \operatorname{Filt}^{\oplus}(\Delta) \subseteq \operatorname{mod}(\Lambda,k) \qquad \textit{and} \qquad Y \in \operatorname{Filt}^{\oplus}(\Delta) \subseteq \operatorname{mod}(\Lambda^{\mathrm{op}},k)$$

we have

$$\operatorname{Tor}^{\Lambda}_p(X,Y) = 0 \quad \textit{for} \quad p > 0.$$

Proof. This follows from Corollary 7.2 with the first isomorphism in Lemma 7.10, since $\mathrm{Hom}_k(-k)$ induces an equivalence $\operatorname{Filt}^{\oplus}(\nabla) \xrightarrow{\sim} \operatorname{Filt}^{\oplus}(\Delta)$; see (6.6). □

Serre duality. Let Λ be a k-algebra that is finitely generated projective over k. Then the Λ-module $\mathrm{Hom}_k(\Lambda,k)$ is an injective cogenerator of $\operatorname{mod}(\Lambda,k)$ and plays the role of a dualising complex.

Lemma 7.12. *Suppose that the Λ-module* $\mathrm{Hom}_k(\Lambda,k)$ *has finite projective dimension. Then*

$$F = - \otimes^{\mathbf{L}}_{\Lambda} \mathrm{Hom}_k(\Lambda,k) \colon \mathbf{D}^{\mathrm{perf}}(\Lambda) \longrightarrow \mathbf{D}^{\mathrm{perf}}(\Lambda)$$

is a Serre functor *in the sense that F is a triangle equivalence and*

$$\mathbf{R}\mathrm{Hom}_k(\mathbf{R}\mathrm{Hom}_{\Lambda}(X,-),k) \cong \mathbf{R}\mathrm{Hom}_{\Lambda}(-,F(X)) \quad \textit{for} \quad X \in \mathbf{D}^{\mathrm{perf}}(\Lambda).$$

Proof. Using the standard isomorphisms from Lemma 7.10 we have

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_k(\mathbf{R}\mathrm{Hom}_\Lambda(X,-),k) &\cong \mathbf{R}\mathrm{Hom}_k(-\otimes^{\mathbf{L}}_\Lambda \mathbf{R}\mathrm{Hom}_\Lambda(X,\Lambda),k)\\ &\cong \mathbf{R}\mathrm{Hom}_\Lambda(-,\mathbf{R}\mathrm{Hom}_k(\mathbf{R}\mathrm{Hom}_\Lambda(X,\Lambda),k))\\ &\cong \mathbf{R}\mathrm{Hom}_\Lambda(-,X\otimes^{\mathbf{L}}_\Lambda \mathrm{Hom}_k(\Lambda,k))\end{aligned}$$

and a quasi-inverse of F is given by $\mathbf{R}\mathrm{Hom}_\Lambda(\mathrm{Hom}_k(\Lambda,k),-)$. □

Proposition 7.13. *Let Λ be a k-split quasi-hereditary algebra. Then*

$$-\otimes^{\mathbf{L}}_\Lambda \mathrm{Hom}_k(\Lambda,k)\colon \mathbf{D}^{\mathrm{perf}}(\Lambda)\longrightarrow \mathbf{D}^{\mathrm{perf}}(\Lambda)$$

is a Serre functor.

Proof. Combine Lemma 7.12 with the fact that $\mathrm{Hom}_k(\Lambda,k)$ has finite projective dimension; see Remark 6.10. □

Serre duality and Ringel duality are closely related for a quasi-hereditary algebra. In order to explain this we need the following lemma.

Lemma 7.14. *Let Λ be a k-split quasi-hereditary algebra with characteristic tilting module T and set $\Gamma=\mathrm{End}_\Lambda(T)$. Then $\mathrm{Hom}_k(T,k)$ is a characteristic tilting module over Γ and Λ^{op} with canonical isomorphisms*

$$\mathrm{End}_\Gamma(\mathrm{Hom}_k(T,k))\cong\Lambda \qquad \textit{and} \qquad \mathrm{End}_{\Lambda^{\mathrm{op}}}(\mathrm{Hom}_k(T,k))\cong\Gamma^{\mathrm{op}}.$$

Moreover, T is a characteristic tilting module over Γ^{op} with $\mathrm{End}_{\Gamma^{\mathrm{op}}}(T)\cong\Lambda^{\mathrm{op}}$.

Proof. For an exact category $\mathcal{A}$ we write $\operatorname{proj}\mathcal{A}$ and $\operatorname{inj}\mathcal{A}$ to denote the full subcategories of projective and injective objects, respectively.

