CENTRE DE RECERCA MATEMÀTICA

Preprint núm. 1194
September 2014

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CONTACT ISOTROPIC REALISATIONS OF JACOBI MANIFOLDS

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Abstract. This paper investigates the local and global theory of contact isotropic realisations of Jacobi manifolds, which are contact realisations of minimal dimension. These arise in the study of integrable Hamiltonian systems on contact manifolds, while also extending the Boothby-Wang construction of regular contact manifolds. The main results of the paper are local smooth and contact normal forms for contact isotropic realisations, which, amongst other things, provide an intrinsic proof of the existence of local action-angle coordinates for integrable Hamiltonian systems, as well as a cohomological criterion to construct such realisations. Moreover, one of the smooth invariants of such realisations is interpreted as providing a type of transversal projective structure on the foliation of the underlying Jacobi structures.

1. Introduction

This paper studies a certain family of contact realisations of Jacobi manifolds. A **contact structure** on a smooth manifold M is a maximally non-integrable smooth hyperplane distribution $H \subset TM$ (cf. Definition 1); these can be thought of as odd-dimensional analogues of symplectic structures (which are closed, non-degenerate 2-forms). This analogy is far-reaching. For instance, as a symplectic manifold (S, ω) gives rise to a Poisson bracket on $C^\infty(S)$ (a Lie bracket which satisfies a Leibniz-type rule) which is transitive, so can contact structures be seen as (odd dimensional) transitive examples of Jacobi manifolds. A **Jacobi structure** on a smooth manifold P is a local Lie bracket $\{\cdot,\cdot\}$ on the space of sections of some fixed real line bundle $L \to P$ (cf. Definition 4 and [24, 26, 27, 28]). A contact manifold (M, H) naturally comes equipped with a Jacobi bracket on the space of sections of the line bundle $TM/H \to M$, which is not necessarily trivial, or even trivialisable, *e.g.* the projectivisation of the cotangent bundle with canonical contact structure (cf. [1, Appendix 4] for more details). Continuing with the analogy

$$\text{symplectic} : \text{Poisson} = \text{contact} : \text{Jacobi},$$

just as symplectic realisations of Poisson manifolds play an important role in understanding the geometry of Poisson structures (cf. [5, 6, 9, 22]), so do **contact realisations** of Jacobi manifolds, *i.e.* surjective submersions $\phi\colon (M, H) \to$

CRM Preprint Series number 1194

2010 *Mathematics Subject Classification.* 53D17, 53D35, 37J35, 37J55.

$(P, L, \{\cdot,\cdot\})$ which are Jacobi morphisms (cf. Remark 4). Intuitively, symplectic/contact realisations contain information regarding the 'integration' of the Lie algebraic structure encoded in a Poisson/Jacobi manifold; as such they should be thought of 'desingularisations' of the underlying Poisson/Jacobi structures (cf. [6, Section 7] and [34, Section 3.2]).

The types of contact realisations considered in this paper are those of minimal dimension, *i.e.* the dimension of the total space (M, H) is as small as possible. These are called **contact isotropic realisations** and are contact realisations $\phi\colon (M, H) \to (P, L, \{\cdot,\cdot\})$ with the extra condition that the fibres are 'isotropic'. Informally, this condition can be described as follows. For each $m \in M$, maximal non-integrability of H induces a non-degenerate bilinear form on H_m, *i.e.* a symplectic form. Then the fibres of ϕ are isotropic if, for each $m \in M$, $\ker D_m\phi \cap H_m$ is an isotropic subspace (cf. Definition 12 for the precise notion). The simplest such example is given by the Boothby-Wang construction of regular contact manifolds (cf. [4, 19]), which can be seen as contact isotropic realisations of symplectic manifolds viewed as Jacobi structures (cf. Example 15). Intuitively, contact isotropic realisations exhibit the most rigid behaviour amongst contact realisations: this observation stems from the results concerning symplectic isotropic realisations of Poisson manifolds. These realisations can be used to study the geometry of integrable Hamiltonian systems on symplectic manifolds (cf. [14, 17]); moreover, recently they have been studied in relation to Poisson manifolds of compact type (cf. the forthcoming [8]). The work in [14] provides local and global classifications of symplectic isotropic realisations; this begs the question of whether analogous results hold for contact isotropic realisations, especially in light of the recently introduced notion of integrable Hamiltonian systems on contact manifolds (cf. [21, 30] and Section 6.1 for further details).

The aim of this paper is to classify contact isotropic realisations of Jacobi manifolds. The approach taken combines the framework of [14, 17] developed to study symplectic isotropic realisations, with new techniques coming from recent work on the integrability of Jacobi structures (cf. [10] and also [11] for even more general methods applicable to Lie groupoids and algebroids). The main difficulty to apply the ideas of [14, 17] directly is the presence of *non-trivial line bundles*, which complicate matters both conceptually and technically. This is precisely why the methods of [10, 11] are employed in this paper; the upshot of this approach is to provide *intrinsic* results on contact/Jacobi manifolds whose underlying line bundles are non-trivial, of which there are not many in the existing literature, while also extending results on the specific types of realisations under consideration in this paper (cf. [2, 4, 21]).

The isotropy condition imposes a restriction on the type of Jacobi manifolds admitting a contact isotropic realisation. Associated to a Jacobi structure

$(P, L, \{\cdot,\cdot\})$ is a **Lie algebroid** structure on the first jet bundle J^1L of $L \to P$, which consists of a Lie bracket on $\Gamma(J^1L)$ and a compatible map $\rho\colon J^1L \to TP$ (cf. Proposition 2 and [10]). This plays a central role in this paper. A necessary condition for $(P, L, \{\cdot,\cdot\})$ to admit a contact isotropic realisation is that the structure be regular with even dimensional leaves, *i.e.* ρ is of constant even rank (cf. Lemma 3). Examples of such Jacobi manifolds are regular Poisson structures, zero Jacobi structures (defined on any line bundle), and locally conformal symplectic manifolds (cf. Example 7). Once this necessary condition is established, the first main result is a *smooth* classification of contact isotropic realisations of a fixed Jacobi manifold $(P, L, \{\cdot,\cdot\})$; as in the case of symplectic isotropic realisations, there are two invariants:

- a **period lattice** $\Sigma \subset \ker\rho \subset J^1L$ (cf. Definition 11), all of whose sections are *holonomic*, *i.e.* of the form j^1u for some $u \in \Gamma(L)$ (cf. Corollary 12);
- the **Chern class** $c \in \mathrm{H}^2(P;\Sigma)$ (cf. Definition 14).

This follows from Theorem 9, whose proof is entirely analogous to that of [32, Theorem 8.15]. In fact, the content of Theorem 9 is that a smooth local model for a contact isotropic realisation with period lattice Σ is given by $\ker\rho/\Sigma \to P$.

The second main result is a *contact* local normal form (in a neighbourhood of a fibre) for contact isotropic realisations. Recall that, given a line bundle $L \to P$, its first jet bundle J^1L comes equipped with a canonical contact structure H, defined as the kernel of the Cartan 1-form $\theta_{\mathrm{can}} \in \Omega^1(J^1L; L)$ (cf. Example 2 for further details).

Contact model. *A contact isotropic realisation $\phi\colon (M,H) \to (P, L, \{\cdot,\cdot\})$ is locally isomorphic to*

$$\pi\colon (\ker\rho/\Sigma, \ker(\theta_0 + \pi^*\beta)) \to (P, L, \{\cdot,\cdot\}),$$

where $\theta_0 \in \Omega^1(\ker\rho/\Sigma; L)$ is induced by θ_{can} and $\beta \in \Omega^1(P; L)$ is a locally defined 1-form determined (not uniquely!) by the Jacobi bracket.

This is the content of Theorem 13, which provides a more precise statement. While there exist similar results in the literature regarding integrable Hamiltonian systems on contact manifolds (cf. the theorems on existence of action-angle coordinates in [2, 21]), the above statement and the proof presented in this paper are more intrinsic, as the various geometric objects related to Jacobi structures are used in a natural fashion to obtain the local normal form.

Given a regular Jacobi manifold $(P, L, \{\cdot,\cdot\})$ whose foliation $\mathcal{F}$ consists solely of even dimensional leaves, a natural question to ask is to construct all its contact isotropic realisations. The third main result of the paper answers this question *once* a suitable $\Sigma \subset \ker\rho$ has been fixed, by providing a cohomological criterion which is analogous to that of [14, Theorems 4.2 and 4.3].

Cohomological criterion. *A cohomology class* $c \in \mathrm{H}^2(P;\Sigma)$ *is the Chern class of a contact isotropic realisation of* $(P, L, \{\cdot,\cdot\})$ *if and only if* $\mathcal{D}c = [\omega]$*, where* $\mathcal{D}\colon \mathrm{H}^2(P;\Sigma) \to \mathrm{H}^2(\mathcal{F};L)$ *is a homomorphism induced by the classical Spencer operator of* J^1L*, and* $[\omega]$ *is a the cohomology class of a foliated 2-form with values in* L *determined by the Jacobi bracket.*

The above criterion provides both a common framework and an extension of various existing results in the literature, such as the criterion for constructing regular contact manifolds (cf. [19, Theorems 7.2.4 and 7.2.5]), and the cohomological criterion for the existence of completely integrable contact forms of toric type (cf. [2]). At the heart of the above result lies the **classical Spencer operator** $\mathrm{D}\colon \Gamma(J^1L) \to \Omega^1(P;L)$, which recently has been discovered to play a central role in Jacobi geometry (cf. Observation 6 for a definition of D and [10] for further results).

In analogy with a symplectic isotropic realisation of a Poisson manifold, associated to which is an integral affine structure transversal to the symplectic foliation, the last main result of this paper characterises the underlying geometric structure on the foliation $\mathcal{F}$ of a Jacobi manifold $(P, L, \{\cdot,\cdot\})$ admitting a contact isotropic realisation. The idea is to use give an intrinsic characterisation of the period lattice $\Sigma \subset J^1L$, *i.e.* which does not rely on the given realisation (cf. Definition 18). As expected from the case $L = P \times \mathbb{R}$, the ensuing geometric structure on $\mathcal{F}$ is **transversally integral projective**, *i.e.* it gives rise to a foliated atlas with values in real projective spaces along with an integrality condition (cf. Definition 21); what is surprising is that this result holds even when L is non-trivial: this is the content of Theorem 19.

The structure of this paper is as follows. Section 2 recalls basic notions and properties of contact and Jacobi manifolds which are used throughout; particular importance is given the Lie algebroid associated to a Jacobi manifold, described from the point of view developed in [10] (cf. Section 2.3). Contact isotropic realisations are introduced and studied in Section 3. After establishing a simple necessary condition on the Jacobi structure for the existence of such a realisation (cf. Section 3.1), there is an intermezzo dealing with properties of the family of Jacobi manifolds that occur: these share many properties with regular Poisson manifolds. This is the content of Section 3.2. The smooth and contact local normal forms of contact isotropic realisations are constructed in Sections 3.3 and 3.4 respectively. The question of constructing contact isotropic realisations, both locally and globally, is dealt with in Section 4. Any regular Jacobi manifold all of whose leaves are even dimensional admits *locally* a contact isotropic realisation (cf. Section 4.1); the cohomological problem of glueing these local constructions together is solved in Section 4.2. The geometric structure induced by a contact isotropic realisation on the foliation of a Jacobi manifold is introduced in Section 5. Some interesting examples and applications of the theory developed are discussed in Section 6. These include integrable Hamiltonian systems on contact manifolds, which are shown to be equivalent to contact isotropic realisations of a special class of Jacobi manifolds (cf. Section 6.1), and contact isotropic realisations of Poisson manifolds and of locally conformal symplectic manifolds (viewed as Jacobi structures on the trivial line bundle) in Sections 6.2 and 6.3 respectively. Throughout this note, there are two types of comments, labelled **Observation** and *Remark* respectively; those with the former label are central to the problems studied in this paper, while those with the latter may be skipped at a first reading.

Notation and conventions. Throughout the paper, whenever a collection of open sets $\{U_i\}$ is considered, $U_{i_1 i_2 \ldots i_N} = U_{i_1} \cap \ldots \cap U_{i_N}$ for any finite set of indices $i_1, \ldots, i_N$. All line bundles considered in this paper are real unless otherwise stated.

Acknowledgements. We would like to thank Camilo Arias Abad for interesting conversations. M.A.S. would like to thank IMPA, CRM and MPIM Bonn for hospitality at various stages of the project. M.A.S. was partly supported by the DevMath programme of the Centre de Recerca Matemàtica and by the Max Planck Institute for Mathematics in Bonn. D.S. was partly supported by ERC starting grant 279729 and by the NWO Veni grant 639.031.345.

2. Generalities on contact and Jacobi manifolds

This section recalls basic definitions, properties and examples of contact and Jacobi manifolds that are used throughout this paper (cf. [3, 10, 15, 18, 28] amongst others for further details and proofs).

2.1. Contact manifolds and their properties.

Definition 1 (Contact manifolds)**.** A **contact manifold** is a pair (M, H) consisting of a manifold M and a smooth hyperplane distribution $H \subset TM$ with the property that the **curvature** map defined on sections by

$$c\colon \Gamma(H) \times \Gamma(H) \to TM/H, \; c(X, Y) := [X, Y] \bmod H, \tag{1}$$

is fibre-wise non-degenerate. The distribution H is said to be a **contact distribution**.

Remark 1. If H is co-orientable, *i.e.* the line bundle $TM/H \to M$ is trivialisable, then a choice of trivialisation for $TM/H \to M$ corresponds to a globally defined 1-form $\theta \in \Omega^1(M)$ satisfying $H = \ker \theta$. In this case, the condition of equation (1) can be rephrased as non-degeneracy of $\mathrm{d}\theta|_H$.

Observation 1. Most works in the literature deal with contact manifolds (M, H) whose associated line bundle $L := TM/H \to M$ is co-oriented (cf. Remark 1). However, there are natural examples of contact manifolds which do not satisfy this hypothesis, *e.g.* the canonical contact structure on the projectivisation of any cotangent bundle (cf. Example 2 for other important examples which are central to this work). In general, given a contact manifold (M, H) as in Definition 1, the contact structure H can be described tautologically as the kernel of $\theta \in \Omega^1(M, L)$, where

$$\theta \colon TM \to L = TM/H$$

is the projection. The condition of equation (1) can be rephrased as fibre-wise non-degeneracy of $(X, Y) \mapsto \theta([X, Y])$, for $X, Y \in H$. In fact, contact manifolds can be equivalently described as pairs (M, θ) consisting of a smooth manifold M and a point-wise surjective 1-form $\theta \in \Omega^1(M, L)$, such that the curvature defined on sections by

$$\Gamma(\ker\theta) \times \Gamma(\ker\theta) \to \Gamma(L),\ (X, Y) \mapsto c_\theta(X, Y) := \theta([X, Y]) \tag{2}$$

is fibre-wise non-degenerate. Throughout this paper, both points of views are used for different purposes and the 1-form satisfying $\theta = \ker H$ is referred to as a **generalised contact form**.

Example 2. Let $\pi \colon L \to P$ a line bundle and denote by $pr \colon J^1L \to P$ the **first jet bundle**

$$J^1L|_x = \{j^1_x u \mid u \in \Gamma(L)\}.$$

Any (local) section $u \in \Gamma(L)$ induces a (local) section $j^1u \in \Gamma(J^1L)$ which is defined by $p \mapsto j^1_p u$; these are called **holonomic** sections. The Cartan contact form

$$\theta_{\mathrm{can}} \in \Omega^1(J^1L, pr^*L)$$

detects holonomic section in the sense that a section ξ of J^1L is holonomic if and only if $\xi^*\theta_{\mathrm{can}} = 0$. The Cartan contact form is defined as follows. By abuse of notation denote by $pr \colon J^1L \to L$ the canonical projection and let $X \in T_{j^1_p u}J^1L$. Then

$$\theta_{\mathrm{can}}(X) := Dpr(X) - Du(D\pi(X)),$$

lies in $\ker D\pi \cong pr^*L$. As its name suggests, the Cartan contact form θ_{can} is a generalised contact form.

Throughout this subsection, fix a contact manifold (M, H) unless otherwise stated, and denote by

$$L = TM/H$$

the induced line bundle and by $\theta \colon TM \to L$ the projection.

Definition 3. (Reeb vector fields) A **Reeb vector field** of the contact manifold (M, H) is any vector field R satisfying

$$[R, \Gamma(H)] \subset \Gamma(H).$$

The vector space of Reeb vector fields is henceforth denoted by $\mathfrak{X}_{\mathrm{Reeb}}(M,H)$.

The following lemma relates $\mathfrak{X}_{\mathrm{Reeb}}(M,H)$ with $\Gamma(L)$ (cf. [10] for its proof).

Lemma 1. *The map*

$$\begin{aligned}\mathfrak{X}_{\mathrm{Reeb}}(M,H) &\to \Gamma(L)\\ R &\mapsto \theta(R)\end{aligned}$$

is a vector space isomorphism.

Observation 2. The vector field R_u associated to $u \in \Gamma(L)$ is called the *Reeb vector field* of u and is uniquely defined by $\theta(R_u) = u$ and $\theta([R_u, H]) = 0$, where $\theta \in \Omega^1(M, L)$ is a generalised contact form. If $f \in C^\infty(M)$, it can be checked that

$$R_{fu} = fR_u + c_\theta^\sharp(-\mathrm{d}f|_H \otimes u),$$

where $\theta([c_\theta^\sharp(\eta), \cdot]) = -\eta$ for any section $\eta \in \Gamma(\mathrm{Hom}(H, L))$ (and the convention followed is the same as in [28]).

Remark 2. The above notion of Reeb vector fields generalises the commonly found notion of *the* Reeb vector field associated to a co-oriented contact structure, which, in the above description, is nothing but R_1.

The isomorphism of Lemma 1 allows to define another geometric structure associated to a contact manifold (M,H), namely the **Reeb bracket** on $\Gamma(L)$, given by

$$\{u, v\} := \theta([R_u, R_v]), \tag{3}$$

for $u, v \in \Gamma(L)$. The triple $(M, L, \{\cdot,\cdot\})$ gives rise to a Jacobi structure, as explained in the next subsection.

2.2. **Jacobi structures.** As symplectic forms are special examples of Poisson structures, so are contact distributions instances of a more general geometric structure, first introduced in [24, 26], and further explained in [28].

Definition 4 (Jacobi structure)**.** A **Jacobi structure** on a manifold P is a pair $(L, \{\cdot,\cdot\})$ consisting of a line bundle $L \to P$, and a Lie bracket $\{\cdot,\cdot\}$ on $\Gamma(L)$ with the property that it is *local*, in the sense that

$$\mathrm{supp}(\{u, v\}) \subset \mathrm{supp}(u) \cap \mathrm{supp}(v) \quad \forall u, v \in \Gamma(L),$$

where $\mathrm{supp}(u)$ denotes the support of u. The triple $(P, L, \{\cdot,\cdot\})$ consisting of a manifold P and a Jacobi structure $(L, \{\cdot,\cdot\})$ is henceforth referred to as a *Jacobi manifold.*

Observation 3 (cf. [26])**.** If $L = P \times \mathbb{R}$, a Jacobi structure $\{\cdot,\cdot\}$ is completely described by a pair $(\Lambda, R) \in \mathfrak{X}^2(P) \times \mathfrak{X}(P)$, satisfying

$$[\![\Lambda, \Lambda]\!] = 2R \wedge \Lambda, \quad [\![\Lambda, R]\!] = 0, \tag{4}$$

where $[\![\cdot,\cdot]\!]$ is the Schouten bracket. The Lie bracket on $\Gamma(L) = C^\infty(P)$ is given by

$$\{f, g\} := \Lambda(\mathrm{d}f, \mathrm{d}g) + f(Rg) - g(Rf),$$

for $f, g \in C^\infty(P)$.

Example 5. A **contact manifold** (M, H) comes equipped with a natural Jacobi structure $(L, \{\cdot,\cdot\})$, where $L = TM/H \to P$ and $\{\cdot,\cdot\}$ is the Reeb bracket (3).

Example 6. A **Poisson manifold** (P, Λ) is a manifold P along with a bivector field $\Lambda \in \mathfrak{X}^2(P)$, which satisfies $[\![\Lambda, \Lambda]\!] = 0$. In light of Observation 3, Poisson manifolds (P, Λ) are precisely Jacobi structures on $P \times \mathbb{R} \to P$ with $\Lambda_P = \Lambda$ and $R_P \equiv 0$.

Example 7. A **locally conformal symplectic manifold** is a triple (P, σ, τ), where P is a manifold, $(\sigma, \tau) \in \Omega^2(P) \times \Omega^1(P)$, σ is non-degenerate, $\mathrm{d}\tau = 0$, and $\mathrm{d}\sigma = -\tau \wedge \sigma$. Following [28], given (P, σ, τ) define a bivector $\Lambda \in \mathfrak{X}^2(P)$ and a vector field $R \in \mathfrak{X}(P)$ uniquely by

$$\iota(R)\sigma = -\tau \qquad \text{and} \qquad \iota(\Lambda^\sharp(\eta))\sigma = -\eta$$

for all $\eta \in \Omega^1(P)$, where ι denotes the interior product. Then (P, Λ_P, R_P) is a Jacobi manifold (cf. [28]).

In the literature on Jacobi manifolds two types of maps between Jacobi manifolds are considered, namely those which are, in some sense, 'strictly' Jacobi and those which are up to some rescaling (conformal). The following definition provides a common framework for both of the above types of maps in the more general setting of Jacobi manifolds in the sense of Definition 4.

Definition 8 (Jacobi maps)**.** Let $(N, L_N, \{\cdot,\cdot\}_N)$ and $(P, L_P, \{\cdot,\cdot\}_P)$ be Jacobi manifolds such that there exists an isomorphism $F\colon \phi^* L_P \to L_N$. A map $\phi\colon N \to P$ is said to be **Jacobi** with **bundle component** F if for all $u, v \in \Gamma(L_P)$

$$\{F \circ \phi^* u, F \circ \phi^* v\}_N = F \circ \phi^* \{u, v\}_P.$$

Observation 4. Fix a Jacobi manifold $(P, L, \{\cdot,\cdot\})$. Let $\{U_i\}$ be an open cover of P with the property that, for each i, there exists a nowhere vanishing section $v_i\colon U_i \to L$, which trivialises $L|_{U_i} \cong U_i \times \mathbb{R}$. Then, for each i, there is a unique Jacobi structure $(U_i \times \mathbb{R}, \{\cdot,\cdot\}_i := \{\cdot,\cdot\}|_{U_i})$. If i, j are such that $U_{ij} \neq \emptyset$, then there exists a nowhere vanishing function $g_{ij}\colon U_{ij} \to \mathbb{R}$ such that $v_j = g_{ij} v_i$ on U_{ij}. The Jacobi manifolds $(U_{ij}, U_{ij} \times \mathbb{R}, \{\cdot,\cdot\}_j|_{U_{ij}})$ and $(U_{ij}, U_{ij} \times \mathbb{R}, \{\cdot,\cdot\}_i|_{U_{ij}})$ are Jacobi isomorphic with bundle component g_{ij} (viewed as a bundle isomorphism $U_{ij} \times \mathbb{R} \to U_{ij} \times \mathbb{R}$). Conversely, given an open cover $\{U_i\}$ of P such that

- for each i, there is a Jacobi structure on $U_i \times \mathbb{R}$;
- for each pair i, j with $U_{ij} \neq \emptyset$, the Jacobi structures on $U_{ij} \times \mathbb{R}$ are isomorphic with bundle component $g_{ij}\colon U_{ij} \to \mathrm{GL}(\mathbb{R})$;
- the collection of nowhere vanishing functions $\{g_{ij}\}$ satisfy the cocycle condition, *i.e.* for all i, j, l for which $U_{ijl} \neq \emptyset$, $g_{il} = g_{jl} g_{ij}$;

then there exists a unique Jacobi structure on the line bundle $L \to P$ whose (isomorphism class) is uniquely defined by the collection of $\{g_{ij}\}$ (cf. [15, Section 1.4]).

Example 9. If, in the above definition, L_N and L_P are trivial, and the bundle component is the identity, the condition that $\phi\colon (N, L_N, \{\cdot,\cdot\}_N) \to (P, L_P, \{\cdot,\cdot\}_P)$ is a Jacobi map is equivalent to asking that Λ_N and R_N are ϕ-related to Λ_P and R_P respectively, where (Λ_N, R_N) and (Λ_P, R_P) are the bivector and vector fields which uniquely determine the Jacobi structures on L_N and L_P respectively (cf. Observation 3).

2.3. **The Lie algebroid associated to a Jacobi manifold.** Given a Jacobi manifold $(P, L, \{\cdot,\cdot\})$, the first jet bundle $J^1L \to P$ can be endowed with the structure of a **Lie algebroid**, which codifies the geometry of $(P, L, \{\cdot,\cdot\})$ – cf. [7] for a definition and examples of Lie algebroids. This Lie algebroid is described in [23] for the trivial line bundle, in [13] in the general case, and more recently from a different point of view in [10]. Its defining properties are given in the proposition below, stated without proof (cf. [10]).

Proposition 2. *Given a Jacobi manifold $(P, L, \{\cdot,\cdot\})$, there exists a Lie algebroid structure on J^1L which is uniquely characterised by the following properties:*

(I) The anchor map $\rho\colon J^1L \to TP$ satisfies

$$\{u, fv\} = f\{u, v\} + L_{\rho(j^1u)}(f)v,$$

for all $u, v \in \Gamma(L)$, $f \in C^\infty(M)$;

(II) The Lie bracket on $\Gamma(J^1L)$ satisfies

$$[j^1u, j^1v] = j^1\{u, v\}, \quad \forall u, v \in \Gamma(L).$$

This Lie algebroid is henceforth referred to as the associated Lie algebroid to the Jacobi manifold $(P, L, \{\cdot,\cdot\})$.

Example 10 (Trivial line bundle, cf. [12])**.** For Jacobi structures on the trivial line bundle $L = P \times \mathbb{R}$, uniquely defined by a pair (Λ, R) as in Observation 3, the Lie algebroid structure on $J^1L = J^1P = T^*P \oplus \mathbb{R}$ has anchor $\rho = \rho_{\Lambda,R}$ given by

$$\rho(\eta, \lambda) = \Lambda_P^\sharp(\eta) + \lambda R_P,$$

for any element $(\eta, \lambda) \in \mathrm{T}^*P \oplus \mathbb{R}$, and Lie bracket defined by

$$[(\eta, f), (\xi, g)] = ([\eta, \xi]_\Lambda - i_R(\eta \wedge \xi) + f L_R(\xi) - g L_R(\eta), \Lambda(\eta, \xi) + L_{\rho(\eta,f)}(g) - L_{\rho(\xi,g)}(f)),$$

for any two sections $(\eta, f), (\xi, g) \in \Gamma(J^1P)$.

Observation 5. For a contact manifold (M, H), where $H = \ker\theta$, with $\theta \in \Omega^1(M, L)$, the Lie algebroid structure of J^1L associated to the underlying Jacobi structure defined in Example 5 can be expressed in terms of the Reeb vector fields, and the curvature map c_θ (cf. equation (2)). Using the defining property

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(I) of the anchor and the definition of the Reeb bracket of equation (3), it follows that $\rho(j^1u) = R_u$. This implies that, given $(u, \eta) \in \Gamma(J^1L)$,

$$\rho(u, \eta) = R_u + c^\sharp_\theta(\eta|_H) = R_u + c^\sharp_\theta(\mathrm{D}(u, \eta)|_H).$$

Observation 6 (The Spencer operator of a Jacobi manifold, cf. [10])**.** The first jet bundle $J^1L \to P$ fits into a short exact sequence of vector bundles

$$0 \to \mathrm{T}^*P \otimes L \xrightarrow{i} J^1L \xrightarrow{pr} L \to 0, \tag{5}$$

where $pr\colon J^1L \to L$ is the standard projection, and $i(df \otimes u) = fj^1(u) - j^1(fu)$ for $f \in C^\infty(P)$ and $u \in \Gamma(L)$. While the above sequence is *not* canonically split, there is a canonical splitting at the level of sections, which is very useful when studying geometric structures on J^1L. This splitting, known as the **Spencer decomposition**, is used extensively in this paper; it is defined by $u \mapsto j^1u$, for $u \in \Gamma(L)$. This induces the Spencer decomposition

$$\Gamma(J^1L) \cong \Gamma(L) \oplus \Gamma(\mathrm{Hom}(TP, L)),$$

where the $C^\infty(P)$-module structure on the right hand is given by

$$f \cdot (u, \phi) = (fu, \phi + df \otimes u).$$

The projection to the second component

$$D\colon \Gamma(J^1L) \to \Omega(P; L), \quad (u, \eta) \mapsto \eta$$

is the so-called **classical Spencer operator**; D is compatible with the Lie algebroid of a Jacobi manifold $(P, L, \{\cdot,\cdot\})$ in the following sense. First, L is a representation of J^1L with flat connection uniquely defined by

$$\nabla\colon \Gamma(J^1L) \times \Gamma(L) \to \Gamma(L), \quad \nabla_{j^1u}(v) = \{u, v\} \tag{6}$$

(and extended using the Leibniz rule, cf. [10] for details). Secondly, D satisfies the following conditions (of Spencer operators on Lie algebroids [11])

$$\mathrm{D}_{\rho(\alpha)}(\alpha') = \nabla_{\alpha'}(pr(\alpha)) + pr([\alpha, \alpha']) \tag{7}$$

$$\mathrm{D}_X[\alpha, \alpha'] = \nabla_\alpha(\mathrm{D}_X\alpha') - \mathrm{D}_{[\rho(\alpha), X]}\alpha' - \nabla_{\alpha'}(\mathrm{D}_X\alpha) + \mathrm{D}_{[\rho(\alpha'), X]}\alpha, \tag{8}$$

for any $\alpha, \alpha' \in \Gamma(J^1L)$ and $X \in \mathfrak{X}(P)$. Henceforth, the above Spencer decomposition is going to be used tacitly unless otherwise stated.

Jacob

Jacobi manifolds whose underlying characteristic distribution $\mathrm{Im}\,\rho \subset TP$ has constant rank play an important role in this paper (cf. Lemma 3).

Definition 11. A Jacobi manifold $(P, L_P, \{\cdot,\cdot\})$ is said to be **regular** of corank k if, for all $p \in P$, $\dim\ker \rho_{P,p} = k$.

Remark 3. There exist other notions of 'regular' Jacobi manifolds in the literature which are different to the one of Definition 11 (cf. [16]).

3. Contact isotropic realisations: definition and invariants

3.1. **Definition and basic properties.**

The following definition introduces the main object studied in this paper.

Definition 12 (Contact isotropic realisations)**.** A **contact isotropic realisation** of a Jacobi manifold $(P, L_P, \{\cdot,\cdot\})$ is a contact manifold (M, H) together with a surjective submersion $\phi\colon (M, H) \to (P, L_P, \{\cdot,\cdot\})$ with connected, compact fibers satisfying the following properties

(IR1) ϕ is a Jacobi map with bundle component $F\colon \phi^*(L_P) \to L_M$ (cf. Definition 8);

(IR2) $H = \ker\theta$ is transversal to ϕ, *i.e.*

$$H + \operatorname{Ker} D\phi = \mathrm{T}M;$$

(IR3) $\operatorname{Ker} D\phi \subset (\operatorname{Ker} D\phi)^\perp$, where

$$(\operatorname{Ker} D\phi)^\perp := \rho_M(\phi^* J^1 L_P)$$

is the *pseudo-orthogonal distribution* of $\ker D\phi$, where $\phi^* J^1 L_P$ is identified with its image in $J^1 L_M$ via the map F.

Henceforth, whenever referring to an isotropic contact realisation $\phi\colon (M, H) \to (P, L_P, \{\cdot,\cdot\})$, the dimensions of M and of P are fixed to be $2n+1$ and $2n+1-k$ respectively, unless otherwise stated.

Remark 4.

- The assumption on compactness of the fibres of a contact isotropic realisation can be weakened for what follows, but it is imposed at this stage to simplify the exposition;
- A surjective submersion $\phi\colon (M, H) \to (P, L_P, \{\cdot,\cdot\})$ with connected fibres satisfying only properties (IR1) and (IR2) is called a *contact realisation.*

Observation 7. Each of the properties in Definition 12 have important implications.

(1) Property (IR1) implies commutativity of the following diagram

$$\begin{array}{ccc} J^1 L_M & \xrightarrow{\rho_M} & TM \\ {\scriptstyle \phi^*}\uparrow & & \downarrow{\scriptstyle D\phi} \\ \phi^* J^1 L_P & \xrightarrow[\rho_M]{} & \phi^* TP, \end{array}$$

where ρ_M, ρ_P are the anchor maps for the Jacobi structures on P and M respectively, and $\phi^*\colon \phi^* J^1 L_P \to J^1 L_M$ is the injective vector bundle morphism given by $j^1 u \mapsto j^1(F\phi^* u)$. This can be seen as follows. Consider $\{u, gv\}_P$ for any sections $u, v \in \Gamma(L)$ and any function $g \in C^\infty(P)$; using the defining property (I) of the anchor of a Lie algebroid associated to a Jacobi structure together with property (IR1), it can be shown that

$$\phi^*(\rho_P(j^1u)g) = \rho_M(j^1(F\phi^*u)\phi^*g),$$

which, in turn, implies the claimed result;

(2) Property (IR2) can be used to show that for all $m \in M$, $\dim(H_m \cap \ker D_m\phi) = k-1$, since

$$\dim(H_m + \ker D_m\phi) - \dim H_m - \dim\ker D_m\phi = \dim(H_m \cap \ker D_m\phi)$$

and by property (IR2), for all $m \in M$, $H_m + \ker D_m\phi = T_mM$;

(3) When $(P, L_P, \{\cdot,\cdot\})$ is Poisson, condition (IR3) is equivalent to requiring that, for all $m \in M$, the vector space $\ker D_m\phi \cap H_m$ is isotropic with respect to the symplectic structure on H_m induced by a suitable choice of contact form (cf. [25] for details). This explains the nomenclature used in the definition.

Not all Jacobi manifolds admit isotropic contact realisations; in fact, a necessary condition is that the structure be regular and all its leaves be even dimensional. This is the content of the next lemma, which should be compared to [32, Proposition 8.10], which proves an analogous statement for isotropic realisations of Poisson manifolds.

Lemma 3. *For a contact isotropic realisation $\phi\colon (M,H) \to (P, L_P, \{\cdot,\cdot\})$ the Jacobi structure on P is regular with corank equal to k, and all its leaves are even dimensional.*

Proof. Fix points $p \in P$ and $m \in M$ such that $m \in \phi^{-1}(p)$. Using item (1) of Observation 7,

$$\text{rk}\,\rho_{P,p}(J^1_p(L_P))) = \text{rk}\,(\rho_{M,m}\circ\phi^*(J^1L_P)) - \text{rk}\,\ker(D\phi|_{\rho_{M,m}\circ\phi^*(J^1L_P)}). \tag{9}$$

On the one hand, property (IR2) implies that $\rho_M\circ\phi^*\colon \phi^*J^1L_P \to TM$ is injective: indeed, Observation 5 gives that for $(u,\eta) \in \Gamma(L_P)\oplus\Omega^1(P,L_P) \simeq \Gamma(J^1L_P)$,

$$\rho_M(\phi^*(u,\eta)) = R_{F(\phi^*u)} - c^\sharp(F\circ(\phi^*\eta|_H)).$$

If $\rho_{M,m}(\phi^*(u,\eta)) = 0$, then $R_{F(\phi^*u)}(m) = c^\sharp(F\circ(\phi^*\eta|_H))(m) \in H$. By definition of the Reeb vector field, this means that

$$F(\phi^*u)(m) = R_{F(\phi^*u)}(m) \bmod H = 0.$$

Hence, $F\circ(\phi^*\eta_m|_H) = 0$ which in turns implies that $\phi^*\eta_m|_H = 0$. Condition (IR2) implies that $D\phi|_{H_m}\colon H_m \to T_{\phi(m)}P$ is surjective (as ϕ is a surjective submersion); thus $\phi^*\eta_m|_H = 0$ holds if and only if $\eta_{\phi(m)} = 0$, thus proving injectivity of $\rho_M\circ\phi^*$.

Condition (IR3) gives that $\text{rk}\,\ker(D\phi|_{\rho_M\circ\phi^*J^1L_P}) = \text{rk}\,\ker D\phi = k$. By definition $\text{rk}\,J^1L_P = 2n-k+2$, while injectivity of $\rho_M\circ\phi^*$ implies that

$$\text{rk}\,\rho_{M,m}(\phi^*J^1L_P) = 2n-k+2;$$

equation (9) yields $\mathrm{rk}\,\rho_{P,p}(J^1_p(L_P))) = 2n+2-k-k = 2n+2-2k$. Since $p \in P$ is arbitrary, the proof of the lemma is completed. □

3.2. **Properties of regular Jacobi manifolds all of whose leaves are even dimensional.** In light of Lemma 3 a necessary condition for a Jacobi manifold to admit a contact isotropic realisation is that it is regular and all its leaves are even dimensional. The aim of this subsection is to prove some properties of Jacobi manifolds of this kind, which generalise those enjoyed by regular Poisson manifolds (which are an example of this family of Jacobi structures). Throughout this section, let $(P, L_P, \{\cdot,\cdot\})$ be a regular Jacobi manifold all of whose leaves are even dimensional, unless otherwise stated. The notation employed below follows from that of Section 3.1.

Proposition 4. *Let $(P, L_P, \{\cdot,\cdot\})$ be a regular Jacobi manifold with even dimensional leaves. Then $\pi\colon \ker\rho_P \to P$ is a bundle of abelian Lie algebras, so that the inclusion $\ker\rho_P \hookrightarrow J^1L_P$ is a morphism of Lie algebroids.*

Proof. Since $\ker\rho_P$ is the kernel of the anchor $\rho\colon J^1L_P \to TP$ and the restriction of $\pi\colon J^1L_P \to P$ to $\ker\rho_P$ is a vector bundle, it follows that $\pi\colon \ker\rho_P \to P$ is a bundle of Lie algebras. It remains to show that each such Lie algebra is abelian.