The equivalence (7.1) given by $\mathrm{Hom}_\Lambda(T,-)$ restricts to an equivalence

$$\operatorname{inj}(\mathrm{Filt}^\oplus(\nabla))\xrightarrow{\sim}\operatorname{inj}(\mathrm{Filt}^\oplus(\Delta))=\operatorname{add}T$$

and sends $\mathrm{Hom}_k(\Lambda,k)$ to

$$\mathrm{Hom}_\Lambda(T,\mathrm{Hom}_k(\Lambda,k))\cong\mathrm{Hom}_k(T,k).$$

Thus $\mathrm{Hom}_k(T,k)$ is a characteristic tilting module over Γ with

$$\mathrm{End}_\Gamma(\mathrm{Hom}_k(T,k))\cong\mathrm{End}_\Lambda(\mathrm{Hom}_k(\Lambda,k))\cong\Lambda.$$

On the other hand, the equivalence (6.6) given by $\mathrm{Hom}_k(-,k)$ restricts to an equivalence

$$(\operatorname{add}T)^{\mathrm{op}}=\operatorname{inj}(\mathrm{Filt}^\oplus(\Delta))^{\mathrm{op}}\xrightarrow{\sim}\operatorname{proj}(\mathrm{Filt}^\oplus(\nabla)).$$

Thus $\mathrm{Hom}_k(T,k)$ is a characteristic tilting module over Λ^{op} with

$$\mathrm{End}_{\Lambda^{\mathrm{op}}}(\mathrm{Hom}_k(T,k))\cong\mathrm{End}_\Lambda(T)^{\mathrm{op}}\cong\Gamma^{\mathrm{op}}.$$

The assertion for the Γ^{op}-module T now follows since $T\cong\mathrm{Hom}_k(\mathrm{Hom}_k(T,k),k)$. □

Theorem 7.15. *Let Λ be a k-split quasi-hereditary algebra with characteristic tilting module T and set $\Gamma = \operatorname{End}_\Lambda(T)$. Then*

$$\operatorname{Hom}_k(T,k) \otimes_\Gamma T \cong \operatorname{Hom}_k(T,k) \otimes^{\mathbf{L}}_\Gamma T \cong \operatorname{Hom}_k(\Lambda,k)$$

as Λ-Λ-bimodules. Therefore the composite

$$\mathbf{D}^{\mathrm{perf}}(\Lambda) \xrightarrow{-\otimes^{\mathbf{L}}_\Lambda \operatorname{Hom}_k(T,k)} \mathbf{D}^{\mathrm{perf}}(\Gamma) \xrightarrow{-\otimes^{\mathbf{L}}_\Gamma T} \mathbf{D}^{\mathrm{perf}}(\Lambda)$$

is a Serre functor.

Proof. We apply Lemma 7.14. The modules T and $\operatorname{Hom}_k(T,k)$ over Γ are characteristic tilting modules; this yields the first isomorphism by Proposition 7.11. The second isomorphism follows from Lemma 7.10. The description of the Serre functor then follows by Lemma 7.12. □

Ringel self-dual algebras. The connection between Serre duality and Ringel duality is of particular interest for a quasi-hereditary algebra that is Ringel self-dual.

Definition 7.16. We say that a quasi-hereditary algebra Λ is *Ringel self-dual* if it satisfies one of the following equivalent conditions:

(1) The highest weight category $\operatorname{mod}(\Lambda,k)$ is equivalent to its Ringel dual.
(2) There is a characteristic tilting module T over Λ and an isomorphism $\Lambda' = \operatorname{End}_\Lambda(T) \xrightarrow{\sim} \Lambda$ which identifies the standard modules over Λ' and Λ.

The following description of Ringel duality as a square root of Serre duality is inspired by a result for strict polynomial functors [23] and a similar result in the context of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ [25].

Let us fix for a Ringel self-dual algebra Λ a characteristic tilting module T as in the above definition and identify $\operatorname{End}_\Lambda(T) = \Lambda$. This turns T and $\operatorname{Hom}_k(T,k)$ into Λ-Λ-bimodules.

Corollary 7.17. *Let Λ be a k-split quasi-hereditary algebra. Suppose that Λ is Ringel self-dual with characteristic tilting module T. Then the following are equivalent:*

(1) $T \cong \operatorname{Hom}_k(T,k)$ *as Λ-Λ-bimodules.*
(2) $T \otimes^{\mathbf{L}}_\Lambda T \cong \operatorname{Hom}_k(\Lambda,k)$ *as Λ-Λ-bimodules.*
(3) $(- \otimes^{\mathbf{L}}_\Lambda T)^2$ *is a Serre functor for $\mathbf{D}^{\mathrm{perf}}(\Lambda)$.*

Proof. Apply Theorem 7.15. □

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