Let $\nu^* \subset \mathrm{T}^*P$ denote the *conormal bundle* of the regular foliation $\mathcal{F}$ whose distribution is $T\mathcal{F} := \mathrm{Im}\,(\rho\colon J^1L_P \to TP)$. By assumption, the corank of the Jacobi structure is k, which implies that the rank of $\nu^* \to P$ is $k-1$. The claim is that $\ker\rho_P$ fits in a short exact sequence of vector bundles

$$0 \to \nu^* \otimes L_P \to \ker\rho_P \to L_P \to 0,$$

where the projection $\ker\rho_P \to L_P$ is nothing but the restriction of $pr\colon J^1L_P \to L_P$ (cf. Observation 6). To this end, it suffices to prove that

$$\nu^* \otimes L_P = \ker\rho_P \cap (\mathrm{T}^*P \otimes L_P) = \ker pr|_{\ker\rho_P}. \tag{10}$$

For, if equation (10) holds, then dimension counting implies that the restriction $pr\colon \ker\rho_P \to L_P$ is surjective. First, it is shown that $\nu^*\otimes L_P \subset \ker\rho_P$. Suppose that $\mathrm{d}f \otimes u \in \Gamma(\nu^*\otimes L_P)$, and fix $g \in C^\infty(P)$ and $v \in \Gamma(L_P)$. Using the $C^\infty(M)$-module structure of the Spencer decomposition of $\Gamma(J^1L_P)$ (cf. Observation 6), have that $i(\mathrm{d}f\otimes u) = fj^1u - j^1(fu)$, and therefore $\rho_P(i(\mathrm{d}f\otimes u)) = f\rho_P(j^1u) - \rho_P(j^1(fu))$. The defining property (I) of the anchor implies that

$$\begin{aligned} \mathcal{L}_{\rho(\mathrm{d}f\otimes u)}(g)v &= f\mathcal{L}_{\rho(j^1u)}(g)v - \mathcal{L}_{\rho(j^1(fu))}(g)v \\ &= f(\{u,gv\} - g\{u,v\}) - (\{fu,gv\} - g\{fu,v\}) \\ &= -\mathcal{L}_{\rho(\mathrm{d}g\otimes v)}(f)u \;=\; 0. \end{aligned} \tag{11}$$

This shows that $\nu^*\otimes L_P \subset \ker\rho_P \cap (T^*P\otimes L_P)$. In order to show that equality holds, it suffices to check that, for each $p \in P$, $\dim\ker\rho_{P,p} \cap (\mathrm{T}^*P\otimes L_P)_p = k-1$. To this end, note that $\ker\rho_P \cap (T^*P \otimes L_P) = \ker\rho_P|_{\mathrm{T}^*P\otimes L_P}$. The map

$\rho_P\colon T^*P\otimes L_P\to TP$ is, in fact, antisymmetric, *i.e.* for any $\eta,\xi\in \mathrm{T}^*P\otimes L_P$, $\eta(\rho(\xi))=-\xi(\rho(\eta))$. This can be checked directly from the defining property of the anchor ρ_P (in the case of trivial coefficients, this map is given by the sharp of a bivector). Thus, for each $p\in P$, the vector space $\rho_{P,p}(\mathrm{T}^*_pP\otimes L_{P,p})\subset T_p\mathcal{F}$ is even dimensional. On the other hand, since $k\geq \dim\ker\rho_{P,p}|_{\mathrm{T}^*_pP\otimes L_{P,p}}\geq k-1$, it follows that

$$2n-1\leq \dim\rho_{P,p}(\mathrm{T}^*_pP\otimes L_{P,p})\leq 2n,$$

with $\dim\rho_{P,p}(\mathrm{T}^*_pP\otimes L_{P,p})=2n$ if and only if $\dim\ker\rho_{P,p}|_{\mathrm{T}^*_pP\otimes L_{P,p}}=k-1$. Since $\dim\rho_{P,p}(\mathrm{T}^*_pP\otimes L_{P,p})$ is even dimensional, it follows that $\dim\ker\rho_{P,p}|_{\mathrm{T}^*_pP\otimes L_{P,p}}=k-1$; as $p\in P$ is arbitrary, this completes the proof that equation (10) holds.

Using the above description of $\ker\rho_P$, it is possible to prove that $\ker\rho_P\to P$ is a bundle of abelian Lie algebras. Let $\alpha,\alpha'\in\Gamma(\ker\rho_P)$; the Spencer decomposition of Observation 6 gives that $[\alpha,\alpha']=(pr[\alpha,\alpha'],\mathrm{D}[\alpha,\alpha'])$, where $\mathrm{D}\colon\Gamma(J^1L_P)\to\Omega^1(M;L_P)$ is the classical Spencer operator. Writing $\alpha=(u,\eta)$ and $\alpha'=(v,\xi)$, where $u,v\in\Gamma(L_P)$ and $\eta,\xi\in\Omega^1(P;L_P)$, the compatibility conditions of equations (7) and (8) yield the following formula for the Lie bracket restricted to sections of $\ker\rho_P$

$$[\alpha,\alpha']=(-\nabla_{\alpha'}u,\nabla_\alpha(\xi(\cdot))-\nabla_{\alpha'}(\eta(\cdot))), \tag{12}$$

where ∇ is the J^1L_P-connection defined by equation (6). There are two cases to consider: the first case is when $u=0=v$, and the second is when $u=0$ and $v\neq 0$. Note that, for dimensional reasons, $\Gamma(\ker\rho_P)$ is (locally) generated by elements of this form, and therefore, as $\ker\rho_P$ is a bundle of Lie algebras it is enough to check that the bracket is zero on these elements. Suppose first that $u=0=v$; using the defining properties of ∇ and of ρ_P (cf. equation (6) and property (I) respectively), the following equality can be proved

$$\nabla_\xi s=-\xi(\rho_P(j^1s)), \tag{13}$$

where $s\in\Gamma(L_P)$ is arbitrary (and analogously for $\nabla_\eta s$). Since $\xi,\eta\in\Gamma(\nu^*\otimes L_P)$, equation (13) implies that $[\eta,\xi]=0$. Thus it only remains to consider the case $u=0$ and $v\neq 0$. Write $\alpha'=j^1z+i(\gamma)$, for $\gamma\in\Omega^1(P,L_P)$; without loss of generality, the section $z\in\Gamma(L_P)$ can be taken to be nowhere vanishing at the cost of restricting its domain of definition, say to $U\subset P$. This does not matter to the purpose at hand, as the aim is to check that a *fibre-wise* Lie algebra structure is abelian and, to do so, there is a freedom in choosing the sections α,α' under consideration. Since $\rho_P(\alpha')=0$, the defining property (I) of ρ_P implies that the operator $\nabla_{\alpha'}(\cdot)\colon\Gamma(L_P|_U)\to\Gamma(L_P|_U)$ is $C^\infty(U)$-linear and hence it defines a vector bundle automorphism $L_P|_U\to L_P|_U$. As $\mathrm{Hom}(L_P,L_P)$ is trivial and of rank 1 (because L_P is of rank 1), then there exists a smooth function $G\in C^\infty(U)$ such that

$$\nabla_{\alpha'}(s)(p)=G(p)s(p),$$

for all $s \in \Gamma(L_P|_U)$ and $p \in U$. Hence,

$$Gz = \nabla_{\alpha'}(z) = \{z, z\} - \gamma(\rho_P(j^1 z)) = -\gamma(\rho_P(\gamma)) = 0,$$

where antisymmetry of $\rho_P\colon \mathrm{T}^*P \otimes L_P \to L_P$ is used to obtain the last equality. As $z \in \Gamma(L_P|_U)$ does not vanish at any point, it follows that $G \equiv 0$. This, in turn, implies that $[\alpha, \alpha'] = 0$ in this case as well, thus completing the proof. □

Observation 8. The previous proof shows that under the hypotheses of Proposition 4, the connection $\nabla\colon \Gamma(J^1 L_P) \times \Gamma(L_P) \to \Gamma(L_P)$ (cf. equation (6)) restricted to $\Gamma(\ker \rho) \subset \Gamma(J^1 L)$ is zero.

The next aim is to construct a cohomology class associated to $(P, L_P, \{\cdot,\cdot\})$ which generalises the foliated cohomology class given by the leafwise symplectic form induced by a regular Poisson manifold. As in the proof of Proposition 4, denote the characteristic foliation of $(P, L_P, \{\cdot,\cdot\})$ by $\mathcal{F}$. Intuitively, the cohomology class defined below is a foliated 2-form with values in L_P. However, in order to formalise this intuition, it is necessary to prove that there is a $T\mathcal{F}$-connection on L_P associated to the Jacobi structure $(L_P, \{\cdot,\cdot\}_P)$, where the foliation $T\mathcal{F} \subset TP$ is viewed as a Lie subalgebroid. Let ∇ be the $J^1 L_P$-connection on L_P defined by equation (6).

Proposition 5. *There is a unique $T\mathcal{F}$-connection $\bar{\nabla}\colon \Gamma(T\mathcal{F}) \times \Gamma(L_P) \to \Gamma(L_P)$ on L_P uniquely defined by*

$$\bar{\nabla}_{\rho_P(j^1 u)} v := \nabla_{j^1 u} v = \{u, v\}. \tag{14}$$

Proof. Observation 8 implies that $\bar{\nabla}$ is well-defined as a connection. Flatness of $\bar{\nabla}$ follows from that of ∇, thus completing the proof.

□

In light of Proposition 4, it makes sense to consider the Lie algebroid cohomology $\mathrm{H}^*(\mathcal{F}; L_P)$, which is defined as the cohomology of the complex of smooth foliated forms with values in L_P with the following (standard) differential $\mathrm{d}_{\mathcal{F}}$

$$\begin{aligned} \mathrm{d}_{\mathcal{F}}\omega(X_1, \ldots, X_{l+1}) := &\sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{l+1}) \\ &+ \sum_i (-1)^{i+1} \bar{\nabla}_{X_i}(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{l+1})), \end{aligned} \tag{15}$$

for $X_1, \ldots, X_{l+1} \in \Gamma(T\mathcal{F})$ and $\omega \in \Omega^l(\mathcal{F}; L_P) := \Gamma(\wedge^l \mathrm{T}^*\mathcal{F} \otimes L_P)$, and where hat denotes omission. The next lemma associates a cohomology class in $\mathrm{H}^2(\mathcal{F}; L_P)$ to $(P, L_P, \{\cdot,\cdot\})$.

Lemma 6. *The foliated 2-form with values in L_P uniquely defined by*

$$\omega(\rho_P(j^1 u), \rho_P(j^1 v)) := \{u, v\}_P, \tag{16}$$

for any $u, v \in \Gamma(L_P)$, is a 2-cocycle, i.e. $\mathrm{d}_{\mathcal{F}}\omega = 0$.

Proof. Since sections of the form j^1u, for $u \in \Gamma(L_P)$ form a $C^\infty(P)$-basis of $\Gamma(J^1L_P)$ and $T\mathcal{F} = \rho_P(J^1L_P)$, equation (16) defines a unique map $\Gamma(T\mathcal{F}) \times \Gamma(T\mathcal{F}) \to \Gamma(L_P)$ which is manifestly antisymmetric. Let $f \in C^\infty(P)$ and fix $u, v \in \Gamma(L_P)$; then

$$\begin{aligned}\omega(f\rho_P(j^1u), \rho_P(j^1v)) &= \omega(\rho_P(fu, \mathrm{d}f \otimes u), \rho_P(j^1v)) = \{fu, v\}_P + \mathrm{d}f \otimes u(\rho_P(j^1v)) \\ &= f\{u, v\}_P - \mathcal{L}_{\rho_P(j^1v)}fu + \mathcal{L}_{\rho_P(j^1v)}fu = f\omega(\rho_P(j^1u), \rho_P(j^1v)),\end{aligned}$$

where the first equality follows from the $C^\infty(P)$-structure on $\Gamma(J^1L_P)$ arising from the Spencer decomposition (cf. Observation 6), the second by definition of ω, and the third by the characterising property (I) of the anchor ρ_P (cf. Proposition 2). The above calculation shows that equation (16) indeed defines a foliated 2-form with values in L_P (also denoted by ω) as it is $C^\infty(P)$-bilinear.

To check that $\mathrm{d}_\mathcal{F}\omega = 0$ it suffices to check that

$$\mathrm{d}_\mathcal{F}\omega(\rho_P(j^1u), \rho_P(j^1v), \rho_P(j^1w)) = 0$$

for any $u, v, w \in \Gamma(L_P)$. This follows from the fact that $\mathrm{d}_\mathcal{F}\omega$ is $C^\infty(P)$-linear in each entry and that sections of the form $\rho_P(j^1u)$ form a $C^\infty(P)$-basis of $\Gamma(T\mathcal{F})$. Then

$$\begin{aligned}&\mathrm{d}_\mathcal{F}\omega(\rho_P(j^1u), \rho_P(j^1v), \rho_P(j^1w)) = \bar{\nabla}_{\rho_P(j^1u)}(\omega(\rho_P(j^1v), \rho_P(j^1w))) + \text{c.p.} \\ &\quad - (\omega([\rho_P(j^1u), \rho_P(j^1v)], \rho_P(j^1w)) + \text{c.p.}) = \{u, \{v, w\}_P\}_P + \text{c.p.} \\ &\quad - (\omega(\rho_P([j^1u, j^1v]), \rho_P(j^1w)) + \text{c.p.}) = -(\omega(\rho_P(j^1\{u, v\}_P), \rho_P(j^1w)) + \text{c.p.}) \\ &= -(\{\{u, v\}_P, w\}_P + \text{c.p.}) = 0,\end{aligned}$$

where c.p. stands for cyclic permutation, the second equality follows from the fact that ρ_P is a map of Lie algebroids, the third by the defining property (II) of the Lie bracket on J^1L_P (cf. Proposition 2) and by the Jacobi identity for $\{\cdot, \cdot\}$, which also implies the last equality. This shows that $\mathrm{d}_\mathcal{F}\omega = 0$, as required. □

Observation 9. The following is an explicit formula for ω

$$\omega(\rho_P(u, \eta), \rho_P(v, \zeta)) := \{u, v\}_P + \eta(\rho_P(v, \zeta)) - \zeta(\rho_P(j^1u)),$$

for $(u, \eta), (v, \zeta) \in \Gamma(J^1P)$.

3.3. **Smooth classification.** Any contact realisation $\phi\colon (M, H) \to (P, L_P, \{\cdot, \cdot\})$ (cf. Remark 4) comes equipped with an **action** of the Lie algebroid J^1L_P, in the sense that there exists a vector bundle map $\psi\colon \Gamma(J^1L_P) \to \mathfrak{X}(M)$ satisfying

(A1) it induces a Lie algebra homomorphism $\Gamma(J^1L_P) \to \mathfrak{X}(M)$;
(A2) for all $m \in M$, $D_m\phi \circ \psi_m = \rho_{\phi(m)}$.

Lemma 7. *Let* $\phi\colon (M, H) \to (P, L_P, \{\cdot, \cdot\})$ *be a contact realisation. The map* $\psi\colon \phi^*J^1L_P \to TM$ *given at the level of sections by*

$$\phi^*j^1u \mapsto R_{F\circ\phi^*u}, \quad \forall u \in \Gamma(L_P)$$

defines a faithful action of J^1L_P *on* $\phi\colon M \to P$*.*

Observation 10. Note that ψ can be alternatively described as the composite $\rho_M \circ \phi^*\colon \phi^* J^1 L_P \to TM$, where the notation is as in item 1 of Observation 7.

Proof of Lemma 7. Since $\phi\colon (M,H) \to (P, L_P, \{\cdot,\cdot\})$ is a Jacobi map, item 1 in Observation 7 implies that

$$D\phi \circ \rho_M \circ \phi^* = \phi^*(\rho_P). \tag{17}$$

In light of Observation 10, it follows that property (A2) holds. It remains to prove that ψ induces a Lie algebra morphism, *i.e.*

$$\psi([\alpha,\alpha']) - [\psi(\alpha),\psi(\alpha')] = 0, \tag{18}$$

for all $\alpha,\alpha' \in \Gamma(J^1L_P)$. This equation holds when $\alpha,\alpha' \in \Gamma(J^1P)$ are holonomic sections, *i.e.* of the form j^1u, j^1v for smooth sections $u, v \in C^\infty(P)$. This is because $\phi\colon M \to P$ is a Jacobi map with bundle component F, which implies that

$$R_{F\circ\phi^*\{u,v\}} = R_{\{F\circ\phi^* u, F\circ\phi^* v\}} = [R_{F\circ\phi^* u}, R_{F\circ\phi^* v}]. \tag{19}$$

for $u, v \in \Gamma(L_P)$, where in the last equality property (3) of the Jacobi bracket of (M,H) is used. In general, notice that the left hand side of equation (18) satisfies

$$\psi([f\alpha,\alpha']) - [\psi(f\alpha),\psi(\alpha')] = (\phi^* f)(\psi([\alpha,\alpha']) - [\psi(\alpha),\psi(\alpha')]);$$

for $f \in C^\infty(P)$ and $\alpha,\alpha' \in \Gamma(J^1L_P)$. This again follows from the fact that $\phi\colon (M,H) \to (P, L_P, \{\cdot,\cdot\})$ is a Jacobi map, so that equation (17) holds. Thus, since equation (18) holds for holonomic sections and these generate $\Gamma(J^1L_P)$ as a $C^\infty(P)$-module, it follows that equation (18) holds for all sections of J^1P, which proves that ψ satisfies property (A1). The proof of Lemma 3 shows that ψ is injective (this follows from property (IR2)). □

The following proposition shows that, for a contact isotropic realisation $\phi\colon (M,H) \to (P, L_P, \{\cdot,\cdot\})$, the restriction of the action of Lemma 7 to the Lie subalgebroid $\ker \rho_P \to P$ yields an action.

Proposition 8. *Let* $\phi\colon (M,H) \to (P, L_P, \{\cdot,\cdot\})$ *be a contact isotropic realisation of a Jacobi manifold. The action* ψ *restricted to the bundle of abelian Lie algebras* $\ker \rho_P$*, is a vector bundle isomorphism over* M*,*

$$\psi\colon \phi^* \ker \rho_P \to \ker D\phi,$$

which at the level of sections $\psi\colon \Gamma(\ker \rho_P) \to \Gamma(\ker D\phi)$ *is a Lie algebra map.*

Proof. Property (A2) implies that the image of $\psi|_{\ker \rho_P}$ consists of *vertical* vector fields, *i.e.* tangent to the fibres of ϕ. On the other hand, property (A1) implies that at the level of sections $\psi\colon \Gamma(\ker \rho_P) \to \Gamma(\ker D\phi)$ is a Lie algebra map. Now, since by Lemma 7, the action $\psi\colon \phi^* \ker \rho_P \to TM$ is injective and for all $p \in P$, $\dim \ker \rho_{P,p} = k = \dim \phi^{-1}(p)$, dimension counting implies that for all $m \in M$,

$$\psi((\phi^* \ker \rho_P)_m) = T_m \phi^{-1}(\phi(m)).$$ □

The action $\psi\colon \phi^* \ker \rho_P \to TM$ should be thought of as being infinitesimal; since the fibres of ϕ are compact, ψ can be integrated to an action of $\pi\colon \ker \rho_P \to P$ considered as a bundle of abelian Lie groups.

Remark 5. In fact, the action ψ can be integrated under a weaker assumption on the fibres, which is generally known as *completeness* (cf. [6]). All following results hold in this more general setting with the appropriate modifications.

Fix a contact isotropic realisation $\phi\colon (M,H) \to (P, L_P, \{\cdot,\cdot\})$ and let $\psi\colon \phi^* \ker \rho_P \to TM$ be the associated infinitesimal $\pi\colon \ker \rho_P \to P$ action. The integrated action is given by

$$\begin{aligned} \Psi\colon\ \ker \rho_P \, {}_{\pi}\!\times_{\phi} M &\to M \\ (\alpha, m) &\mapsto \varphi^1_\alpha(m), \end{aligned} \tag{20}$$

where

$$\ker \rho_P \, {}_{\pi}\!\times_{\phi} M := \{(\alpha, m) \in \ker \rho_P \times M \mid \pi(\alpha) = \phi(m)\}$$

is a smooth manifold, and φ^1_α is the time-1 flow of $\psi(\phi^*\alpha)$.

Observation 11. For each $p \in P$, the action of equation (20) restricts to an action of the abelian Lie group $\ker \rho_{P,p} \cong \mathbb{R}^k$ on $\phi^{-1}(p)$. Connectivity of $\phi^{-1}(p)$ and Observation 8 imply that the action of $\ker \rho_{P,p}$ is transitive. Moreover, since $\ker \rho_{P,p}$ is an abelian Lie group, if $j^1_p u \in \ker \rho_{P,p}$ is such that $\Psi(j^1_p u, m) = m$, then $\Psi(j^1_p u, m') = m'$ for *any* other $m' \in \phi^{-1}(p)$. This is because the isotropy of the action at any two points on $\phi^{-1}(p)$ are equal. Therefore, if

$$\Sigma_p := \{j^1_p u \in \ker \rho_{P,p} \mid \varphi^1_{j^1_p u} = \mathrm{id}\},$$

then $\phi^{-1}(p) \cong \ker \rho_{P,p}/\Sigma_p$; since $\phi^{-1}(p)$ is compact by assumption, it follows that $\phi^{-1}(p)$ is diffeomorphic to $\mathbb{T}^k$ and that Σ_p is a cocompact lattice in $\ker \rho_{P,p}$ and, therefore, isomorphic to $\mathbb{Z}^k$.

Just as in the theory of symplectic isotropic realisations of Poisson manifolds, the isotropy of the action of equation (20) plays an important role in the classification of contact isotropic realisations of Jacobi manifolds.

Definition 13 (Period lattice)**.** The subset

$$\Sigma := \coprod_{p \in P} \Sigma_p \subset \ker \rho_{P,p}$$

is called the **period lattice** of the isotropic contact realisation $\phi\colon (M,H) \to (P, L_P, \{\cdot,\cdot\})$.

Remark 6. The above notion of period lattice extends that of a *Legendre lattice* introduced in [2].

The following theorem provides a local model for isotropic realisations

$$\phi\colon (M,H) \to (P, L_P, \{\cdot,\cdot\});$$

its proof is omitted as it is entirely analogous to that of [14, Theorem 2.1].

Theorem 9. *Let* $\phi\colon (M,H) \to (P, L_P, \{\cdot,\cdot\})$ *be an isotropic realisation with associated period lattice* Σ*. Then*

(I) Σ *is a closed submanifold of* $\ker \rho_P$*;*

(II) *the quotient* $\ker \rho_P/\Sigma$ *is a smooth manifold and the projection*

$$\pi\colon\ \ker \rho_P/\Sigma \to P$$

is a fibre bundle with fibre $\mathbb{T}^k$*;*

(III) *upon a choice of a local section* $\sigma\colon U \subset M \to \phi^{-1}(U)$*, the map*

$$\Psi_\sigma\colon\ \ker \rho_P/\Sigma|_U \to \phi^{-1}(U), \quad [\alpha] \mapsto \varphi^1_\alpha(\sigma(\pi(\alpha))) \tag{21}$$

is an isomorphism of fibres bundles, making the following diagram

$$\begin{array}{ccc} \ker \rho_P/\Sigma|_U & \xrightarrow{\Psi_\sigma} & \phi^{-1}(U) \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\phi} & \\ & U & \end{array}$$

commute.

Fix a good open cover $\mathcal{U} := \{U_i\}$ of P, *i.e.* all finite intersections of the elements of $\mathcal{U}$ are contractible, and, for each i, fix a section $\sigma_i\colon U_i \to M$. Suppose that $U_{ij} \neq \emptyset$; then the composite $\Psi^{-1}_{\sigma_j} \circ \Psi_{\sigma_i}\colon\ \ker \rho_P/\Sigma|_{U_{ij}} \to \ker \rho_P/\Sigma|_{U_{ij}}$ is given by translation along the fibres by a section $t_{ij}\colon U_{ij} \to \ker \rho_P/\Sigma$, *i.e.*

$$\Psi^{-1}_{\sigma_j} \circ \Psi_{\sigma_i}([\alpha]) = [\alpha] + t_{ij}(\pi(\alpha)),$$

where t_{ij} is uniquely defined by $\sigma_j(p) = \varphi^1_{t_{ij}(p)}(\sigma_i(p))$ for all $p \in U_{ij}$. The collection $\{t_{ij}\}$ defines a Čech cohomology class

$$t \in \mathrm{H}^1(\mathcal{U}; \mathcal{C}^\infty(\ker \rho_P/\Sigma)) \cong \mathrm{H}^1(P; \mathcal{C}^\infty(\ker \rho_P/\Sigma)),$$

where $\mathcal{C}^\infty(\ker \rho_P/\Sigma)$ is the sheaf of smooth sections of $\ker \rho_P/\Sigma \to P$, and the isomorphism follows from the fact that open cover $\mathcal{U}$ is good.

The short exact sequence of sheaves

$$1 \to \Sigma \hookrightarrow \mathcal{C}^\infty(\ker \rho_P) \to \mathcal{C}^\infty(\ker \rho_P/\Sigma) \to 1,$$

where, by abuse of notation, Σ denotes the sheaf of sections of $\Sigma \to P$, induces a long exact sequence in cohomology whose connecting morphisms

$$\delta\colon\ \mathrm{H}^k(P; \mathcal{C}^\infty(\ker \rho_P/\Sigma)) \to \mathrm{H}^{k+1}(P; \Sigma)$$

are isomorphisms for all $k \geq 1$, since $\mathcal{C}^\infty(\ker \rho_P)$ is fine.

Definition 14. The cohomology class $c = \delta(t) \in \mathrm{H}^2(P;\Sigma)$ is called the **Chern class** of the contact isotropic realisation $\phi\colon (M,H) \to (P, L_P, \{\cdot,\cdot\})$.

In conclusion, this section proves that a contact isotropic realisation

$$\phi\colon (M,H) \to (P, L_P, \{\cdot,\cdot\})$$

is classified smoothly by

- its period lattice $\Sigma \subset \ker \rho_P \subset J^1 L_P$;
- its Chern class $c \in \mathrm{H}^2(P;\Sigma)$;

this is, two contact isotropic realisations $\phi\colon (M,H) \to (P, L_P, \{\cdot,\cdot\})$ and $\phi'\colon (M',H') \to (P, L_P, \{\cdot,\cdot\})$ are smoothly equivalent if and only if their period lattices and Chern classes coincide, where smooth equivalence means the existence of a diffeomorphism $B\colon (M,H) \to (M',H')$ satisfying $B \circ \phi = \phi'$.

3.4. **Contact classification.** This section studies the problem of finding a (local) *contact* classification of contact isotropic realisations, which complements the results of Section 3.3. The main result in this section is Theorem 13 which provides a local normal form for generalised contact forms arising in contact isotropic realisations. Before proving Theorem 13, some preparatory results are required. Throughout, fix a contact isotropic realisation $\phi\colon (M,H) \to (P, L_P, \{\cdot,\cdot\})$ with compact fibres, generalised contact form $\theta \in \Omega^1(M, L_M)$, and bundle component denoted by $F\colon \phi^* L_P \to TM/H$. Denote by $\bar{\theta} \in \Omega^1(M; \phi^* L_P)$ the one form given by the composition $F^{-1} \circ \theta$.

Recall that for any section $\alpha \in \Gamma(\ker \rho_P)$, $\psi(\phi^*\alpha) \in \Gamma(\ker D\phi)$, where $\psi\colon \phi^* \ker \rho_P \to TM$ is defined as in Lemma 7. Therefore, the flow $\varphi^t_\alpha\colon M \to M$ of $\psi(\phi^*\alpha)$ preserves the fibres, *i.e.* $\phi \circ \varphi^t_\alpha = \phi$ for all $t \in \mathbb{R}$. Hence, $(\varphi^t_\alpha)^* \phi^* L_P \cong \phi^* L_P$ canonically. This allows to define a 'Lie derivative' type operator on $\Omega^*(M; \phi^* L_P)$ by

$$(\mathcal{L}_\alpha \omega)_m = \frac{d}{dt}\Big|_{t=0} ((\varphi^t_\alpha)^* \omega)_m \tag{22}$$

for any $m \in M$ and $\omega \in \Omega^l(M, \phi^* L_P)$. It obeys rules which are analogous to that of the standard Lie derivative, for instance,

$$\frac{d}{dt}\Big|_{t=s} ((\varphi^t_\alpha)^* \omega)_m = ((\varphi^s_\alpha)^* (\mathcal{L}_\alpha \omega))_m. \tag{23}$$

The following lemma, stated below without proof, shows that the above Lie derivative also satisfies a familiar identity involving the interior product (cf. [11, Lemma 3.8]).

Lemma 10. *For any $\omega \in \Omega^l(M; \phi^* L_P)$ and $X \in \mathfrak{X}(M)$,*

$$[i_X, \mathcal{L}_\alpha]\omega = (i_{[X, \psi(\phi^*\alpha)]}\omega).$$

These differential geometric tools allow to prove an important property of the flows φ^t_α, which generalises [14, Lemma 2.1].

Lemma 11. *For any section* $\alpha = (u,\eta) \in \Gamma(\ker \rho_P)$,

$$(\varphi^1_{(u,\eta)})^*\bar\theta - \bar\theta = \phi^*\eta.$$

Proof. Fix points $p \in P$ and $m \in \phi^{-1}(p)$, a tangent vector $X_m \in T_mM$, and a section $(u,\eta) \in \Gamma(\ker \rho_P)$ defined near p. Extend X_m to a (locally defined) vector field $\tilde X$. Since $\phi \circ \varphi^1_{(u,\eta)} = \phi$, the following expression makes sense

$$((\varphi^1_{(u,\eta)})^*\bar\theta - \bar\theta)(\tilde X) = \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}t}(\varphi^t_{(u,\eta)})^*\bar\theta\right)(\tilde X)\mathrm{d}t. \tag{24}$$

Consider the integrand

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\varphi^t_{(u,\eta)})^*\bar\theta\right)(\tilde X) &= ((\varphi^t_{(u,\eta)})^*(\mathcal{L}_{(u,\eta)}\bar\theta))(\tilde X)\\ &= (\varphi^t_{(u,\eta)})^*((\mathcal{L}_{(u,\eta)}\bar\theta)(D\varphi^t_{(u,\eta)}(\tilde X)))\\ &= (\varphi^t_{(u,\eta)})^*(\mathcal{L}_{(u,\eta)}(\bar\theta(D\varphi^t_{(u,\eta)}(\tilde X))) + \bar\theta([D\varphi^t_{(u,\eta)}(\tilde X), \psi(\phi^*u,\phi^*\eta)])),\end{aligned} \tag{25}$$

where the first equality follows from equation (23) and the last by Lemma 10; write

$$D\varphi^t_{(u,\eta)}(\tilde X) = \sum_i f_{t,i}A_{t,i} + \tilde X_t,$$

where $f_{t,i} \in C^\infty(M)$, $A_{t,i} := \psi(\phi^*u_{t,i}, \phi^*\eta_{t,i}) \in \Gamma(\ker D\phi)$, and $\tilde X_t \in \Gamma(H)$ are all time-dependent, with $D\phi(\tilde X_t) = D\phi(\tilde X)$. This decomposition follows from property (IR2) and the fact that the action $\psi\colon \Gamma(\phi^*\ker\rho_P) \to \Gamma(\ker D\phi)$ is an isomorphism of Lie algebras. On the one hand,

$$\bar\theta(D\varphi^t_{(u,\eta)}(\tilde X)) = \bar\theta\Big(\sum_i f_{t,i}\psi(\phi^*u_{t,i},\phi^*\eta_{t,i})\Big) = \sum_i f_{t,i}\phi^*u_{t,i},$$

since $\psi(\phi^*u_{t,i},\phi^*\eta_{t,i})) = \rho_M(F\circ(\phi^*u_{t,i},\phi^*\eta_{t,i})) = R_{F\circ\phi^*u_{t,i}} + c^\sharp_\theta(F\circ\phi^*\eta_{t,i}|_H)$, $c^\sharp_\theta(F\circ\phi^*\eta_{t,i}|_H) \in \Gamma(H)$ for all t,i, and by definition of Reeb vector fields (cf. Observation 2). Thus

$$\mathcal{L}_{(u,\eta)}(\bar\theta(D\varphi^t_{(u,\eta)}(\tilde X))) = \mathcal{L}_{(u,\eta)}\Big(\sum_i f_{t,i}\phi^*u_{t,i}\Big) = \sum_i(\mathcal{L}_{\psi(u,\eta)}f_{t,i})\phi^*u_{t,i} \tag{26}$$

since $\psi(\phi^*u,\phi^*\eta)\in\Gamma(\ker D\phi)$. On the other hand,

(27)
$$\begin{aligned}\bar\theta([D\varphi^t_{(u,\eta)}(\tilde X),\psi(\phi^*u,\phi^*\eta)]) &= \bar\theta([\tilde X_t,\psi(\phi^*u,\phi^*\eta)])+\sum_i\bar\theta([f_{t,i}A_{t,i},\psi(\phi^*u,\phi^*\eta)])\\ &= \bar\theta([\tilde X_t,\psi(\phi^*u,\phi^*\eta)])-\sum_i\mathcal{L}_{\psi(\phi^*u,\phi^*\eta)}f_{t,i}\bar\theta(A_{t,i})\\ &= \bar\theta([\tilde X_t,\psi(\phi^*u,\phi^*\eta)])-\sum_i\mathcal{L}_{\psi(\phi^*u,\phi^*\eta)}f_{t,i}\phi^*u_{t,i},\end{aligned}$$

where the second equality follows from the fact that $A_{t,i},\psi(\phi^*u,\phi^*\eta)\in\Gamma(\ker D\phi)$ and this is a bundle of abelian Lie algebras (since $\ker\rho_P\to P$ is and $\ker D\phi\cong\phi^*\ker\rho_P$ as bundles of Lie algebras), and the third by definition of $A_{t,i}$. Plugging equations (26) and (27) into equation (25), obtain that

(28)
$$\left(\frac{\mathrm{d}}{\mathrm{d}t}(\varphi^t_{(u,\eta)})^*\bar\theta\right)(\tilde X)=((\varphi^t_{(u,\eta)})^*(\mathcal{L}_{(u,\eta)}\bar\theta))(\tilde X)=(\varphi^t_{(u,\eta)})^*(\bar\theta([\tilde X_t,\psi(\phi^*u,\phi^*\eta)]));$$

the right hand side of equation (28) can be computed to be

$$\bar\theta([\tilde X_t,c^\sharp_\theta(\phi^*\eta|_H)])=\phi^*\eta|_H(\tilde X_t)=\phi^*(\eta(D\phi(\tilde X_t))))=\phi^*(\eta(D\phi(\tilde X)),$$

where the second equality follows from property (IR2), and the last from $D\phi(\tilde X_t)=D\phi(\tilde X)$. Therefore, equation (28) becomes

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}(\varphi^t_{(u,\eta)})^*\bar\theta\right)(\tilde X)=(\varphi^t_{(u,\eta)})^*\phi^*(\eta D\phi(\tilde X))=\phi^*(\eta(D\phi\tilde X))=(\phi^*\eta)(\tilde X),\tag{29}$$

where the second equality follows from $\phi\circ\varphi^t_{(u,\eta)}=\phi$. The above equation implies that the integrand of equation (24) is independent of t, thus yielding that

$$((\varphi^1_{(u,\eta)})^*\bar\theta-\bar\theta)(\tilde X)=(\phi^*\eta)(\tilde X).$$

This implies that at the point $m\in M$ the 1-forms $(\varphi^1_{(u,\eta)})^*\bar\theta-\bar\theta$ and $\phi^*\eta$ are equal; since m is arbitrary, this completes the proof. □

As in the case of symplectic isotropic realisations of Poisson manifolds, Lemma 10 provides a local description of the sections of the period lattice $\Sigma\to P$.

Corollary 12. *Any local section of* $\Sigma\to P$ *is of the form* j^1u*, for some locally defined section* $u\in\Gamma(L_P)$.

Proof. Suppose that $(u,\eta)\in\Gamma(\Sigma)$ (possibly locally defined). By definition, $\varphi^1_{(u,\eta)}=\mathrm{id}$. Fix $p\in P$, $X_p\in T_pP$, $m\in\phi^{-1}(p)$ and $\tilde X_m\in H_m$ with $D_m\phi(\tilde X_m)=X_p$. Applying Lemma 11 to the section (u,η), obtain that $\eta(X_p)=0$. Since $p\in P$ and $X_p\in T_pP$ are arbitrary, it follows that $\eta=0$. This means precisely that (u,η) is holonomic, *i.e.* of the form j^1u as required. □

Observation 12. Consider the canonical contact form $\theta_{\text{can}} \in \Omega^1(J^1L_P)$ described in Example 2. By Corollary 12, translations by elements of Σ preserve θ_{can} (as $(u,\eta)^*\theta_{\text{can}} = \eta$, for $u \in \Gamma(L_P), \eta \in \Omega^1(P, L_P)$, and sections of Σ are holonomic), and therefore its restriction to $\ker \rho_P \subset J^1L_P$ descends to a 1-form

$$\theta_0 \in \Omega^1(\ker \rho_P/\Sigma; \pi^* L_P),$$

which does not necessarily define a contact distribution (unless $\ker \rho_P = J^1L_P$).

The following theorem gives a local normal form for sufficiently small open ϕ-saturated neighbourhoods of the fibres of $\phi\colon (M,H) \to (P, L_P, \{\cdot,\cdot\})$; as such, it can be thought of as providing local **action-angle** variables in analogy with [14, Corollary 2] in the case of complete isotropic realisations of Poisson manifolds, and [2, Theorem 1] and [21, Theorems 4 and 5] in the case of non-commutative integrable systems on contact manifolds (the former with elliptic-type singularities) *once* a suitable local contact form has been fixed (cf. Section 6.1 for the relation between integrable Hamiltonian systems on contact manifolds and contact isotropic realisations).

Theorem 13. *[Action-angle coordinates] Given a local section $\sigma\colon U \subset P \to (M,H)$, then*

$$\Psi_\sigma^*\bar{\theta} = \theta_0 + \pi^*\sigma^*\bar{\theta},$$

where $\Psi_\sigma\colon \ker \rho_P/\Sigma|_U \to \phi^{-1}(U)$ is defined by equation (21).

It is important to remark that all forms in the above theorem take values in π^*L_P.

Proof. First, it is shown that the difference $\Psi_\sigma^*\bar{\theta} - \theta_0 \in \Omega^1(\ker \rho_P/\Sigma|_U; \pi^*L_P)$ is basic, *i.e.*

$$\Psi_\sigma^*\bar{\theta} - \theta_0 = \pi^*\beta \tag{30}$$

for some 1-form $\beta \in \Omega^1(U; L_P)$. This is the case if $i_Z(\Psi_\sigma^*\bar{\theta} - \theta_0) = 0$ and $\mathcal{L}_Z(\Psi_\sigma^*\bar{\theta} - \theta_0) = 0$ for any Z tangent to $\ker(\pi\colon \ker \rho_P/\Sigma|_U \to U)$, (where $\mathcal{L}_Z\omega \in \Omega^k(\ker \rho_P/\Sigma; \pi^*L_P)$ is defined as in (22) for any $\omega \in \Omega^k(\ker \rho_P/\Sigma; \pi^*L_P)$). In order to prove this, identify (canonically) $T^\pi_{[j^1_p v]}(\ker_{P,p}/\Sigma_p)$ with $\ker \rho_{P,p}$ via the isomorphism $T^\pi(\ker \rho_P/\Sigma) \simeq \pi^* \ker \rho_P$. Observe that for sections $(u',\eta'), (u,\eta) \in \Gamma(\ker \rho_P)$, $\Psi_\sigma((u',\eta') + t(u,\eta)) = \phi^t_{(u,\eta)}(\Psi_\sigma(u',\eta'))$ for any $t \in \mathbb{R}$, where $\varphi^t_{(u,\eta)}\colon M \to M$ is the flow of $\psi(\phi^*(u,\eta))$. Equivalently,

$$\Psi_\sigma \circ \varphi^t_{(u,v)} = \varphi^t_{(u,v)} \circ \Psi_\sigma, \tag{31}$$

where on the left hand side $\varphi^t_{(u,v)}\colon \ker \rho_P/\Sigma \to \ker \rho_P/\Sigma$ stands for the flow of $\pi^*(u,\eta)$, given by $j^1_p v \mapsto j^1_p v + t(u,\eta)(p)$. Differentiating equation (31), obtain that

$$D_{[j^1_p v]}\Psi_\sigma(\pi^*(u,\eta)) = \psi(\phi^*u, \phi^*\eta)_{\Psi_\sigma(j^1_p v)}.$$

Consider $i_Z(\Psi_\sigma^*\bar{\theta} - \theta_0)$; the above yields

$$i_Z\Psi_\sigma^*\bar{\theta}(j^1_p v) = \bar{\theta}_{\Psi_\sigma(j^1_v)}(\psi(\phi^* u, \phi^*\eta)) = \bar{\theta}_{\Psi_\sigma(j^1_v)}(R_{F\phi^* u} + c(F\phi^*\eta|_H)) = u_p.$$

for any $p \in P$. On the other hand, θ_0 restricted to $\pi^* \ker \rho_P$ is equal to the projection $pr\colon \ker \rho_P \to L$ which implies that $i_Z\theta_0 = \theta_{0,j^1_p v}(u,\eta) = u_p$. Thus $i_Z(\Psi_\sigma^*\bar{\theta} - \theta_0) = 0$ follows. To compute the Lie derivative along Z, observe that

$$\mathcal{L}_Z\Psi_\sigma^*\bar{\theta} = \frac{d}{dt}(\varphi^t_{(u,\eta)})^*(\Psi_\sigma^*\bar{\theta})|_{t=0} = \frac{d}{dt}\Psi_\sigma^* \circ (\varphi^t_{(u,\eta)})^*\bar{\theta}|_{t=0}$$

where the second equality uses equation (31). By equation (29), have that

$$(\varphi^t_{(u,\eta)})^*\bar{\theta}|_{t=0} = \phi^*\eta,$$

which implies that $\mathcal{L}_Z\Psi_\sigma^*\bar{\theta} = \pi^*\eta$. On the other hand,

$$\mathcal{L}_Z\theta_0 = \frac{d}{dt}(\varphi^t_{(u,\eta)})^*\theta_0|_{t=0} = \frac{d}{dt}(id^* + t\pi^* \circ (u,\eta)^*)\theta_0|_{t=0} = \frac{d}{dt}(\theta_0 + t\pi^*\eta)|_{t=0} = \pi^*\eta,$$

where the third equality uses that for a section $(u,\eta)\colon P \to J^1L$, $(u,\eta)^*\theta_{\text{can}} = \eta$. With this, $\mathcal{L}_Z(\Psi_\sigma^*\bar{\theta} - \theta_0) = 0$. As vector fields of the form $\pi^*(u,v)$ generate $\pi^* \ker \rho_P \simeq T^\pi(\ker \rho_P/\Sigma)$ as a $C^\infty(\ker \rho_P/\Sigma)$-module, this implies that equation (30) holds.

To show that $\beta = \sigma^*\bar{\theta}$, consider the section $z\colon P \to \ker \rho_P/\Sigma, p \mapsto [\Sigma_p] = 0$ of $\pi\colon \ker/\Sigma \to P$. Then $z^*\text{pr}^*\beta = \beta$, $z^*\Psi_\sigma^*\bar{\theta} = \sigma^*\bar{\theta}$ as $\Psi_\sigma \circ z = \sigma$, and $z^*\theta_0 = 0$ as for any $s \in \Gamma(\Sigma)$, $s^*\theta_{\text{can}} = 0$ (see Observation 12). Therefore, $\beta = \sigma^*\bar{\theta}$. □

4. Constructing contact isotropic realisations

In analogy with the classification of symplectic isotropic realisations of Poisson manifolds carried out in [14], the classification of contact isotropic realisations over a given regular Jacobi manifold $(P, L_P, \{\cdot,\cdot\})$ of corank k, all of whose leaves are even dimensional, consists of the following two parts:

Part 1: Classify closed submanifolds $\Sigma \subset \ker \rho_P$ for which the composite $\Sigma \hookrightarrow \ker \rho_P \to P$ is a $\mathbb{Z}^k$-bundle and all of its sections are holonomic. Such a submanifold is henceforth called **period lattice** even though it may not necessarily arise from a contact isotropic realisation.

Part 2: Having fixed a period lattice $\Sigma \subset \ker \rho_P$, determine which cohomology classes in $\mathrm{H}^2(P;\Sigma)$ give rise to contact isotropic realisations, *i.e.* whether the total space of the (isomorphism class of the) principal $\ker \rho_P/\Sigma$-bundle associated to c admits a contact structure which makes the projection to P into a contact isotropic realisation.

Part 1, while very interesting in its own right, is beyond the scope of this paper; Section 5 gives an alternative geometric interpretation of a period lattice. This section concentrates on solving the problem outlined in Part 2. To this end, it is important to remark that there already exist partial solutions to this question, as the following examples show.

Example 15 (Regular contact manifolds)**.** Suppose that $(P, L_P, \{\cdot,\cdot\})$ is a closed symplectic manifold, so that $L_P = P \times \mathbb{R}$ and the Jacobi bracket is nothing but the Poisson bracket on $C^\infty(P)$ induced by a symplectic form ω. Then $\ker \rho_P = P \times \mathbb{R} \subset J^1 L_P$, which implies that any period lattice Σ is trivial. Let $\Sigma \subset \ker \rho_P$ be a period lattice and suppose that $\alpha \in \Gamma(\Sigma)$ is a nowhere vanishing section. Corollary 12 gives that $\alpha = j^1 f$ for some smooth function f, which implies that f is a non-zero constant function. Without loss of generality, fix $f \equiv 1$. A contact isotropic realisation $\phi\colon (M, \theta) \to (P, \omega)$ is nothing but a principal S^1-bundle whose infinitesimal action is generated by the Reeb vector field R_M associated to the connection 1-form (which is a contact form!) $\theta \in \Omega^1(M)$ (cf. Remark 2). A co-oriented contact manifold (M, θ) as above is known in the literature as a **regular contact manifold** (cf. [4, 19]). Such a bundle exists if and only if the cohomology class $[\omega] \in \mathrm{H}^2(P; \mathbb{R})$ is integral and the Chern class of the bundle must then be equal to $[\omega]$ (cf. [19, Theorems 7.2.4 and 7.2.5]). Reformulating this result in this paper's language, (P, ω) endowed with the period lattice $\Sigma \to P$ admits a contact isotropic realisation if and only if $[\omega]$ is integral and the corresponding Chern class equals $[\omega]$, where $\mathrm{H}^2(P; \Sigma)$ is identified with $\mathrm{H}^2(P; \mathbb{Z})$. In particular, not all elements of $\mathrm{H}^2(P; \Sigma)$ give rise to a contact isotropic realisation (in fact, there may be none that does).

Example 16 (Zero Jacobi structure)**.** Suppose that $(P, L_P, \{\cdot,\cdot\})$ is endowed with the zero Jacobi structure, so that $L_P = P \times \mathbb{R}$ and the Jacobi bracket is identically zero. In this case, $\ker \rho_P = J^1 L_P := J^1 P$. Given any period lattice $\Sigma \subset J^1 P$, it can be shown that *every* cohomology class in $\mathrm{H}^2(P; \Sigma)$ gives rise to a contact isotropic realisation, as proved in [2].

The main result of this section provides a cohomological criterion that solves Part 2 in the spirit of [14, Theorems 4.2 and 4.3] which unifies and extends Examples 15 and 16 above (cf. Theorem 17). The rest of the section is structured as follows. First, it is shown that *locally* any regular Jacobi manifold $(P, L_P, \{\cdot,\cdot\})$ all of whose leaves are even dimensional with a period lattice $\Sigma \subset \ker \rho_P$ admits a contact isotropic realisation (cf. Section 4.1). The global obstruction to constructing these objects is investigated in Section 4.2.

4.1. **Local construction.** Fix notation as in Theorem 13 and its proof. Let $\phi\colon (M, H) \to (P, L_P, \{\cdot,\cdot\})$ be a contact isotropic realisation with characteristic foliation denoted by $\mathcal{F}$, and let $\sigma\colon U \to M$ be a local section. Theorem 13 proves that $\Psi_\sigma^* \bar{\theta} = \theta_0 + \pi^* \sigma^* \bar{\theta}$, where $\theta_0 \in \Omega^1(\ker \rho_P/\Sigma; \pi^* L_P)$ is the 1-form constructed in Observation 12. The next lemma shows that $\sigma^* \bar{\theta} \in \Omega^1(P; \sigma^* \phi^* L_P) = \Omega^1(P; L_P)$ is related to the foliated 2-form uniquely defined by $\omega(\rho_P(j^1 u), \rho_P(j^1 v)) = \{u, v\}_P$ (cf. Proposition 6).

Lemma 14. *The restriction of the 1-form $\sigma^*\bar\theta$ to the foliation $\mathcal{F}$ satisfies $\mathrm{d}_{\mathcal{F}}\sigma^*\bar\theta = \omega$, i.e. it is a local primitive for ω.*

Proof. As in the proof of Proposition 6, it suffices to check that

$$\mathrm{d}_{\mathcal{F}}\sigma^*\bar\theta(\rho_P(j^1u),\rho_P(j^1v)) = \omega(\rho_P(j^1u),\rho_P(j^1v)).$$

By definition,

$$\begin{aligned}\mathrm{d}_{\mathcal{F}}\sigma^*\bar\theta(\rho_P(j^1u),\rho_P(j^1v)) = -\sigma^*\bar\theta([\rho_P(j^1u),\rho_P(j^1v)]) + \bar\nabla_{\rho_P(j^1u)}(\sigma^*\bar\theta(\rho_P(j^1v)))\\ - \bar\nabla_{\rho_P(j^1v)}(\sigma^*\bar\theta(\rho_P(j^1u))).\end{aligned} \tag{32}$$

First, consider

$$\bar\nabla_{\rho_P(j^1u)}(\sigma^*\bar\theta(\rho_P(j^1v))) = \bar\nabla_{\rho_P(j^1u)}(\sigma^*(\bar\theta(D\sigma(\rho_P(j^1v)))));$$

since $\phi\circ\sigma = id$ and $\phi\colon (M,H)\to(P,L_P,\{\cdot,\cdot\})$ is a Jacobi map with bundle morphism F, it follows that

$$D\sigma(\rho_P(j^1v)) = \rho_M(j^1(F\circ\phi^*v)) + X_v,$$

where $X_v \in \Gamma(\ker D\phi|_{\sigma(U)})$ (and similarly for u). Note that since $\ker D\phi = \rho_M\circ F(\phi^*\ker\rho_P)$ and the vector fields X_u, X_v are only defined on $\sigma(U)$, it follows that there exist local sections $(w,\eta),(z,\zeta)\in\Gamma(\ker\rho_P)$ with

$$X_u = \rho_M(F(\phi^*w,\phi^*\eta)) \text{ and } X_v = \rho_M(F(\phi^*z,\phi^*\zeta)).$$

Thus by definition of ρ_M (cf. Observation 5)

$$\bar\theta(D\sigma(\rho_P(j^1v))) = \bar\theta(\rho_M(j^1(F\circ\phi^*v)) + \rho_M(F(\phi^*z,\phi^*\zeta))) = \phi^*(v+z),$$

and, similarly, $\bar\theta(D\sigma(\rho_P(j^1u))) = \phi^*(u+w)$. Therefore, by definition of $\bar\nabla$,

$$\bar\nabla_{\rho_P(j^1u)}(\sigma^*\bar\theta(\rho_P(j^1v))) - \bar\nabla_{\rho_P(j^1v)}(\sigma^*\bar\theta(\rho_P(j^1u))) = \{u,v+z\}_P - \{v,u+w\}_P. \tag{33}$$

On the other hand,

$$\begin{aligned}(\sigma^*\bar\theta)([\rho_P(j^1u),\rho_P(j^1v)]) &= \sigma^*(\bar\theta(D\sigma([\rho_P(j^1u),\rho_P(j^1v)])))\\ &= \sigma^*(\bar\theta(([D\sigma(\rho_P(j^1u)),D\sigma(\rho_P(j^1v))])))\\ &= \sigma^*(\bar\theta([R_{F(\phi^*(u+w))} + c_\theta^\sharp(F\circ\phi^*\eta|_H), R_{F(\phi^*(v+z))} + c_\theta^\sharp(F\circ\phi^*\zeta|_H)]))\\ &= \sigma^*(\bar\theta(R_{\{F(\phi^*(u+w)),F(\phi^*(v+z))\}_M} + [c_\theta^\sharp(F\circ\phi^*\eta|_H), c_\theta^\sharp(F\circ\phi^*\zeta|_H)]))\\ &= \sigma^*(\phi^*(\{u+w,v+z\}_P) - \phi^*\eta(\rho_M(F\circ\phi^*\zeta)))\\ &= \{u+w,v+z\}_P - \eta(D\phi(\rho_M(F\circ\phi^*\zeta))) = \{u+w,v+z\}_P - \eta(\rho_P(\zeta)),\end{aligned} \tag{34}$$

where the third equality follows from the definition of ρ_M and the above conventions, the fourth from the defining property of Reeb vector fields (cf. Definition 3), the fifth by definition of Reeb vector fields and the curvature map, and the last from the fact that ϕ is a Jacobi map with bundle component F. Observe that since $(w,\eta),(z,\zeta)\in\Gamma(\ker\rho_P)$,

$$\{w,z\}_P-\eta(\rho_P(\zeta))=\{w,z\}_P+\eta(\rho_P(j^1z))=\omega(\rho_P(w,\eta),\rho_P(z,\zeta))=0,$$

where the general definition of ω is used (cf. Observation 9). Thus equation (34) yields that

$$(\sigma^*\bar{\theta})([\rho_P(j^1u),\rho_P(j^1v)])=\{u,v\}_P+\{u,z\}_P+\{w,v\}_P.$$

Using this identity together with equation (33) in equation (32) yields the required result. □

The condition of Lemma 14 is, in fact, sufficient to construct a local contact isotropic realisation of a regular Jacobi manifold all of whose leaves are even dimensional; this is the content of the next proposition, stated below without proof (it can be checked using local coordinates, for instance). Fix one such $(P,L_P,\{\cdot,\cdot\})$ and a period lattice $\Sigma\subset\ker\rho_P$.

Proposition 15. *Let $U\subset P$ be an open set and fix $\beta\in\Omega^1(U,L)$ with the property that $\mathrm{d}_{\mathcal{F}}\beta=\omega$. Then*

$$\pi:(\ker\rho/\Sigma|_U,\theta_0+\pi^*\beta)\to(U,L_U,\{\cdot,\cdot\})$$

is a contact isotropic realisation.

4.2. **A cohomological criterion.** Fix a regular Jacobi manifold $(P,L_P,\{\cdot,\cdot\})$ all of whose leaves are even dimensional, whose underlying foliation is denoted by $\mathcal{F}$, and a period lattice $\Sigma\subset J^1L_P$. By assumption, $\Gamma(\Sigma)$ consists solely of holonomic sections (cf. Corollary 12); thus the restriction of the Spencer operator $\mathrm{D}\colon\Gamma(\ker\rho_P)\to\Omega^1(P;L_P)$ (cf. Observation 6) descends to a homomorphism $\hat{\mathrm{D}}\colon\Gamma(\ker\rho_P/\Sigma)\to\Omega^1(P;L_P)$.

The next lemma shows that the image $\hat{\mathrm{D}}(\Gamma(\ker\rho_P/\Sigma))$ consists solely of $\mathrm{d}_{\mathcal{F}}$-closed 1-forms, where $\mathrm{d}_{\mathcal{F}}$ is defined as in equation (15).

Lemma 16. *For any $\alpha\in\Gamma(\ker\rho_P)$, $\mathrm{d}_{\mathcal{F}}\mathrm{D}(\alpha)=0$.*

Proof. It suffices to check that $\mathrm{d}_{\mathcal{F}}\mathrm{D}(\alpha)(\rho_P(j^1u),\rho_P(j^1v))=0$ for any $u,v\in\Gamma(L_P)$ (cf. the proof of Lemma 6). Fix two such sections of L_P; by definition of $\mathrm{d}_{\mathcal{F}}$,

$$\begin{aligned}\mathrm{d}_{\mathcal{F}}\mathrm{D}(\alpha)(\rho_P(j^1u),\rho_P(j^1v))&=-\mathrm{D}_{[\rho_P(j^1u),\rho_P(j^1v)]}(\alpha)\\&\quad+\bar{\nabla}_{\rho_P(j^1u)}(\mathrm{D}_{\rho_P(j^1v)}(\alpha))-\bar{\nabla}_{\rho_P(j^1v)}(\mathrm{D}_{\rho_P(j^1u)}(\alpha))\\&=-\mathrm{D}_{\rho_P(j^1\{u,v\}_P)}(\alpha)+\{u,\mathrm{D}_{\rho_P(j^1v)}(\alpha)\}_P-\{v,\mathrm{D}_{\rho_P(j^1u)}(\alpha)\}_P,\end{aligned}\tag{35}$$

where the second equality uses properties of the anchor map ρ_P, of the Lie bracket on $\Gamma(J^1L_P)$, and the definition of $\bar{\nabla}$ (cf. equation (14)). Let $z\in\Gamma(L_P)$ be any section; then

$$\begin{aligned}\mathrm{D}_{\rho_P(j^1z)}(\alpha)&=\nabla_\alpha z+pr([j^1z,\alpha])=pr([j^1z,\alpha])=-pr([\alpha,j^1z])\\&=\nabla_{j^1z}(pr(\alpha))-\mathrm{D}_{\rho(\alpha)}(j^1z)=\{z,pr(\alpha)\}_P,\end{aligned}$$

where the first and fourth equalities use the compatibility condition for the Spencer operator of equation (7), the second uses that $\alpha \in \Gamma(\ker \rho_P)$, the third exploits anti-symmetry of the Lie bracket, and the last follows by definition of the J^1L_P-connection ∇ (cf. equation (6)). Applying the above equality to the right hand side of equation (35), it follows that

$$\mathrm{d}_{\mathcal{F}}\mathrm{D}(\alpha)(\rho_P(j^1u), \rho_P(j^1v)) = -(\{\{u,v\}_P, pr(\alpha)\}_P + \text{c.p.}) = 0,$$

where c.p. stands for cyclic permutation and the last equality follows from the Jacobi identity. This proves that $\mathrm{d}_{\mathcal{F}}D(\alpha) = 0$, thus completing the proof. □

Let $\mathcal{C}^\infty(\ker \rho_P/\Sigma)$ and $\mathcal{Z}^1(\mathcal{F}; L_P)$ denote the sheaves of smooth sections of $\ker \rho_P/\Sigma \to P$ and of foliated closed 1-forms with values in L_P respectively. Lemma 16 implies that there is a morphism of sheaves

$$\hat{\mathrm{D}} : \mathcal{C}^\infty(\ker \rho_P/\Sigma) \to \mathcal{Z}^1(\mathcal{F}; L_P), \tag{36}$$

which induces a homomorphism at the level of cohomology with values in the above sheaves,

$$\mathcal{D} : \mathrm{H}^1(P; \mathcal{C}^\infty(\ker \rho_P/\Sigma)) \to \mathrm{H}^1(P; \mathcal{Z}^1(\mathcal{F}; L_P)). \tag{37}$$

Recall that $\mathrm{H}^1(P; \mathcal{C}^\infty(\ker \rho_P/\Sigma)) \cong \mathrm{H}^2(P;\Sigma)$ (cf. the discussion preceding Definition 14), while a standard double Čech-Lie algebroid differential complex argument implies that $\mathrm{H}^1(P; \mathcal{Z}^1(\mathcal{F}; L_P)) \cong \mathrm{H}^2(\mathcal{F}; L_P)$. Using these identifications tacitly, the above yields a homomorphism $\mathcal{D} : \mathrm{H}^2(P;\Sigma) \to \mathrm{H}^2(\mathcal{F}; L_P)$ which plays a central role in the following theorem, which is the main result of this section.

Theorem 17. *A Jacobi manifold $(P, L_P, \{\cdot,\cdot\})$ admits a contact isotropic realisation with period lattice Σ and Chern class $c \in \mathrm{H}^2(P;\Sigma)$ if and only if $\mathcal{D}c = [\omega]$, where $\omega \in \Omega^2(\mathcal{F}; L_P)$ is the closed foliated 2-form constructed in Lemma 6.*

Proof. The strategy is similar to that of [14, Theorems 4.2 and 4.3]. Throughout the proof, let $\mathcal{U} := \{U_i\}$ denote a good open cover of P unless otherwise stated. Suppose first that $\phi : (M, H) \to (P, L_P, \{\cdot,\cdot\})$ is a contact isotropic realisation with Chern class $c \in \mathrm{H}^2(P;\Sigma)$, and denote by $\bar{\theta} \in \Omega^1(M; \phi^* L_P)$ the induced generalised contact form. For each i, let $\sigma_i : U_i \to M$ be a local smooth section of ϕ; as in Theorem 9, the section σ_i induces a local trivialisation $\Psi_i : \ker \rho_P/\Sigma|_{U_i} \to \phi^{-1}(U_i)$. By Definition 14, a Čech cocycle representing c is given by the smooth maps $t_{ij} : U_{ij} \to \ker \rho_P/\Sigma$ defined by

$$\Psi_j^{-1} \circ \Psi_i([\alpha]) = [\alpha] + t_{ij}(\pi(\alpha)),$$

where $\pi : \ker \rho_P/\Sigma \to P$ is the projection (cf. the discussion following Theorem 9). Thus a Čech cocycle representing $\mathcal{D}c$ is given by $\{t_{ij}^*\theta_0 = \hat{\mathrm{D}}(t_{ij})\}$, where the equality follows by definition of θ_0 (cf. Observation 12). By definition of t_{ij}, $\Psi \circ t_{ij} = \sigma_j$ on U_{ij}; thus

$$\sigma_j^*\bar{\theta} = t_{ij}^* \circ \Psi_i^*\bar{\theta} = t_{ij}^*(\theta_0 + \pi^*\sigma_i^*\bar{\theta}) = t_{ij}^*\theta_0 + \sigma_i^*\bar{\theta}, \tag{38}$$

where the second equality uses Theorem 13, and the last follows by noticing that $\pi \circ t_{ij} = id$. Equation (38) gives that $t_{ij}^*\theta_0 = \sigma_j^*\bar{\theta} - \sigma_i^*\bar{\theta}$. By Lemma 14, $\mathrm{d}_{\mathcal{F}}(\sigma_j^*\bar{\theta}) = \omega = \mathrm{d}_{\mathcal{F}}(\sigma_i^*\bar{\theta})$; this implies that the cohomology class corresponding to the Čech cocycle $\{\sigma_j^*\bar{\theta} - \sigma_i^*\bar{\theta}\}$ is precisely $[\omega]$, as it is the difference of two primitives of ω.

Conversely, suppose that $\mathcal{D}c = [\omega]$. Choose the good open cover $\mathcal{U}$ so that, for each i, there exists a primitive for $\omega|_{U_i}$, *i.e.* there exists $\beta_i \in \Omega^1(\mathcal{F}|_{U_i}; L_P)$ with $\mathrm{d}_{\mathcal{F}}\beta_i = \omega|_{U_i}$. Since $\mathcal{F}$ is regular, the foliated 1-forms β_i can be extended to honest 1-forms (*i.e.* elements of $\Omega^1(U_i; L_P)$), which are henceforth also denoted by β_i by abuse of notation. Since $\mathcal{D}c = [\omega]$, there exists a Čech cocycle $\{t_{ij}\}$ representing c (subordinate to the above good open cover) which satisfies $t_{ij}^*\theta_0 = \beta_j - \beta_i$ on U_{ij}. For each i, consider the contact isotropic realisation $\pi : (\ker \rho_P/\Sigma|_{U_i}, \theta_0 + \pi^*\beta_i) \to (U_i, L_P|_{U_i}, \{\cdot,\cdot\}_P|_{U_i})$ (cf. Proposition 15). The principal $\ker \rho_P/\Sigma$-bundle over P with Chern class c is constructed (up to isomorphism) by glueing the above local models using the translations t_{ij}. The condition $t_{ij}^*\theta_0 = \beta_j - \beta_i$ ensures that the locally defined generalised contact forms $\theta_0 + \pi^*\beta_i$ patch together to give a globally defined generalised contact form $\theta \in \Omega^1(M; \phi^* L_P)$, where $\phi : M \to P$ is the principal $\ker \rho_P/\Sigma$-bundle over P with Chern class c constructed above. Setting $H := \ker\theta$, have that $\phi : (M, H) \to (P, L_P, \{\cdot,\cdot\})$ is a contact isotropic realisation as it suffices to check that the properties (IR1), (IR2) and (IR3) hold locally and they do by construction, since the bundle is obtained by glueing contact isotropic realisations. This completes the proof of the theorem. □

Example 17. Theorem 17 can be used to construct examples of contact isotropic realisations all of whose invariants are non-trivial, *i.e.* with non-trivial line bundle, with period lattice Σ defining a non-trivial local system of coefficients and with non-zero Chern class. To this end, consider $\mathbb{R}\mathrm{P}^2$ with the tautological line bundle $\tau \to \mathbb{R}\mathrm{P}^2$ and the zero Jacobi bracket. Example 20 below shows how to construct a period lattice $\Sigma_{\mathbb{R}\mathrm{P}^2} \subset J^1\tau$. It can be checked that $\mathrm{H}^2(\mathbb{R}\mathrm{P}^2; \Sigma_{\mathbb{R}\mathrm{P}^2}) \cong \mathbb{Z}^3$ (cf. [31] for the explicit calculation). Since the Jacobi structure under consideration is trivial, the induced characteristic foliation is by points; in other words, $\mathrm{H}^2(\mathcal{F}; \tau) = 0$. This implies that the homomorphism $\mathcal{D} : \mathrm{H}^2(\mathbb{R}\mathrm{P}^2; \Sigma_{\mathbb{R}\mathrm{P}^2}) \to \mathrm{H}^2(\mathcal{F}; \tau)$ is identically zero; Theorem 17 therefore gives that all $c \in \mathrm{H}^2(\mathbb{R}\mathrm{P}^2; \Sigma_{\mathbb{R}\mathrm{P}^2})$ give rise to contact isotropic realisations. Taking $c \neq 0$, obtain an example of contact isotropic realisation defined on a non-trivial line bundle with non-trivial invariants.

5. Transversal integral L-projective structures

The period lattice Σ associated to a contact isotropic realisation $\phi : (M, H) \to (P, L_P, \{\cdot,\cdot\})$ induces a geometric structure on the characteristic foliation of the Jacobi manifold $(P, L_P, \{\cdot,\cdot\})$. The aim of this section is to introduce this structure, which, in the case of the zero Jacobi structure on the trivial line bundle,

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has already been noticed (cf. [2]). Intuitively, this is the analogue of the geometric structure induced on the symplectic (regular!) foliation of on the base of a symplectic isotropic realisation of a Poisson manifold.

Throughout this section, let $(N, \mathcal{F})$ denote a foliated manifold whose foliation has codimension l, and let $\pi\colon L \to N$ be a line bundle, unless otherwise stated.

Definition 18. A **transversal integral L-projective structure** (TIP-structure for short) on $(N, \mathcal{F})$ is a choice of

- flat $T\mathcal{F}$-connection on $\pi\colon L \to N$, $\nabla : \Gamma(T\mathcal{F}) \times \Gamma(L) \to \Gamma(L)$;
- an embedded smooth submanifold $\Sigma \subset J^1L$ satisfying

(T1) the composite $\Sigma \hookrightarrow J^1L \to N$ is a fibre bundle with fibre $\mathbb{Z}^{l+1}$ and structure group $\mathrm{GL}(l+1;\mathbb{Z})$ (henceforth referred to as a $\mathbb{Z}^{l+1}$-*bundle*);

(T2) any locally defined section $\xi\colon U \subset N \to \Sigma$ is holonomic, i.e. $\xi = j^1u$ for some $u \in \Gamma(L)$;

(T3) any locally defined section $j^1u : U \subset N \to \Sigma$ is flat, i.e. $\nabla u = 0$.

Before giving some examples of the above structure, it is useful to reformulate the existence of a flat $T\mathcal{F}$-connection and property (T1) in Definition 18 in equivalent terms; this is done in the following lemma.

Lemma 18. *The foliated manifold $(N, \mathcal{F})$ admits a transversal integral L-projective structure if and only if there exists an embedded submanifold $\Sigma \subset J^1L$ satisfying properties (T1), (T2), and*

(T4) *$\Sigma^{\mathbb{R}} := \Sigma \otimes \mathbb{R}$ fits in a short exact sequence of vector bundles*

$$0 \to \nu^* \otimes L \to \Sigma^{\mathbb{R}} \to L \to 0, \tag{39}$$

where $\nu^* \subset \mathrm{T}^*N$ is the conormal bundle to the foliation, and $\Sigma^{\mathbb{R}} \to L$ is the restriction of the canonical projection $J^1L \to L$.

Proof. Suppose that $(N, \mathcal{F})$ admits a transversal integral L-projective structure, so that there exist ∇ and Σ as in Definition 18. The aim is to prove that the short exact sequence of equation (39) holds. First, it is shown that the map $\Sigma^{\mathbb{R}} \to L$ is pointwise surjective. Fix a point $p \in P$, an open neighbourhood $p \in U$ and a trivialisation $L|_U \cong U \times \mathbb{R}$. By shrinking U if needed, it may be assumed that $\Sigma|_U$ is also trivialisable; properties (T1) and (T2) imply that there exist smooth functions $u_1, \ldots, u_{l+1} \in C^\infty(U)$ with the property that $j^1u_1, \ldots, j^1u_{l+1}$ are a frame of $\Sigma|_U$. It suffices to prove that there exists i such that $u_i(p) \neq 0$. Suppose the contrary; since ∇ is a flat connection, it can be locally written by

$$\nabla_X u := \mathrm{d}u(X) + \beta(X)u,$$

where $X \in \Gamma(T\mathcal{F}|_U)$, $u \in C^\infty(U)$ and $\beta \in \Omega^1(\mathcal{F})$ is a closed foliated 1-form. Property (T3) implies that, for all i, and for all $X \in T_p\mathcal{F}$, $\mathrm{d}_pu_i(X_p) = \beta_p(X_p)u_i(p)$ $= 0$. Thus, for all i, $\mathrm{d}_pu_i \in \nu^*_p$. Using the identification $J^1U \cong \mathrm{T}^*U \oplus \mathbb{R}$, have that $j^1_pu_1, \ldots, j^1_pu_{l+1} \in \nu^*_p \oplus 0 \subset J^1U$. However, $\dim \nu^*_p = l$, which contradicts the

independence of $j^1u_1, \dots, j^1u_{l+1}$ on U. Since $p \in P$ is arbitrary, this argument shows that $\Sigma^{\mathbb{R}} \to L$ is pointwise surjective.
To complete the proof that equation (39) holds, it suffices to show that $\nu^* \otimes L = (\mathrm{T}^*P \otimes L) \cap \Sigma^{\mathbb{R}}$. To this end, fix $p \in P$, let U be an open neighbourhood over which both L and J^1L can be trivialised as above. Given any element of $\alpha_p \in (\mathrm{T}^*_pP \otimes L) \cap \Sigma^{\mathbb{R}}$, (by possibly shrinking U if needed) there exists a smooth section $\alpha\colon U \to (\mathrm{T}^*P \otimes L) \cap \Sigma^{\mathbb{R}}$ with $\alpha(p) = \alpha_p$, since $(\mathrm{T}^*P \otimes L) \cap \Sigma^{\mathbb{R}} = \ker(\Sigma^{\mathbb{R}} \to L) \to P$ is a vector bundle of rank l. As above, fix a frame $j^1u_1, \dots, j^1u_{l+1}$ of $\Sigma|_U$; then there exist smooth functions $f_1, \dots f_{l+1} \in C^\infty(U)$ with $\alpha = \sum_{i=1}^{l+1} f_i j^1 u_i$. Since $\alpha \in \Gamma(\ker(J^1L \to L)|_U)$, it follows that $\sum_{i=1}^{l+1} f_i u_i = 0$ on U. Using property (T3), for all $X \in \Gamma(T\mathcal{F}|_U)$,

$$0 = \nabla_X \left(\sum_{i=1}^{l+1} f_i u_i \right) = \sum_{i=1}^{l+1} \mathrm{d}f_i(X) u_i = \mathrm{D}(\alpha)(X),$$

where D is the Spencer operator. This implies that $\mathrm{D}(\alpha) \in \Gamma(\nu^* \otimes L|_U)$, which, in turn, gives that $\alpha \in \Gamma(\nu^* \otimes L|_U)$, since $\alpha \in \Gamma(\ker(J^1L \to L)|_U)$. Thus $\alpha_p \in \nu^*_p \otimes L_p$; since both p and α_p are arbitrary, this proves that $(\mathrm{T}^*P \otimes L) \cap \Sigma^{\mathbb{R}} \subset \nu^* \otimes L$. Equality follows by dimension counting; this shows that $\Sigma^{\mathbb{R}}$ fits in the short exact sequence of equation (39).

Conversely, suppose that $\Sigma \subset J^1L$ satisfies properties (T1) and (T2), and that the short exact sequence of equation (39) holds. Let $v \in \Gamma(L)$ and $p \in P$; since $\Sigma^{\mathbb{R}} \to L$ is surjective, there exists an open neighbourhood U of p and a section $j^1u \in \Gamma(\Sigma|_U)$ with $u(q) \neq 0$ for all $q \in U$. Then, locally, there exists a smooth function $f \in C^\infty(U)$ with $v = fu$. For $X \in \Gamma(T\mathcal{F}|_U)$, define

$$\nabla_X v := \mathcal{L}_X(f)u. \tag{40}$$

The above is well-defined, for if $v = f'u'$ for some other smooth nowhere vanishing section $u' \in \Gamma(L|_U)$ with $j^1u' \in \Gamma(\Sigma|_U)$, then, by definition, $fj^1u - f'j^1u' \in \Gamma(\Sigma^{\mathbb{R}})$, and

$$\begin{aligned} fj^1u - f'j^1u' &= (j^1(fu), \mathrm{D}(fj^1u)) - (j^1(f'u'), \mathrm{D}(f'j^1u')) \\ &= (0, \mathrm{D}(fj^1u - f'j^1u')) \in \Gamma(\mathrm{T}^*P \otimes L). \end{aligned}$$

Thus $fj^1u - f'j^1u' \in \nu^* \otimes L$, which implies that, for all $X \in \Gamma(T\mathcal{F}|_U)$, $\mathcal{L}_X(f)u = \mathcal{L}_X(f')u'$, giving that the formula (40) is independent of the choices made. It can be checked that equation (40) defines a flat $T\mathcal{F}$-connection and that Σ satisfies property (T3) as required. This completes the proof of the lemma. □

Example 19 (Radiant transversal integral affine structures)**.** Let $\Xi \subset \nu^*$ be a transversal integral affine structure on the foliated manifold $(N, \mathcal{F})$, *i.e.* Ξ is an embedded submanifold which is a full rank lattice of ν^* and such that any (locally defined) smooth section $\sigma \in \Gamma(\Xi)$ is closed. Such structures arise naturally in the study of isotropic realisations of Poisson manifolds (cf. [14]). The existence of a Ξ as above is equivalent to the existence of an atlas $\{(U_i, \varkappa_i)\}$ of submersions $\varkappa_i \colon U_i \to \mathbb{R}^l$ locally defining the foliation $\mathcal{F}$ such that for all i, j with $U_{ij} \neq \emptyset$, there exists $h_{ij} := (A_{ij}, c_{ij}) \in \mathrm{GL}(l;\mathbb{Z}) \ltimes \mathbb{R}^l$ with $\varkappa_j = h_{ij} \circ \varkappa_i$ on U_{ij} (cf. [33] and Theorem 19 below for an analogous statement for TIP-structures). Suppose further that Ξ is **radiant**, *i.e.* for all indices i, j as above, $c_{ij} = 0$ (cf. [20, 33] for a more intrinsic definition). Then Ξ gives rise to a TIP-structure on the trivial line bundle $L := N \times \mathbb{R} \to \mathbb{R}$ with flat connection $\nabla := \mathrm{d}_{\mathcal{F}}$, where $\mathrm{d}_{\mathcal{F}}$ denotes the restriction of the exterior derivative to $T\mathcal{F}$, as follows. Let $\{(U_i, \varkappa_i)\}$ be an atlas of submersions as above; for each i, set $\varkappa_i := (f_i^1, \ldots, f_i^l)$. The 1-forms $\mathrm{d}f_i^1, \ldots, \mathrm{d}f_i^l$ are linearly independent over U_i and $\Xi|_{U_i} = \mathbb{Z}\langle \mathrm{d}f_i^1, \ldots, \mathrm{d}f_i^l \rangle$. Locally Ξ can be extended to a lattice of $J^1L|_{U_i}$ by considering

$$\mathbb{Z}\langle j^1 f_i^1, \ldots j^1 f_i^l \rangle;$$

since Ξ is radiant, these locally defined lattices patch together to yield a globally defined lattice $\bar{\Xi} \subset J^1L$. The lattice

$$\Sigma := \bar{\Xi} \oplus \mathbb{Z}\langle j^1 1 \rangle$$

satisfies properties (T1), (T2) and (T4) by construction. In light of Lemma 18, it defines a TIP-structure.

The following example is, in some sense, 'universal' (cf. Definition 21 and Theorem 19).

Example 20 (The tautological line bundle over $\mathbb{R}\mathrm{P}^l$)**.** Let $\mathbb{R}\mathrm{P}^l$ be the real projective space and denote by $\tau \to \mathbb{R}\mathrm{P}^l$ the **tautological line bundle**, whose fibre at a point $[\mathfrak{l}] \in \mathbb{R}\mathrm{P}^l$ is given by $\tau_{[\mathfrak{l}]} = \mathfrak{l}$. The aim is to construct a transversal integral τ-projective structure on $\mathbb{R}\mathrm{P}^l$ endowed with the trivial foliation. Thus it suffices to construct a full-rank lattice $\Sigma_{\mathbb{R}\mathrm{P}^l} \subset J^1\tau$ all of whose sections are holonomic, *i.e.* so that it satisfies properties (T1) and (T2).

The universal cover of $\mathbb{R}\mathrm{P}^l$ can be identified with $\mathbb{R}^{l+1} \setminus 0$ and the pull-back of τ to $\mathbb{R}^{l+1} \setminus 0$, denoted by $\tilde{\tau}$ is trivialisable. In fact, it can be described as follows: it is the line bundle whose fibre over $r \in \mathbb{R}^{l+1} \setminus 0$ is given by $\mathbb{R}\langle r \rangle$. A nowhere vanishing section of τ is given by $z : r \mapsto (r, r)$. The bundle $J^1\tilde{\tau}$ admits a full-rank lattice given by

$$\tilde{\Sigma} := \mathbb{Z}\langle j^1 u_1, \ldots, j^1 u_{l+1} \rangle,$$

where $u_i := \frac{r_i}{\|r\|} z$, and $r = (r_1, \ldots, r_{l+1})$. It can be checked that $\tilde{\Sigma}$ defines a transversal integral $\tilde{\tau}$-projective structure on $\mathbb{R}^{l+1} \setminus 0$. The quotient of $\mathbb{R}^{l+1} \setminus 0$ by the $\mathbb{R}^*$-action by homotheties can be identified with $\mathbb{R}\mathrm{P}^l$; the quotient of $\tilde{\tau}$ by the obvious lifted action yields τ. By construction, the lifted $\mathbb{R}^*$-action on $J^1\tilde{\tau}$ preserves $\tilde{\Sigma}$, which implies that $\tilde{\Sigma}$ induces a full-rank lattice $\Sigma_{\mathbb{R}\mathrm{P}^l} \subset J^1\tau$

which satisfies property (T2), as $\tilde{\Sigma}$ does. Thus $\Sigma_{\mathbb{RP}^l}$ defines a transversal integral τ-projective structure.

Another way of constructing TIP-structures is through the following special class of foliated atlases.

Definition 21. A **transversal integral projective atlas** of $(N, \mathcal{F})$ (TIP-atlas for short), is an atlas $\mathcal{A} := \{(U_i, \chi_i)\}$ of submersions $\chi_i : U_i \to \mathbb{RP}^l$, locally defining $\mathcal{F}$, and smooth maps $A_{ij} \colon U_{ij} \to \mathrm{GL}(l+1; \mathbb{Z})$ for all i, j with $U_{ij} \neq 0$ satisfying

- $\chi_j = [A_{ij}] \circ \chi_i$ on U_{ij}, where $[A_{ij}] \in \mathrm{PGL}(l+1; \mathbb{R})$ is the equivalence class of A_{ij};
- $\{A_{ij}\}$ satisfies the cocycle condition, *i.e.* for all i, j, k with $U_{ijk} \neq \emptyset$, $A_{ik} = A_{jk} A_{ij}$.

Remark 7. There is a natural notion of two TIP-atlases being compatible (essentially inducing the same structure) and, thus, of *maximal* TIP-atlases. Henceforth, whenever two TIP-atlases are said to be equal, it means that they are compatible (and hence contained in the same maximal one).

Observation 13. A TIP-atlas $\{(U_i, \chi_i)\}$ on $(N, \mathcal{F})$ gives rise to a TIP-structure as follows. The line bundle $L \to N$ is defined by

$$L = \chi^*(\tau) := \coprod_i \chi_i^*(\tau)/\sim, \tag{41}$$

where $(m, v_{[\mathfrak{l}]}) \in \chi_i^*(\tau)$ is related to $(m', v_{[\mathfrak{l}']}) \in \chi_j^*(\tau)$ if $m = m' \in N$ (hence A_{ij} maps the line $\mathfrak{l}$ to $\mathfrak{l}' \subset \mathbb{R}^{l+1}$) and $v_{[\mathfrak{l}']} = A_{ij}(v_{[\mathfrak{l}']}) \in \mathbb{R}^{l+1}$. The reason why equation (41) yields a well-defined smooth line bundle is that $\{A_{ij}\}$ satisfies the cocycle condition. With this, there is a canonical inclusion of vector bundles

$$\chi^* : \chi^*(J^1\tau) \hookrightarrow J^1 L,$$

where $\chi^*(J^1\tau) := \coprod_i \chi_i^*(J^1\tau)/\sim$, and $\gamma : T_{\chi_i(m)}\mathbb{RP}^l \to T_{v_{\chi_i(m)}}\tau \in \chi_i^*(J^1\tau)_m$ is related to $\gamma' : T_{\chi_j(m')}\mathbb{RP}^l \to T_{v_{\chi_j(m')}}\tau \in \chi_j^*(J^1\tau)_{m'}$ if $m = m'$ and $DA_{ij} \circ \gamma = \gamma' \circ DA_{ij}$. Explicitly, χ is defined by sending the class of $\gamma \in \chi_i^*(J^1\tau)_m$ to the linear map

$$\chi^*(\gamma) : T_m N \to T_{[m, v_{\chi_i(m)}]} L, \quad X \mapsto [X, \gamma(D\chi_i(X))].$$

As $A_{ij} \in \mathrm{GL}(l+1; \mathbb{Z})$, then

$$\Sigma := \chi^*(\Sigma_{\mathbb{RP}^l}) = \coprod_i \chi_i^*(\Sigma_{\mathbb{RP}^l})/\sim \subset J^1 L$$

is a well defined lattice sitting inside $J^1 L$ via the map χ. Note that $\mathrm{rk}\,(\Sigma) = \mathrm{rk}\,(\Sigma_{\mathbb{RP}^l}) = l+1$, and that Σ has properties (T1) and (T2). Finally, Lemma 18 implies that $\Sigma \subset J^1 L$ defines a TIP-structure on $(N, \mathcal{F})$, since $\Sigma^{\mathbb{R}}$ fits into the exact sequence (39) by construction.

In fact, Observation 13 can be strengthened, since every TIP-structure also gives rise to a TIP-atlas. This is the content of the following theorem.

Theorem 19. *There is a 1-1 correspondence between*

- *a transversal integral L-projective structure* $\Sigma \subset J^1L$ *of* $(N, \mathcal{F})$*;*
- *a transversal integral projective atlas* $\{(U_i, \chi_i)\}$ *of* $(N, \mathcal{F})$*.*

The correspondence is given by

$$L = \chi^*(\tau), \quad \Sigma = \chi^*(\Sigma_{\mathbb{RP}^l}),$$

where the notation is as in Observation 13.

Proof. Observation 13 provides a way to construct a TIP-structure starting from a TIP-atlas. It remains to prove that that a TIP-structure gives rise to a TIP-atlas and that the two procedures are inverse to one another.

Fix a TIP-structure $\Sigma \subset J^1L$ of $(N, \mathcal{F})$. Choose an open cover $\{U_i\}$ of N with the property that over each U_i the line bundle $\pi : L \to N$ trivialises via the nowhere vanishing section $z_i \colon U_i \times \mathbb{R} \to \pi^{-1}(U_i)$, and denote the transition functions by $c_{ij} : U_{ij} \to \mathrm{GL}(\mathbb{R})$. Without loss of generality, assume that all U_i are small enough so that $\Sigma|_{U_i}$ can be trivialised with local frame $\alpha^i_1, \dots \alpha^i_{l+1} \in \Gamma(\Sigma|_{U_i})$ and transition functions denoted by $A_{ij} : U_{ij} \to \mathrm{GL}(l+1; \mathbb{Z})$. Property (T2) implies that for all i and each $r = 1, \dots, l+1$, there exist smooth functions $g^i_r : U_i \to \mathbb{R}$ with $\alpha^i_r = j^1(g^i_r z_i)$. The TIP-atlas is going to be constructed using these functions g^i_r; before proceeding to the construction, two preparatory claims need to be proved.

Claim 1. For each i, the smooth map $\bar{\chi}_i : U_i \to \mathbb{R}^{l+1}, \bar{\chi}_i := (g^i_1, \dots, g^i_{l+1})$ takes values on $\mathbb{R}^{l+1} \setminus 0$ and is transversal to the Euler vector field $E = \sum_r x_r \frac{\partial}{\partial x_r}$.

Proof of Claim 1. Indeed, as $j^1 g^i_1, \dots, j^1 g^i_{l+1}$ are a frame of $\Sigma|_{U_i}$ and $\Sigma^{\mathbb{R}} \to L$ is pointwise onto by property (T4), it follows that $\bar{\chi}_i \neq 0$. Secondly, to prove that $\bar{\chi}_i$ is transversal to E, *i.e.* $\mathrm{Im}\, D\bar{\chi}_i + E = T\mathbb{R}^{l+1}$, it suffices to show that $\bar{\chi}^*_i \colon \mathrm{Ann}(E) \to \nu^*$ is injective. Using the same argument of Lemma 18, have that $dg^i_r \mid_{\mathcal{F}} = g^i_r \beta \mid_{\mathcal{F}}$, for some fixed closed, foliated 1-form β; hence

$$D\bar{\chi}_i|_{\mathcal{F}} = \beta \otimes E, \tag{42}$$

which implies that $\chi^*_i(\mathrm{Ann}(E)) \subset \nu^*$. To show that $\bar{\chi}^*_i \colon \mathrm{Ann}(E)_{\bar{\chi}_i(p)} \to \nu^*_p$ is injective for any $p \in U_i$, it suffices to show that it is surjective (which is, of course, equivalent by dimension counting). Let $\gamma_p \in \nu^*_p$ and extend it to a locally defined smooth section $\gamma \in \Gamma(\nu^*|_{U_i})$. By property (T4) there exist smooth functions $f^1, \dots, f^{l+1} \in C^\infty(U_i)$ with

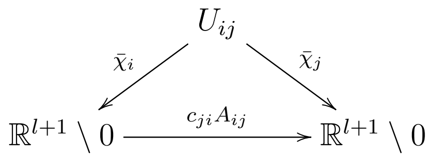
$$\gamma = \sum_r f^r j^1 g^i_r \in \nu^* \subset J^1L \mid_{U_i} .$$

Using the Spencer decomposition, it follows that $\sum_r f^r g_r^i = 0$ and $\sum_r f^r dg_r^i = \gamma$. Therefore, $\chi_i^*(\gamma') = \gamma$ for

$$\gamma' = \sum_r f^r(p)\mathrm{d}x_r \in \Gamma(Ann(E)|_{\bar{\chi}_i(U_i)}),$$

and hence surjectivity follows. □

Claim 2. The diagram



commutes.

Proof of Claim 2. This follows by definition of the maps $\bar{\chi}_i$. □

Using Claims 1 and 2, a TIP-atlas can be defined as follows. Set $\chi_i : U_i \to \mathbb{R}\mathrm{P}^l$ to be the composite

$$\chi_i : U_i \xrightarrow{\bar{\chi}_i} \mathbb{R}^{l+1} \setminus 0 \xrightarrow{q} \mathbb{R}\mathrm{P}^l,$$

where $q\colon \mathbb{R}^{l+1} \setminus 0 \to \mathbb{R}\mathrm{P}^l$ is the quotient map. By Claim 1, χ_i is a submersion, and Claim 2 implies that $\chi_j = [A_{ij}] \circ \chi_i$ on U_{ij}. Moreover, since $Dq(E) = 0$, equation (42) implies that $D\chi_i|_{\mathcal{F}} = 0$, hence $\mathcal{F}$ is tangent to the fibres of χ_i on the one hand, and on the other, dimension counting shows that for $p \in U_i$, $T\mathcal{F}_p = \ker D_p\chi_i$. This shows that the atlas $\{(U_i, \chi_i)\}$ defines an TIP-atlas for $(N, \mathcal{F})$.

It remains to show that these constructions are inverse to each other. Start with a TIP-structure $\Sigma \subset J^1L$, and construct the TIP-atlas $\{(U_i, \chi_i)\}$ as above. First, it is shown that L is isomorphic to $\chi^*(\tau)$, as constructed in Observation 13. Observe that, for each i, $\chi_i^*\tau$ is trivialisable; a nowhere vanishing section is given by $p \mapsto (p, \chi_i(p))$; this follows from the fact that $\chi_i = q \circ \bar{\chi}_i$ and $q^*\tau$ is trivialisable. For each i, define a local trivialisation of $\chi^*(\tau)|_{U_i}$ by

$$\varphi_i : U_i \times \mathbb{R} \to \chi^*(\tau)|_{U_i}, \quad (p,t) \mapsto (p, t\bar{\chi}_i(p)) \in \chi_i^*(\tau).$$

The transition functions with respect to these trivialisations are given by

$$\begin{aligned}\varphi_i(p,t) &= (p, t\bar{\chi}_i(p)) \sim (p, A_{ij}(t\bar{\chi}_i(p))) \\ &= (m, t\frac{c_{ji}}{c_{ji}}A_{ij}(\bar{\chi}_i(p))) = (p, tc_{ij}\bar{\chi}_j(p)) = \varphi_j(p, tc_{ij}),\end{aligned}$$

for $p \in U_{ij}$. Since the cocycle of transition functions of $\chi^*(\tau)$ and L are equal over the same open cover, it follows that they are canonically isomorphic. This identification can be used to show that the TIP-structure obtained from the TIP-atlas $\{(U_i, \chi_i)\}$ as outlined in Observation 13 is the same as the original TIP-structure. Moreover, it is straightforward to check that given a TIP-structure obtained from a TIP-atlas as in Observation 13, the induced TIP-atlas is compatible with the original one. In both cases, details are left to the reader to check. □

6. Some examples

6.1. **Integrable systems on contact manifolds.** Let (M,H) be a $2n+1$-dimensional contact manifold. A vector field $X \in \mathfrak{X}(M)$ is said to be an **infinitesimal automorphism** of (M,H) if its flow preserves H, or, equivalently, if $[X,Y] \in H$ for any $Y \in \Gamma(H)$. Suppose that $\phi\colon (M,H) \to P$ is a proper surjective submersion onto a $2n+1-k$-dimensional smooth manifold P with connected fibres (*i.e.* a fibre bundle).

Definition 22. The quadruple $(M,H,\phi,\mathcal{X})$, where $\mathcal{X}$ is an Abelian Lie algebra of infinitesimal automorphisms of (M,H) of dimension k, is a **complete pre-isotropic contact structure** if

- $\ker D\phi + H = TM$;
- the subbundle $\ker D\phi \cap H$ is isotropic or, equivalently, it is involutive (cf. [21, Lemma 2]);
- the orbits of $\mathcal{X}$ coincide with the fibres of ϕ, *i.e.* the orbits of the action of the simply connected abelian Lie group integrating $\mathcal{X}$ are the fibres of ϕ.

Remark 8. While connectedness of the fibres of ϕ is not required in [21], it is assumed here for simplicity. Many of the arguments below carry over to the case of disconnected fibres with small modifications.

The aim of this section is to prove the following result, which relates the notion of Definition 22 to contact isotropic realisations.

Proposition 20. *Given a complete pre-isotropic contact structure* $(M,H,\phi,\mathcal{X})$*, there exists a unique line bundle* $L_P \to P$ *(up to isomorphism), a bundle isomorphism* $F : \phi^*L_P \to L$*, and a unique Jacobi structure* $(L_P,\{\cdot,\cdot\})$ *so that* $\phi\colon (M,H) \to (P,L_P,\{\cdot,\cdot\})$ *is a contact isotropic realisation with bundle component* F*.*

The strategy of the proof is, first, to obtain the result locally (with the trivial line bundle), and then globalise. In order to prove the local statement, recall the following theorem, stated below without proof (cf. [21, Theorem 3]).

Theorem 21. *Given a complete pre-isotropic contact structure* $(M,H,\phi,\mathcal{X})$*, for any* $m \in M$ *there exists an open* ϕ*-saturated set* U *with* $m \in U$*, a contact form* θ_U *such that* $\mathcal{X}$ *preserves* θ_U*, and the foliation* $\ker D\phi|_U$ *is* θ_U**-complete***, i.e. first integrals of* $\ker D\phi|_U$ *are closed under the Jacobi bracket induced by* θ_U*.*

The key idea behind the proof of Theorem 21 is to construct θ_U by fixing an element of $\ker D\phi$ to be its Reeb vector field. This can be achieved as follows. Fix $m \in M$ and consider a small enough neighbourhood of m such that

- there exists a local section X of $\ker D\phi$ which is transversal to H;
- the contact distribution is locally defined as $\ker\theta$, for θ a locally defined contact form.

The 1-form $\theta_U := \theta/\theta(X)$ is again a contact form which locally defines the contact distribution H and it has the property that X is its Reeb vector field (cf. [21, proof of Theorem 3]). Theorem 21 implies the local form of Proposition 20, as stated in the following corollary, which, in fact, proves a slightly stronger statement:

Corollary 22. *There exists a unique Poisson structure* Λ_W *on* $W = \phi(U) \subset P$ *which makes* $\phi\colon (U,\theta_U) \to (W,\Lambda_W, W\times\mathbb{R})$ *into a contact isotropic realisation with the identity as bundle component.*

Proof. Since ϕ is a submersion, it is an open map, which implies that $W = \phi(U) \subset P$ is an open set. Let $f,g \in C^\infty(W)$ be smooth functions. Then,

$$\{\phi^* f, \phi^* g\} := \Lambda_U^\sharp(\phi^*\mathrm{d}f, \phi^*\mathrm{d}g) + (\phi^* f)X(\phi^* g) - (\phi^* g)X(\phi^* f) = \Lambda_U^\sharp(\phi^*\mathrm{d}f,\phi^*\mathrm{d}g), \tag{43}$$

where Λ_U is the bivector defined by θ_U, and the equality follows from the fact that $X = R_U \in \Gamma(\ker D\phi)$. Since first integrals of $\ker D\phi|_U$ are closed under the above Jacobi bracket (cf. Theorem 21 above), equation (43) implies that there exists a unique smooth bivector Λ_W on W which is defined by the following equation

$$\phi^*(\Lambda_W(\mathrm{d}f,\mathrm{d}g)) = \Lambda_U^\sharp(\phi^*\mathrm{d}f,\phi^*\mathrm{d}g). \tag{44}$$

Note that equation (44) implies that $D\phi(\Lambda_U) = \Lambda_W$. The bivector Λ_W is Poisson since

$$[\![\Lambda_W,\Lambda_W]\!] = [\![D\phi(\Lambda_U), D\phi(\Lambda_U)]\!] = D\phi([\![\Lambda_U,\Lambda_U]\!]) = D\phi(2X\wedge\Lambda_U) = 0,$$

since $D\phi(X) = 0$ by Theorem 21. As the Jacobi structure defined by Λ_W is completely determined by the pair $(\Lambda_W, 0) \in \mathfrak{X}^2(W)\times\mathfrak{X}(W)$, and $D\phi(\Lambda_U) = \Lambda_W$ and $D\phi(X) = 0$, this implies that $\phi\colon (U,\theta_U) \to (W,\Lambda_W)$ is Jacobi with the identity as bundle component (cf. Example 9); this proves property (IR1). Property (IR2) follows from the fact that $X \in \Gamma(\ker D\phi)$. The distribution $\ker D\phi$ contains the Reeb vector field of θ_U and is *pre-isotropic* in the sense of [21, Definition 1]; by [21, part (i) of Theorem 2], it follows that $\ker D\phi \subset (\ker D\phi)^\perp$, so that property (IR3) holds. This completes the proof of the corollary. □

Proof of Proposition 20. Let $\{U_i\}$ be an open cover of M, where each U_i is an open set of the type described in the statement of Theorem 21, so that, for each i, there exists a $\mathcal{X}$-invariant contact form θ_{U_i}. For each i, Corollary 22 gives a Poisson (and hence Jacobi) structure on $W_i := \phi(U_i)$. Suppose that i,j are such that $U_{ij} \neq \emptyset$; since both θ_{U_i} and θ_{U_j} are contact forms on U_{ij}, there exists a nowhere vanishing function $G_{ji}\colon U_{ij} \to \mathbb{R}$ with $\theta_j = G_{ji}\theta_i$. As both θ_{U_i} and θ_{U_j} are $\mathcal{X}$-invariant, it follows that for all $X \in \mathcal{X}$, $\mathcal{L}_X G_{ji} = 0$. Given that the orbits of $\mathcal{X}$ coincide with the fibres of ϕ, this implies that G_{ji} is *basic*, *i.e.* there

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exists a unique nowhere vanishing function $g_{ji}\colon W_{ij} \to \mathbb{R}$, where $W_i = \phi(W_i)$ and $W_{ij} = W_i \cap W_j = \phi(U_{ij})$, with $G_{ji} = \phi^* g_{ji}$. If Λ_i denotes the Poisson bivector induced on W_i as in Corollary 22 (and similarly for j), it follows from the construction that $\Lambda_j = g_{ji}^{-1}\Lambda_i = g_{ij}\Lambda_i$ on W_{ij}, where the last equality follows from the fact that $G_{ij} = G_{ji}^{-1}$. Thus the Poisson bivectors Λ_i and Λ_j are isomorphic on W_{ij} up to a conformal factor; equivalently, the induced Jacobi structures on the trivial line bundle over W_{ij} defined by Λ_i and Λ_j are isomorphic with bundle component g_{ij}. Moreover, since the collection of functions $\{G_{ji}\}$ satisfy the cocycle condition, it follows that $\{g_{ij}\}$ also do. Therefore, by Observation 4, there exists a line bundle $L_p \to P$ (determined up to isomorphism), and a unique Jacobi structure $(L_P, \{\cdot,\cdot\}_P)$ with the property that there exists a bundle isomorphism $F\colon \phi^* L_P \to L_M := TM/H$ (the latter depending on the explicit isomorphism used to identify $L_P \to P$ as a representative of the isomorphism class of line bundles determined by the cocycle $\{g_{ij}\}$). The properties in the statement of the proposition are local in nature, but these hold locally by Corollary 22, thus completing the proof. □

Remark 9. A closer look at the proof of Proposition 20 yields further restrictions on (the isomorphism class of) the line bundle $L_P \to P$. Using the notation of the above proof, let Λ_i, Λ_j be Poisson bivectors on W_i, W_j respectively; then there exists a nowhere vanishing function $g_{ij}\colon W_{ij} \to \mathbb{R}$ with $\Lambda_j = g_{ij}\Lambda_i$ on W_{ij}. Since the self-commutator of both Λ_i, Λ_j under the Schouten bracket vanishes, the above equality implies that

$$-2g_{ij}\Lambda_j^\sharp(\mathrm{d}g_{ij})\Lambda_j = 0 \qquad (45)$$

on W_{ij}, where the sign convention used is that of [29]. Since $\phi\colon (U_j, \theta_{U_j}) \to (W_j, \Lambda_j)$ is a complete isotropic realisation, it follows that Λ_j is a regular Poisson bivector on W_j (and, thus, on W_{ij}, which is an open subset) – cf. Lemma 3. Therefore, it can be checked that equation (45) gives that $\Lambda_j^\sharp(\mathrm{d}g_{ij}) = 0$, *i.e.* the function g_{ij} is a local Casimir for Λ_j. However, since $\Lambda_i = g_{ij}^{-1}\Lambda_j$, it follows that g_{ij} is also a local Casimir for Λ_i. In other words, the transition functions of the line bundle $L_P \to P$ can be chosen to be *basic* functions with respect to the foliation $\mathcal{F}$ which is locally defined as the symplectic foliation of the Poisson bivectors Λ_i.

Complete pre-isotropic contact structures lie at the heart of non-commutative Hamiltonian integrability in the contact setting (cf. [21, Definition 3]).

Definition 23. Given an infinitesimal automorphism X of (M, H), the associated differential equation

$$\dot{x} = X$$

is said to be **non-commutatively Hamiltonian integrable** if there exists an open dense subset $M_{\mathrm{reg}} \subset M$, a proper surjective submersion $\phi\colon M_{\mathrm{reg}} \to P$ onto a smooth manifold P with X tangent to its fibres, and an Abelian Lie algebra $\mathcal{X}$

of infinitesimal automorphisms of (M, H) such that $(M, H, \phi, \mathcal{X})$ is a complete pre-isotropic contact structure.

In light of Proposition 20, (the regular part of) a non-commutatively Hamiltonian integrable system on a contact manifold gives rise to a contact isotropic realisation. In particular, the structure theory for the latter developed in Sections 3 and 4 can be applied to find invariants of the former. For instance, Theorem 13 provides local action-angle coordinates for non-commutatively integrable Hamiltonian systems on contact manifolds (cf. [2, 21] for alternative proofs).

6.2. **The case of Poisson manifolds.** Since a Jacobi manifold $(P, L_P, \{\cdot,\cdot\})$ admits a contact isotropic realisation only if it is regular and all of its leaves are even dimensional (cf. Lemma 3), it is worth rephrasing some of the results of Sections 3 and 4 for regular Poisson manifolds, as many of the above objects simplify in this case. Henceforth, let (P, Λ_P) be a regular Poisson manifold with symplectic foliation denoted by $\mathcal{F}$; the induced Jacobi structure is defined on the trivial line bundle $L_P := P \times \mathbb{R}$ (cf. Example 6). Following Example 10, the anchor $\rho_P \colon \mathrm{T}^*P \oplus \mathbb{R} \to TP$ is given by $\rho_P(\eta, f) = \Lambda_P^\sharp(\eta)$, for any $(\eta, f) \in \mathrm{T}^*P \oplus \mathbb{R}$. Thus

$$\ker \rho_P = \nu^* \oplus \mathbb{R} \subset \mathrm{T}^*P \oplus \mathbb{R},$$

while the Spencer operator is given by

$$\mathrm{D}(\eta, f) := \mathrm{d}f - \eta.$$

Moreover, the $T\mathcal{F}$-connection given by Proposition 5 is

$$\bar{\nabla} \colon \Gamma(T\mathcal{F}) \times C^\infty(P) \to C^\infty(P), \qquad (X, f) \mapsto \mathrm{d}f(X);$$

thus the differential $\mathrm{d}_\mathcal{F}$ defined by equation (15) is the restriction of the exterior derivative to the foliation $\mathcal{F}$. The cohomology class $[\omega] \in \mathrm{H}^2(\mathcal{F}; L_P) = \mathrm{H}^2(\mathcal{F})$ defined by Lemma 6 is that of the foliated symplectic form defined by Λ_P.

Without loss of generality, consider a contact isotropic realisation $\phi \colon (M, \theta) \to (P, \Lambda_P)$ with the identity as bundle component, where $\theta \in \Omega^1(M)$ is a contact form. Let $\Sigma \subset \ker \rho_P$ denote the period lattice of ϕ; by Corollary 12, any local section of Σ is of the form $j^1 f$, for some $f \in C^\infty(P)$. However, since $\Sigma \subset \nu^* \oplus \mathbb{R}$,

$$j^1 f \in \Gamma(\Sigma) \quad \Rightarrow \quad \mathrm{d}f \in \Gamma(\nu^*); \tag{46}$$

it is important to notice that this does not hold in general, even for regular Jacobi structures defined on the trivial line bundle. The equality $\ker \rho_P = \nu^* \oplus \mathbb{R}$ can be exploited to give another way to construct the homomorphism of equation (37) which underpins the cohomological criterion of Theorem 17. Denote the sheaf of **basic** smooth functions on $(P, \mathcal{F})$ by $\mathcal{C}^\infty_{\text{basic}}(P; \mathcal{F})$, *i.e.* it consists of smooth functions which are locally constant on the leaves of $\mathcal{F}$. There is a short exact sequence of sheaves

$$1 \to \mathcal{C}^\infty_{\text{basic}}(P; \mathcal{F}) \hookrightarrow \mathcal{C}^\infty(P) \xrightarrow{\mathrm{d}_\mathcal{F}} \mathcal{Z}^1(\mathcal{F}) \to 1,$$

where $\mathcal{C}^\infty(P)$ and $\mathcal{Z}^1(\mathcal{F})$ are the sheaves of smooth functions on P and of closed foliated 1-forms respectively, and $\mathrm{d}_{\mathcal{F}}$ is the foliated exterior derivative.

Lemma 23. *The following is a commutative diagram of short exact sequences of sheaves*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Sigma & \hookrightarrow & \mathcal{C}^\infty(\ker \rho_P) & \longrightarrow & \mathcal{C}^\infty(\ker \rho_P/\Sigma) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle \hat{\mathrm{D}}} & & \\ 1 & \longrightarrow & \mathcal{C}^\infty_{\mathrm{basic}}(P;\mathcal{F}) & \hookrightarrow & \mathcal{C}^\infty(P) & \xrightarrow[\mathrm{d}_{\mathcal{F}}]{} & \mathcal{Z}^1(\mathcal{F}) & \longrightarrow & 1, \end{array} \tag{47}$$

where pr denotes the homomorphism of sheaves induced by the projection $pr\colon$ $\mathrm{T}^*P \oplus \mathbb{R} \to P \times \mathbb{R}$, $\hat{\mathrm{D}}$ is the homomorphism of sheaves defined by equation (36) *which is induced by the Spencer operator* D, *and Σ denotes the sheaf of smooth sections of $\Sigma \to P$.*

Proof. First, observe that equation (46) implies that the image $pr(\Sigma)$ of the sheaf homomorphism $pr\colon \mathcal{C}^\infty(\ker \rho_P) \to \mathcal{C}^\infty$ lies in the sheaf $\mathcal{C}^\infty_{\mathrm{basic}}(P;\mathcal{F})$. The only non-trivial fact that needs checking is commutativity on the right hand side of the diagram (47). Let $[\alpha] \in \Gamma(\ker \rho_P/\Sigma)$ and set $\alpha = (\eta, f) \in \Gamma(\ker \rho_P)$ be a lift of $[\alpha]$. Then

$$\hat{\mathrm{D}}([\alpha]) = \mathrm{D}(\eta, f)|_{\mathcal{F}} = (\mathrm{d}f - \eta)|_{\mathcal{F}} = \mathrm{d}_{\mathcal{F}} f = \mathrm{d}_{\mathcal{F}} \circ pr(\alpha),$$

where the third equality follows from the fact that $\ker \rho_P = \nu^* \oplus \mathbb{R}$. □

Since both $\mathcal{C}^\infty(P)$ and $\mathcal{C}^\infty(\ker \rho_P)$ are fine sheaves, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1(P;\mathcal{C}^\infty(\ker \rho_P/\Sigma)) & \xrightarrow{\cong} & \mathrm{H}^2(P;\Sigma) \\ {\scriptstyle \mathcal{D}}\downarrow & & \downarrow \\ \mathrm{H}^1(P;\mathcal{Z}^1(\mathcal{F})) & \xrightarrow{\cong} & \mathrm{H}^2(\mathcal{F}), \end{array} \tag{48}$$

where the vertical maps are induced by the outer vertical maps of diagram (47) (cf. equation (37) also for the definition of $\mathcal{D}$), and the horizontal isomorphisms are the connecting morphisms induced by the short exact sequences in equation (47). Therefore, the following corollary holds.

Corollary 24. *The map in cohomology*

$$\mathrm{H}^2(P;\Sigma) \to \mathrm{H}^2(P;\mathcal{C}^\infty_{\mathrm{basic}}(P;\mathcal{F})) \cong \mathrm{H}^2(\mathcal{F})$$

induced by the sheaf homomorphism $pr\colon \Sigma \to \mathcal{C}^\infty_{\mathrm{basic}}(P;\mathcal{F})$ equals $\mathcal{D}$ via the identifications of equation (48).

Corollary 24 indicates how to calculate the map $\mathcal{D}$, which is central to understanding whether a regular Poisson manifold (P, Λ_P) with given period lattice $\Sigma \subset \nu^* \oplus \mathbb{R}$ admits a contact isotropic realisation (cf. Theorem 17). For instance, suppose that $\Sigma = \mathbb{Z}\langle j^1 f_1, \dots, j^1 f_k\rangle$ for some basic functions f_i, so that it induces the trivial $\mathbb{Z}^k$-system of coefficients. Then (modulo torsion), an element in $\mathrm{H}^2(P;\Sigma)$ can be written as

$$\sum_{i=1}^{k} [\omega_i] \otimes j^1 f_i,$$

where, for each i, $\omega_i \in \Omega^2(P)$ is a 2-form with integral cohomology class. Since the map $\mathcal{D}\colon \mathrm{H}^2(P;\Sigma) \to \mathrm{H}^2(\mathcal{F})$ is induced by the projection $pr\colon \Sigma \to \mathcal{C}^\infty_{\text{basic}}(P;\mathcal{F})$ and Σ is a trivial bundle,

$$\mathcal{D}\Big(\sum_{i=1}^{k} [\omega_i] \otimes j^1 f_i\Big) = \sum_{i=1}^{k} [f_i \omega_i],$$

where, for each i, $f_i\omega_i$ is a closed foliated 2-form since f_i is basic.

6.3. **Locally conformal symplectic manifolds.** As a final example, the question of which locally conformal symplectic (lcs for short) manifolds admit contact isotropic realisations is tackled. Let (P,σ,τ) be a lcs manifold; without loss of generality, it suffices to consider contact isotropic realisations of the form $\phi\colon (M,\theta) \to (P,\sigma,\tau)$, where $\theta \in \Omega^1(P)$ is a contact form. Denote by (Λ_M, R_M) the bivector and vector fields which uniquely define the Jacobi structure on (M,θ) (cf. Observation 3).

Proposition 25. *A lcs manifold (P,σ,τ) admits a contact isotropic realisation if and only if the following conditions hold*

1. *(P,σ,η) is a globally conformal symplectic manifold, i.e. $\tau = \mathrm{d}f$, for some $f \in C^\infty(P)$;*
2. *the cohomology class of the symplectic form $\bar\sigma = e^f\sigma$ is integral, i.e. $[\bar\sigma] \in H^2(P;\mathbb{Z})$.*

Moreover, a contact isotropic realisation $\phi\colon (M,\theta) \to (P,\sigma,\mathrm{d}f)$ is the same thing as a principal S^1-bundle whose infinitesimal action is generated by

$$e^{-\phi^* f}(R_M - \Lambda^\sharp_M(\phi^*\mathrm{d}f)),$$

whose Chern class is $[\bar\sigma] \in H^2(P;\mathbb{Z})$ and satisfying $\mathrm{d}\theta = \phi^*\sigma - \phi^*\mathrm{d}f \wedge \theta$.

The proof of Proposition 25 relies on the following two results.

Lemma 26. *A necessary condition for a lcs manifold (P,σ,τ) to admit a contact isotropic realisation is that $\tau = \mathrm{d}f$ for some smooth function $f \in C^\infty(P)$. Moreover, $\Sigma = \mathbb{Z}\langle Be^{-f}(\tau,-1)\rangle$ for some constant $B \in \mathbb{R}$.*

Proof. Let $\phi\colon (M,\tau) \to (P,\sigma,\tau)$ be a contact isotropic realisation. The dimension of its fibres is 1 as it is equal to the corank of the Jacobi structure on P. By definition, $\ker \rho_P = \mathbb{R}\langle(\tau,-1)\rangle$; since the fibres of ϕ are compact and by smoothness of the period lattice, $\Sigma = \mathbb{Z}\langle g(\tau,-1)\rangle$ for some positive smooth function

$f \in C^\infty(P)$. Corollary 12 implies that there exists a smooth function λ such that $j^1\lambda = (g\tau, -g)$; in other words, $\tau = \mathrm{d}(-\log g)$, as required. □

The following result is stated without proof, as it follows from a simple calculation.

Lemma 27. *Let $(P, \sigma, \mathrm{d}f)$ be a globally conformal symplectic manifold. Then $\phi\colon (M, \theta) \to (P, \sigma, \mathrm{d}f)$ is a contact isotropic realisation if and only if $\phi\colon (M, \bar\theta) \to (P, \bar\sigma)$ is a contact isotropic realisation, where $\bar\theta \in \Omega^1(M)$ is the contact form $\bar\theta := e^f\theta$ and $\bar\sigma \in \Omega^2(P)$ is the symplectic form $\bar\sigma = e^f\sigma$.*

Proof of Proposition 25. Lemma 26 proves that property (25) in the statement of the proposition is a necessary condition for (P, σ, τ) to admit a contact isotropic realisation. Henceforth, without loss of generality, assume that $\tau = \mathrm{d}f$ and that the aim is to construct a complete isotropic realisation of $(P, \sigma, \mathrm{d}f)$ with period net $\Sigma = \mathbb{Z}\langle j^1 e^{-f}\rangle$ (all other cases follow by multiplying by a suitable constant). In this case, Lemma 27 gives that $\phi\colon (M, \theta) \to (P, \sigma, \mathrm{d}f)$ is a contact isotropic realisation if and only if $\phi\colon (M, \bar\theta) \to (P, \bar\sigma)$ is; by Example 15, such a realisation exists if and only if $[\bar\sigma]$ is integral, and $[\bar\sigma]$ is the Chern class of the S^1-bundle $\phi\colon M \to P$. Moreover, $\mathrm{d}\bar\theta = \phi^*\bar\sigma$, *i.e.*

$$\phi^*(e^f)(\mathrm{d}\theta + \phi^*\mathrm{d}f \wedge \theta) = \mathrm{d}\bar\theta = \phi^*\bar\sigma = \phi^*(e^f)\phi^*\sigma$$

which is equivalent to $\mathrm{d}\theta = \phi^*\sigma - \phi^*\mathrm{d}f a \wedge \theta$, and the infinitesimal generator of the action is given by the Reeb vector field of $\bar\theta$, which, written in terms of the bivector and Reeb vector field of θ is given by $e^{-f}(R_M - \Lambda_M^\sharp(\phi^*\mathrm{d}f))$ as required. □

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